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New Relaxation Method for Mathematical Programs with Complementarity Constraints[1]

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Abstract. In this paper, we present a new relaxation method for mathematical programs with complementarity constraints. Based on the fact that a variational inequality problem defined on a simplex can be represented by a finite number of inequalities, we use an expansive simplex instead of the nonnegative orthant involved in the complementarity constraints. We then remove some inequalities and obtain a standard nonlinear program. We show that the linear independence constraint qualification or the Mangasarian–Fromovitz constraint qualification holds for the relaxed problem under some mild conditions. We consider also a limiting behavior of the relaxed problem. We prove that any accumulation point of stationary points of the relaxed problems is a weakly stationary point of the original problem and that, if the function involved in the complementarity constraints does not vanish at this point, it is C-stationary. We obtain also some sufficient conditions of B-stationarity for a feasible point of the original problem. In particular, some conditions described by the eigenvalues of the Hessian matrices of the Lagrangian functions of the relaxed problems are new and can be verified easily. Our limited numerical experience indicates that the proposed approach is promising.

Key Words. Mathematical programs with equilibrium constraints, linear independence constraint qualification, nondegeneracy, weak stationarity, B-stationarity, C-stationarity, M-stationarity, second-order necessary conditions, upper level strict complementarity.

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1. Introduction

A mathematical program with equilibrium constraints (MPEC) is a constrained optimization problem in which the essential constraints are defined by a parametric variational inequality or complementarity system. This problem plays an important role in many fields such as engineering design, economic equilibrium, and multilevel games (see Ref. 1) and has attracted much attention in the recent literature (see Refs. 2–15).

The MPEC considered in this paper is a mathematical program with complementarity constraints,

$$\min \quad f(x, y), \tag{1a}$$

$$\text{s.t.} \quad g(x, y) \leq 0, \qquad h(x, y) = 0, \tag{1b}$$

$$F(x, y) \geq 0, \qquad y \geq 0, \tag{1c}$$

$$y^T F(x, y) = 0, \tag{1d}$$

where

$$f: R^{n+m} \to R, \qquad g: R^{n+m} \to R^p, \qquad h: R^{n+m} \to R^q, \qquad F: R^{n+m} \to R^m$$

are all twice continuously differentiable functions. The major difficulty in solving (1) is that its constraints fail to satisfy a standard constraint qualification at any feasible point (Ref. 16), which is necessary for the regularity of a nonlinear program, so that standard methods are likely to fail for this problem. There have been proposed several approaches such as sequential quadratic programming (SQP) approach, implicit programming approach, penalty function approach, and reformulation approach (Refs. 2–10). In particular, Fukushima and Pang (Ref. 2) considered a smoothing continuation method and showed that, under the MPEC linear independence constraint qualification and an additional condition called asymptotic weak nondegeneracy, an accumulation point of KKT points satisfying the second-order necessary conditions for the perturbed problems is a B-stationary point of the original problem. Subsequently, Scholtes (Ref. 3) presented a regularization scheme and proved, under the MPEC linear independence constraint qualification and the upper level strict complementarity condition, that an accumulation point of stationary points satisfying the second-order necessary conditions for the relaxed problems is a B-stationary point of the original problem. In addition, Fukushima, Luo, and Pang (Ref. 4) proposed a SQP algorithm and gave their convergence result under nondegeneracy. Recently, Fletcher et al. (Ref. 5) observed that SQP solvers can be applied effectively to solve MPECs and showed that the convergence is superlinear near a strongly stationary point under reasonable assumptions.

In this paper, we study problem (1) from another point of view. We use an expansive simplex instead of the nonnegative orthant involved in the complementarity constraints. In other words, our method replaces the complementarity constraints by a variational inequality defined on an expansive simplex. It is well known that such a variational inequality problem can be represented by a finite number of inequalities. We remove some inequalities and obtain a standard nonlinear program. We will show that the linear independence constraint qualification (LICQ) or the Mangasarian–Fromovitz constraint qualification (MFCQ) holds for the relaxed problem under some mild conditions. We consider also a limiting behavior of the relaxed problem. We will prove that any accumulation point of stationary points of the relaxed problems is a weakly stationary point of the original problem and, if the function F does not vanish at this point, it is C-stationary. Furthermore, if the Hessian matrices of the Lagrangian functions of the relaxed problems are uniformly bounded below on the corresponding tangent space, it is M-stationary. We obtain also some sufficient conditions of B-stationarity for a feasible point of the original problem. In particular, some conditions described by the eigenvalues of the Hessian matrices mentioned above are new and can be verified easily.

The rest of this paper is organized as follows. In Section 2, we present the relaxed problem for problem (1) and establish the convergence of global optimal solutions of the relaxed problems. In Section 3, we give some results about constraint qualifications and the convergence of stationary points of the relaxed problems. We report some numerical results in Section 4; then, in Section 5, we make some remarks to conclude the paper.

2. Relaxed Problem and Some Properties

Let

$$e = (1, 1, \ldots, 1)^T.$$

For $i = 1, 2, \ldots, m$, $e_i \in R^m$ denotes the ith column of the $m \times m$ identity matrix. Also, we let $e_0 \in R^m$ denote the zero vector, i.e.,

$$e_0 = (0, 0, \ldots, 0)^T.$$

Then, the expansive simplex mentioned in Section 1 is defined by

$$\Omega_k = \mathrm{co}\{e_0^k, e_1^k, \ldots, e_m^k\},$$

where co stands for the convex hull, k is a positive integer, and

$$e_i^k = (1/k)e + ke_i, \quad i = 0, 1, \ldots, m. \tag{2}$$

For a fixed $x \in R^n$, the variational inequality problem $\mathrm{VI}(F(x, \cdot), \Omega_k)$ is to find a vector $y \in \Omega_k$ such that

$$(y' - y)^T F(x, y) \geq 0, \qquad \forall y' \in \Omega_k,$$

which is equivalent to finding a $y \in R^m$ such that

$$y \in \Omega_k, \qquad (e_i^k - y)^T F(x, y) \geq 0, \qquad i = 0, 1, \ldots, m,$$

or equivalently,

$$\sum_{j=1}^{m} y_j \leq m/k + k, \qquad y_i \geq 1/k, \qquad (e_i^k - y)^T F(x, y) \geq 0,$$
$$i = 0, 1, \ldots, m.$$

In order to simplify the relaxed problem, we replace the condition $y \in \Omega_k$ by $y \geq 0$ and consider the following problem as an approximation of problem (1):

$$\min \quad f(x, y), \tag{3a}$$
$$\text{s.t.} \quad g(x, y) \leq 0, \qquad h(x, y) = 0, \qquad y \geq 0, \tag{3b}$$
$$(e_i^k - y)^T F(x, y) \geq 0, \qquad i = 0, 1, \ldots, m. \tag{3c}$$

Let $\mathscr{F}$ and $\mathscr{F}_k$ denote the feasible sets of problems (1) and (3), respectively, and let

$$\phi_i^k(x, y) = (e_i^k - y)^T F(x, y), \qquad i = 0, 1, \ldots, m. \tag{4}$$

By (2), we have

$$\phi_i^k(x, y) = \phi_0^k(x, y) + kF_i(x, y), \qquad i = 1, 2, \ldots, m. \tag{5}$$

Then, we have the following results.

Theorem 2.1. For problems (1) and (3), the following statements hold:

(i) for any k, $\mathscr{F} \subseteq \mathscr{F}_{k+1} \subseteq \mathscr{F}_k$;

(ii) $\mathscr{F} = \bigcap_{k=1}^{\infty} \mathscr{F}_k$; together with the continuity of the functions involved, this implies that any accumulation point of a sequence $\{(x^k, y^k): (x^k, y^k) \in \mathscr{F}_k\}$ belongs to $\mathscr{F}$.

Proof.

(i) The fact that $\mathscr{F} \subseteq \mathscr{F}_{k+1}$ is clear. Let $(x, y) \in \mathscr{F}_{k+1}$. Then, since for each $i = 0, 1, \ldots, m$, e_i^k can be represented as

$$e_i^k = \sum_{j=0}^{m} t_{ij} e_j^{k+1}, \qquad \sum_{j=0}^{m} t_{ij} = 1, \qquad t_{ij} \geq 0, \qquad j = 0, 1, \ldots, m,$$

we have

$$(e_i^k - y)^T F(x, y) = \sum_{j=0}^{m} t_{ij}(e_j^{k+1} - y)^T F(x, y) \geq 0, \qquad i = 0, 1, \ldots, m,$$

i.e., $(x, y) \in \mathscr{F}_k$. Hence, $\mathscr{F}_{k+1} \subseteq \mathscr{F}_k$.

(ii) From (i), we need only to prove that $\bigcap_{k=1}^{\infty} \mathscr{F}_k \subseteq \mathscr{F}$. Let $(x, y) \in \bigcap_{k=1}^{\infty} \mathscr{F}_k$. Then, we have

$$g(x, y) \leq 0, \qquad h(x, y) = 0, \qquad y \geq 0$$

and, for every $i = 1, 2, \ldots, m$,

$$\phi_i^k(x, y) = ((1/k)e + ke_i - y)^T F(x, y) \geq 0, \qquad \forall k,$$

which implies

$$((1/k^2)e + e_i - (1/k)y)^T F(x, y) \geq 0, \qquad \forall k. \tag{6}$$

Letting $k \to \infty$ in (6), we have

$$F_i(x, y) = e_i^T F(x, y) \geq 0,$$

and hence $F(x, y) \geq 0$. On the other hand,

$$(e_0^k - y)^T F(x, y) \geq 0, \qquad \forall k,$$

implies

$$-y^T F(x, y) \geq 0.$$

So, we have

$$y^T F(x, y) = 0.$$

Therefore, $(x, y) \in \mathscr{F}$ and so $\bigcap_{k=1}^{\infty} \mathscr{F}_k \subseteq \mathscr{F}$. This completes the proof. □

Theorem 2.2. Suppose that (x^k, y^k) is a global optimal solution of problem (3) and that (x^*, y^*) is an accumulation point of the sequence $\{(x^k, y^k)\}$ as $k \to \infty$. Then, (x^*, y^*) is a global optimal solution of problem (1).

Proof. Taking a subsequence if necessary, we assume without loss of generality that

$$\lim_{k \to \infty} (x^k, y^k) = (x^*, y^*).$$

By Theorem 2.1, $(x^*, y^*) \in \mathscr{F}$. Since $\mathscr{F} \subseteq \mathscr{F}_k$ for all k, then

$$f(x^k, y^k) \leq f(x, y), \qquad \forall (x, y) \in \mathscr{F}, \qquad \forall k.$$

Letting $k \to \infty$, we have from the continuity of f that

$$f(x^*, y^*) \le f(x, y), \qquad \forall (x, y) \in \mathcal{F};$$

i.e., (x^*, y^*) is a global optimal solution of problem (1). □

In a similar way, we can prove the next theorem.

Theorem 2.3. Let $\{\epsilon_k\} \subseteq (0,+\infty)$ be convergent to 0 and let $(x^k, y^k) \in \mathcal{F}_k$ be an approximate solution of problem (3) satisfying

$$f(x^k, y^k) - \epsilon_k \le f(x, y), \qquad \forall (x, y) \in \mathcal{F}_k.$$

Then, any accumulation point of $\{(x^k, y^k)\}$ is a global optimal solution of problem (1).

The following result shows that problem (3) may satisfy some constraint qualification at its feasible points. This is in contrast with problem (1), for which a standard contraint qualification fails to hold at any feasible point.

Theorem 2.4. For any $(\bar{x}, \bar{y}) \in \mathcal{F}$ with $F(\bar{x}, \bar{y}) \ne 0$, we have

$$\phi_i^k(\bar{x}, \bar{y}) > 0, \qquad i = 0, 1, \ldots, m, \qquad \forall k,$$

and so they are inactive constraints at $(\bar{x}, \bar{y})$ in problem (3). In this case, if the system

$$g(x, y) \le 0, \qquad h(x, y) = 0, \qquad y \ge 0$$

satisfies some constraint qualification such as LICQ or MFCQ at $(\bar{x}, \bar{y})$, then for any fixed k, there exists a neighborhood $U_k(\bar{x}, \bar{y})$ of $(\bar{x}, \bar{y})$ such that problem (3) satisfies the same constraint qualification at any point $(x, y) \in U_k(\bar{x}, \bar{y})$.

Proof. We obtain the first part from the definition (4) of ϕ_i^k and

$$e_i^k > 0, \qquad 0 \ne F(\bar{x}, \bar{y}) \ge 0, \qquad \bar{y}^T F(\bar{x}, \bar{y}) = 0,$$

immediately. The second part follows from the continuity of g, h, ϕ_i^k, $i = 0, 1, \ldots, m$, and their gradients directly. □

3. Limiting Behavior of Stationary Points

In this section, we consider the behavior of a stationary point of problem (3) as $k \to \infty$. We let $G(x, y) = y$ and

$$\phi^k(x, y) = (\phi_0^k(x, y), \phi_1^k(x, y), \ldots, \phi_m^k(x, y))^T,$$

where ϕ_i^k are defined by (4). As in the previous section, we denote the feasible sets of problems (1) and (3) by $\mathcal{F}$ and $\mathcal{F}_k$, respectively. Note that the gradients of $G_j, j=1,\dots,m$, are constant vectors. Nevertheless, we will often write $\nabla G_j(x,y)$, etc., to specify the point under consideration. In addition, for a function H: $R^{n+m}\to R^m$ and a fixed vector $z\in R^{n+m}$, we let

$$\mathcal{I}_H(z)=\{i: H_i(z)=0\}$$

denote the active index set of H at z.

Theorem 3.1. For any $(\bar{x},\bar{y})\in\mathcal{F}$, if the set of vectors

$$\{\nabla F_i(\bar{x},\bar{y}), \nabla G_i(\bar{x},\bar{y}), \nabla g_l(\bar{x},\bar{y}), \nabla h_r(\bar{x},\bar{y}):$$
$$i=1,2,\dots,m,\ l\in\mathcal{I}_g(\bar{x},\bar{y}),\ r=1,2,\dots,q\}$$

is linearly independent, then there exist a neighborhood $U(\bar{x},\bar{y})$ of $(\bar{x},\bar{y})$ and a positive integer K such that, for any $(x,y)\in U(\bar{x},\bar{y})$ and any $k\geq K$, the following conditions hold:

(i) $\mathcal{I}_F(x,y)\subseteq\mathcal{I}_F(\bar{x},\bar{y})$, $\mathcal{I}_G(x,y)\subseteq\mathcal{I}_G(\bar{x},\bar{y})$, $\mathcal{I}_g(x,y)\subseteq\mathcal{I}_g(\bar{x},\bar{y})$;
(ii) the set of vectors $\{\nabla F_i(x,y), \nabla G_i(x,y), \nabla g_l(x,y), \nabla h_r(x,y)$: $i=1,2,\dots,m,\ l\in\mathcal{I}_g(\bar{x},\bar{y}),\ r=1,2,\dots,q\}$ is linearly independent;
(iii) $\mathcal{I}_{\phi^k}(x,y)\subseteq\{0\}\cup\mathcal{I}_F(\bar{x},\bar{y})$.

Proof. It is obvious that there exists a neighborhood $U_1(\bar{x},\bar{y})$ such that conditions (i) and (ii) hold for any $(x,y)\in U_1(\bar{x},\bar{y})$ by the continuity of $F, G, g, \nabla F, \nabla g, \nabla h$. Now, we show that there exist a neighborhood $U_2(\bar{x},\bar{y})$ and a positive integer K satisfying condition (iii) for any $(x,y)\in U_2(\bar{x},\bar{y})$ and any $k\geq K$. Otherwise, there must be an $i_0\notin\{0\}\cup\mathcal{I}_F(\bar{x},\bar{y})$, a subsequence $\{k_j\}$ of $\{k\}$, and a sequence $\{(x^j,y^j)\}$ converging to $(\bar{x},\bar{y})$ such that

$$\phi_{i_0}^{k_j}(x^j,y^j)=0, \qquad \forall j.$$

Since

$$(1/k_j)\phi_{i_0}^{k_j}(x^j,y^j)$$
$$=F_{i_0}(x^j,y^j)+(1/k_j^2)\sum_{i=1}^m F_i(x^j,y^j)-(1/k_j)(y^j)^T F(x^j,y^j),$$

we have

$$\lim_{j\to\infty}(1/k_j)\phi_{i_0}^{k_j}(x^j,y^j)=F_{i_0}(\bar{x},\bar{y})>0.$$

This implies that

$$\lim_{j\to\infty} \phi_{i_0}^{k_j}(x^j, y^j) = +\infty.$$

This is a contradiction and so the neighborhood $U_2(\bar{x}, \bar{y})$ and positive integer K mentioned above exist. Let

$$U(\bar{x}, \bar{y}) = U_1(\bar{x}, \bar{y}) \cap U_2(\bar{x}, \bar{y}).$$

Then, conditions (i)–(iii) hold for any $(x, y) \in U(\bar{x}, \bar{y})$ and any $k \geq K$. □

As we mentioned in Section 1, the nondegeneracy condition has often been assumed in the literature on MPECs. In general, a point $(x, y) \in \mathcal{F}$ is said to be nondegenerate if

$$\mathcal{I}_F(x, y) \cap \mathcal{I}_G(x, y) = \emptyset.$$

Then, we have the following result about constraint qualifications.

Theorem 3.2. Let $(\bar{x}, \bar{y}) \in \mathcal{F}$ be nondegenerate and satisfy

$$F(\bar{x}, \bar{y}) \neq 0. \tag{7}$$

If the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$
$$i = 1, 2, \ldots, m, l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent, then there exists a neighborhood $U(\bar{x}, \bar{y})$ of $(\bar{x}, \bar{y})$ such that, for any sufficiently large k, problem (3) satisfies the standard LICQ at any point $(x, y) \in U(\bar{x}, \bar{y}) \cap \mathcal{F}_k$.

Proof. By Theorem 3.1, there exist a neighborhood $U(\bar{x}, \bar{y})$ and a positive integer K such that Theorem 3.1 (i)–(iii) hold for any $(x, y) \in U(\bar{x}, \bar{y})$ and any $k \geq K$. Now, we let $k \geq K$ and choose an arbitrary point $(x, y) \in U(\bar{x}, \bar{y}) \cap \mathcal{F}_k$.

Suppose that the LICQ does not hold at (x, y) for problem (3). This means that the set of vectors

$$\{\nabla \phi_i^k(x, y), \nabla G_j(x, y), \nabla g_l(x, y), \nabla h_r(x, y):$$
$$i \in \mathcal{I}_{\phi^k}(x, y), j \in \mathcal{I}_G(x, y), l \in \mathcal{I}_g(x, y), r = 1, 2, \ldots, q\},$$

which is, by (5),

$$\{\nabla \phi_0^k(x, y), k\nabla F_i(x, y) + \nabla \phi_0^k(x, y), \nabla G_j(x, y), \nabla g_l(x, y), \nabla h_r(x, y):$$
$$0 \neq i \in \mathcal{I}_{\phi^k}(x, y), j \in \mathcal{I}_G(x, y), l \in \mathcal{I}_g(x, y), r = 1, 2, \ldots, q\},$$

in the case where $0 \in \mathscr{I}_{\phi^k}(x, y)$, or

$$\{k\nabla F_i(x, y) + \nabla\phi_0^k(x, y), \nabla G_j(x, y), \nabla g_l(x, y), \nabla h_r(x, y):$$

$$i \in \mathscr{I}_{\phi^k}(x, y), j \in \mathscr{I}_G(x, y), l \in \mathscr{I}_g(x, y), r = 1, 2, \ldots, q\},$$

in the case where $0 \notin \mathscr{I}_{\phi^k}(x, y)$, is linearly dependent. Hence, by Theorem 3.1 (ii), $\nabla\phi_0^k(x, y)$ can be represented as a linear combination of the vectors

$$\{\nabla F_i(x, y), \nabla G_j(x, y), \nabla g_l(x, y), \nabla h_r(x, y):$$

$$i \in \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}, j \in \mathscr{I}_G(x, y), l \in \mathscr{I}_g(x, y), r = 1, 2, \ldots, q\}.$$

Therefore, there exist numbers

$$\{\lambda_i, \mu_j, u_l, v_r: i \in \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}, j \in \mathscr{I}_G(x, y), l \in \mathscr{I}_g(x, y), r = 1, 2, \ldots, q\}$$

such that

$$\nabla\phi_0^k(x, y) = \sum_{i \in \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}} \lambda_i \nabla F_i(x, y) + \sum_{j \in \mathscr{I}_G(x, y)} \mu_j \nabla G_j(x, y)$$
$$+ \sum_{l \in \mathscr{I}_g(x, y)} u_l \nabla g_l(x, y) + \sum_{r=1}^{q} v_r \nabla h_r(x, y).$$

Since

$$\phi_0^k(x, y) = ((1/k)e - y)^T F(x, y),$$

we then have

$$\sum_{i \in \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}} (y_i - 1/k + \lambda_i)\, \nabla F_i(x, y) + \sum_{i \notin \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}} (y_i - 1/k)\, \nabla F_i(x, y)$$
$$+ \sum_{j \in \mathscr{I}_G(x, y)} (\mu_j + F_j(x, y))\, \nabla G_j(x, y) + \sum_{j \notin \mathscr{I}_G(x, y)} F_j(x, y)\, \nabla G_j(x, y)$$
$$+ \sum_{l \in \mathscr{I}_g(x, y)} u_l \nabla g_l(x, y) + \sum_{r=1}^{q} v_r \nabla h_r(x, y) = 0.$$

By Theorem 3.1 (ii), we have

$$y_i = \begin{cases} 1/k - \lambda_i & i \in \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}, \\ 1/k, & i \notin \mathscr{I}_{\phi^k}(x, y)\backslash\{0\}, \end{cases} \tag{8}$$

and

$$F_i(x, y) = 0, \; i \notin \mathscr{I}_G(x, y). \tag{9}$$

Suppose that $\phi_0^k(x, y) = 0$. Then, we have by (9) that

$$\phi_i^k(x, y) = k\, F_i(x, y) + \phi_0^k(x, y) = 0, \qquad i \notin \mathscr{I}_G(x, y). \tag{10}$$

On the other hand, we have

$$\mathcal{I}_G(x, y) = \mathcal{I}_G(\bar{x}, \bar{y}). \tag{11}$$

Otherwise, since

$$\mathcal{I}_G(x, y) \subseteq \mathcal{I}_G(\bar{x}, \bar{y}),$$

there exists an $i_0 \in \mathcal{I}_G(\bar{x}, \bar{y}) \setminus \mathcal{I}_G(x, y)$. Then, we must have $\phi^k_{i_0}(x, y) = 0$ by (10). But, by the nondegenerate property of $(\bar{x}, \bar{y})$, $i_0 \in \mathcal{I}_G(\bar{x}, \bar{y})$ means $i_0 \notin \mathcal{I}_F(\bar{x}, \bar{y})$. So, by Theorem 3.1 (iii), we have $\phi^k_{i_0}(x, y) > 0$. This is a contradiction, and so (11) holds. This means that

$$y_i = 0, \qquad i \in \mathcal{I}_G(\bar{x}, \bar{y}). \tag{12}$$

Note that $\mathcal{I}_G(\bar{x}, \bar{y}) \neq \emptyset$ by (7). So, if $i \in \mathcal{I}_G(\bar{x}, \bar{y})$, then $i \notin \mathcal{I}_F(\bar{x}, \bar{y})$ by the nondegeneracy assumption and so $i \notin \mathcal{I}_{\phi^k}(x, y) \setminus \{0\}$ by Theorem 3.1 (iii). Hence, from (8), we have $y_i = 1/k$. This contradicts (12). Therefore, we have $\phi^k_0(x, y) \neq 0$ and so, by (9),

$$\phi^k_i(x, y) = kF_i(x, y) + \phi^k_0(x, y) \neq 0, \qquad i \notin \mathcal{I}_G(x, y). \tag{13}$$

By Theorem 3.1 (iii) and the fact that $0 \notin \mathcal{I}_{\phi^k}(x, y)$, $\phi^k_i(x, y) = 0$ means that $i \in \mathcal{I}_F(\bar{x}, \bar{y})$. On the other hand, for any $i \in \mathcal{I}_F(\bar{x}, \bar{y})$, by the nondegeneracy of $(\bar{x}, \bar{y})$, we have $i \notin \mathcal{I}_G(\bar{x}, \bar{y})$ and so $i \notin \mathcal{I}_G(x, y)$ by Theorem 3.1 (i), which implies $\phi^k_i(x, y) \neq 0$ by (13). Hence, $\mathcal{I}_{\phi^k}(x, y) = \emptyset$; i.e., the last $m+1$ inequality constraints in problem (3) are all inactive at (x, y) and so, by Theorem 3.1 (ii), the LICQ holds at (x, y). This also contradicts our assumption. Therefore, the LICQ holds at (x, y) for problem (3). This completes the proof. □

Now, we consider the limiting behavior of stationary points of problem (3). We will use the standard definition of stationarity for problem (3), i.e., $(x^k, y^k) \in \mathcal{F}_k$ is a stationary point of problem (3) if there exist Lagrange multiplier vectors λ^k, μ^k, δ^k, and γ^k such that

$$\nabla f(x^k, y^k) - \sum_{i \in \mathcal{I}_{\phi^k}(x^k, y^k)} \lambda^k_i \nabla \phi^k_i(x^k, y^k) - \sum_{j \in \mathcal{I}_G(x^k, y^k)} \mu^k_j \nabla G_j(x^k, y^k)$$
$$+ \sum_{l \in \mathcal{I}_g(x^k, y^k)} \delta^k_l \nabla g_l(x^k, y^k) + \sum_{r=1}^{q} \gamma^k_r \nabla h_r(x^k, y^k) = 0 \tag{14}$$

and

$$\lambda^k \geq 0, \qquad \mu^k \geq 0, \qquad \delta^k \geq 0. \tag{15}$$

For problem (1), $(\bar{x}, \bar{y}) \in \mathcal{F}$ is said to be a B-stationary point if it satisfies

$$d^T \nabla f(\bar{x}, \bar{y}) \geq 0, \qquad \forall d \in \mathcal{T}((\bar{x}, \bar{y}), \mathcal{F}), \tag{16}$$

where $\mathcal{T}((\bar{x}, \bar{y}), \mathcal{F})$ stands for the tangent cone of $\mathcal{F}$ at $(\bar{x}, \bar{y})$. As in Ref. 3, a feasible point $(\bar{x}, \bar{y})$ is called weakly stationary to problem (1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}, \bar{\delta} \geq 0$ such that

$$\nabla f(\bar{x}, \bar{y}) - \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} \bar{\lambda}_i \nabla F_i(\bar{x}, \bar{y}) - \sum_{j \in \mathcal{I}_G(\bar{x}, \bar{y})} \bar{\mu}_j \nabla G_j(\bar{x}, \bar{y})$$

$$+ \sum_{l \in \mathcal{I}_g(\bar{x}, \bar{y})} \bar{\delta}_l \nabla g_l(\bar{x}, \bar{y}) + \sum_{r=1}^{q} \bar{\gamma}_r \nabla h_r(\bar{x}, \bar{y}) = 0. \tag{17}$$

If the MPEC–LICQ holds at $(\bar{x}, \bar{y})$, which means that the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_j(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$

$$i \in \mathcal{I}_F(\bar{x}, \bar{y}), j \in \mathcal{I}_G(\bar{x}, \bar{y}), l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent, then the B-stationarity (16) is equivalent to the strong stationarity to problem (1). In general, $(\bar{x}, \bar{y})$ is said to be a strongly stationary point of problem (1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}, \bar{\delta} \geq 0$ such that (17) holds with

$$\bar{\lambda}_i \geq 0, \qquad \bar{\mu}_i \geq 0, \qquad i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y}). \tag{18}$$

Two other kinds of stationarity concepts for MPECs, called C-stationarity and M-stationarity (Ref. 15), which are weaker than B-stationarity, are employed also often. We say $(\bar{x}, \bar{y})$ is C-stationary to problem (1) if there exist multiplier vectors $\bar{\lambda}, \bar{\mu}, \bar{\gamma}, \bar{\delta} \geq 0$ such that (17) holds and

$$\bar{\lambda}_i \bar{\mu}_i \geq 0, \qquad i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y}), \tag{19}$$

and we say $(\bar{x}, \bar{y})$ is M-stationary to problem (1) if, furthermore, either $\bar{\lambda}_i > 0, \bar{\mu}_i > 0$ or $\bar{\lambda}_i \bar{\mu}_i = 0$ for all $i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y})$. In addition, a weakly stationary point $(\bar{x}, \bar{y}) \in \mathcal{F}$ of problem (1) is said to satisfy the upper level strict complementarity condition if there exist multiplier vectors $\bar{\lambda}$, $\bar{\mu}$, $\bar{\gamma}$, $\bar{\delta} \geq 0$ satisfying (17) and

$$\bar{\lambda}_i \bar{\mu}_i \neq 0, \qquad i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y}). \tag{20}$$

Theorem 3.3. Let $(x^k, y^k) \in \mathcal{F}_k$ be a stationary point of problem (3) with Lagrange multiplier vectors $\lambda^k, \mu^k, \delta^k, \gamma^k$ satisfying (14)–(15), and let

$(\bar{x}, \bar{y})$ be an accumulation point of the sequence $\{(x^k, y^k)\}$. Suppose that the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$
$$i = 1, 2, \ldots, m, l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent. Then, we have the following statements.

(a) $(\bar{x}, \bar{y})$ is a weakly stationary point of problem (1) and, if $F(\bar{x}, \bar{y}) \neq 0$, $(\bar{x}, \bar{y})$ is C-stationary. Especially, if $(\bar{x}, \bar{y})$ is nondegenerate, it is B-stationary.
(b) If $(x^k, y^k) \in \mathcal{F}$ for some k, then (x^k, y^k) is B-stationary to problem (1) and, if $(x^k, y^k) \in \mathcal{F}$ for infinitely many k, $(\bar{x}, \bar{y})$ is B-stationary.
(c) If $0 \notin \mathcal{I}_{\phi^k}(x^k, y^k)$ for infinitely many k, then $(\bar{x}, \bar{y})$ is a B-stationary point to problem (1).

Proof. Without loss of generality, we assume that

$$\lim_{k \to \infty} (x^k, y^k) = (\bar{x}, \bar{y}). \tag{21}$$

Then, by Theorem 2.1, we have $(\bar{x}, \bar{y}) \in \mathcal{F}$. By Theorem 3.1, for any sufficiently large k, we have

$$\mathcal{I}_F(x^k, y^k) \subseteq \mathcal{I}_F(\bar{x}, \bar{y}), \qquad \mathcal{I}_G(x^k, y^k) \subseteq \mathcal{I}_G(\bar{x}, \bar{y}),$$
$$\mathcal{I}_g(x^k, y^k) \subseteq \mathcal{I}_g(\bar{x}, \bar{y}), \qquad \mathcal{I}_{\phi^k}(x^k, y^k) \subseteq \{0\} \cup \mathcal{I}_F(\bar{x}, \bar{y}),$$

and the set of vectors

$$\{\nabla F_i(x^k, y^k), \nabla G_i(x^k, y^k), \nabla g_l(x^k, y^k), \nabla h_r(x^k, y^k):$$
$$i = 1, 2, \ldots, m, l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent. Note that the MPEC–LICQ holds at $(\bar{x}, \bar{y})$ for problem (1).

Since

$$\nabla \phi_i^k(x^k, y^k) = \nabla \phi_0^k(x^k, y^k) + k \nabla F_i(x^k, y^k), \qquad i = 1, 2, \ldots, m,$$

and

$$\nabla \phi_0^k(x^k, y^k) = \sum_{i=1}^m (1/k - y_i^k) \nabla F_i(x^k, y^k) - \sum_{j=1}^m F_j(x^k, y^k) \nabla G_j(x^k, y^k), \tag{22}$$

we have from (14) that

$$
\begin{aligned}
&\nabla f(x^k, y^k)\\
&= \sum_{i\in \mathcal{I}_{\phi^k}(x^k,y^k)} \lambda_i^k \nabla \phi_i^k(x^k, y^k) + \sum_{j\in \mathcal{I}_G(x^k,y^k)} \mu_j^k \nabla G_j(x^k, y^k)\\
&\quad - \sum_{l\in \mathcal{I}_g(x^k,y^k)} \delta_l^k \nabla g_l(x^k, y^k) - \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k)\\
&= \sum_{0\neq i\in \mathcal{I}_{\phi^k}(x^k,y^k)} k\lambda_i^k \nabla F_i(x^k, y^k) + a_k \nabla \phi_0^k(x^k, y^k) + \sum_{j\in \mathcal{I}_G(x^k,y^k)} \mu_j^k \nabla G_j(x^k, y^k)\\
&\quad - \sum_{l\in \mathcal{I}_g(x^k,y^k)} \delta_l^k \nabla g_l(x^k, y^k) - \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k) \qquad (23)\\
&= \sum_{i\in \mathcal{I}_F(\bar{x},\bar{y})} u_i^k \nabla F_i(x^k, y^k) + \sum_{i\notin \mathcal{I}_F(\bar{x},\bar{y})} a_k(1/k - y_i^k)\nabla F_i(x^k, y^k)\\
&\quad + \sum_{j\in \mathcal{I}_G(\bar{x},\bar{y})} v_j^k \nabla G_j(x^k, y^k) - \sum_{j\notin \mathcal{I}_G(\bar{x},\bar{y})} a_k F_j(x^k, y^k)\nabla G_j(x^k, y^k)\\
&\quad - \sum_{l\in \mathcal{I}_g(\bar{x},\bar{y})} w_l^k \nabla g_l(x^k, y^k) - \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k), \qquad (24)
\end{aligned}
$$

where

$$
a_k = \begin{cases} \sum_{i\in \mathcal{I}_{\phi^k}(x^k,y^k)} \lambda_i^k, & \mathcal{I}_{\phi^k}(x^k, y^k) \neq \emptyset,\\ 0, & \mathcal{I}_{\phi^k}(x^k, y^k) = \emptyset, \end{cases} \tag{25}
$$

$$
u_i^k = \begin{cases} k\lambda_i^k + a_k(1/k - y_i^k), & 0\neq i\in \mathcal{I}_{\phi^k}(x^k, y^k),\\ a_k(1/k - y_i^k), & i\in \mathcal{I}_F(\bar{x}, \bar{y})\backslash \mathcal{I}_{\phi^k}(x^k, y^k), \end{cases} \tag{26}
$$

$$
v_j^k = \begin{cases} \mu_j^k - a_k F_j(x^k, y^k), & j\in \mathcal{I}_G(x^k, y^k),\\ -a_k F_j(x^k, y^k), & j\in \mathcal{I}_G(\bar{x}, \bar{y})\backslash \mathcal{I}_G(x^k, y^k), \end{cases} \tag{27}
$$

and

$$
w_l^k = \begin{cases} \delta_l^k, & l\in \mathcal{I}_g(x^k, y^k),\\ 0, & l\in \mathcal{I}_g(\bar{x}, \bar{y})\backslash \mathcal{I}_g(x^k, y^k). \end{cases}
$$

Since the set of vectors

$$
\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):
$$

$$
i = 1, 2, \ldots, m, l\in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}
$$

is linearly independent, it follows from (21) and (24) that the multiplier sequences

$$\{u_i^k : i \in \mathcal{I}_F(\bar{x}, \bar{y})\}, \qquad \{v_j^k : j \in \mathcal{I}_G(\bar{x}, \bar{y})\}, \tag{28a}$$

$$\{a_k(1/k - y_i^k) : i \notin \mathcal{I}_F(\bar{x}, \bar{y})\}, \tag{28b}$$

$$\{-a_k F_j(x^k, y^k) : j \notin \mathcal{I}_G(\bar{x}, \bar{y})\}, \qquad \{w_l^k : l \in \mathcal{I}_g(\bar{x}, \bar{y})\}, \tag{28c}$$

$$\{\gamma_r^k : r = 1, 2, \ldots, q\} \tag{28d}$$

are convergent. Next, we will consider several cases to prove statements (a)–(c).

(I) First, we show that, if $(x^k, y^k) \in \mathcal{F}$ for some k, then it is a B-stationary point of problem (1). In fact, if $F(x^k, y^k) \neq 0$, then, from Theorem 2.4, $\mathcal{I}_{\phi^k}(x^k, y^k) = \emptyset$ and so (14)–(15) mean that (x^k, y^k) is a B-stationary point of problem (1). If $F(x^k, y^k) = 0$, then $\mathcal{I}_{\phi^k}(x^k, y^k) = \{0, 1, \ldots, m\}$ and so, it follows from (22) and (23) that

$$\begin{aligned}
0 &= \nabla f(x^k, y^k) - \sum_{i=1}^{m} k\lambda_i^k \nabla F_i(x^k, y^k) - a_k \nabla \phi_0^k(x^k, y^k) \\
&\quad - \sum_{j \in \mathcal{I}_G(x^k, y^k)} \mu_j^k \nabla G_j(x^k, y^k) + \sum_{l \in \mathcal{I}_g(x^k, y^k)} \delta_l^k \nabla g_l(x^k, y^k) \\
&\quad + \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k) \\
&= \nabla f(x^k, y^k) - \sum_{i \in \mathcal{I}_F(x^k, y^k)} (a_k(1/k - y_i^k) + k\lambda_i^k) \nabla F_i(x^k, y^k) \\
&\quad - \sum_{j \in \mathcal{I}_G(x^k, y^k)} \mu_j^k \nabla G_j(x^k, y^k) + \sum_{l \in \mathcal{I}_g(x^k, y^k)} \delta_l^k \nabla g_l(x^k, y^k) \\
&\quad + \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k).
\end{aligned}$$

For $i \in \mathcal{I}_F(x^k, y^k) \cap \mathcal{I}_G(x^k, y^k)$, we have from (15) and (25) that

$$a_k(1/k - y_i^k) + k\lambda_i^k = (1/k)a_k + k\lambda_i^k \geq 0, \qquad \mu_i^k \geq 0,$$

and hence, comparing with (17) and (18), we see that (x^k, y^k) is a B-stationary point of problem (1). This shows the first half of statement (b). Next, we suppose that $(x^{k'}, y^{k'}) \in \mathcal{F}$ for infinitely many k' and show that $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1). In fact, since for any sufficiently large k',

$$\mathcal{I}_F(x^{k'}, y^{k'}) \subseteq \mathcal{I}_F(\bar{x}, \bar{y}), \qquad \mathcal{I}_G(x^{k'}, y^{k'}) \subseteq \mathcal{I}_G(\bar{x}, \bar{y}),$$
$$\mathcal{I}_g(x^{k'}, y^{k'}) \subseteq \mathcal{I}_g(\bar{x}, \bar{y}),$$

we have

$$\nabla f(x^{k'}, y^{k'}) = \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} \hat{u}_i^{k'} \nabla F_i(x^{k'}, y^{k'}) + \sum_{j \in \mathcal{I}_G(\bar{x}, \bar{y})} \hat{v}_j^{k'} \nabla G_j(x^{k'}, y^{k'})$$
$$- \sum_{l \in \mathcal{I}_g(\bar{x}, \bar{y})} \hat{w}_l^{k'} \nabla g_l(x^{k'}, y^{k'}) - \sum_{r=1}^{q} \hat{\gamma}_r^{k'} \nabla h_r(x^{k'}, y^{k'}).$$

By the assumptions of the theorem, the multiplier sequences converge. Letting $k' \to \infty$, we have the B-stationarity of $(\bar{x}, \bar{y})$. This shows the second half of statement (b).

(II) Next, we assume that $(x^k, y^k) \notin \mathcal{F}$ for all sufficiently large k.

(IIa) We consider the case where $\mathcal{I}_F(\bar{x}, \bar{y}) \neq \emptyset$.

(i) We first prove statement (c); i.e., if there is a subsequence $\{k_l\}$ of $\{k\}$ such that $0 \notin \mathcal{I}_{\phi^{k_l}}(x^{k_l}, y^{k_l})$ for all l, then $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1). In fact, noting that, by (25) and (26),

$$\sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} u_i^{k_l} = \sum_{i \in \mathcal{I}_{\phi^{k_l}}(x^{k_l}, y^{k_l})} (k_l \lambda_i^{k_l} + a_{k_l}(1/k_l - y_i^{k_l}))$$
$$+ \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y}) \setminus \mathcal{I}_{\phi^{k_l}}(x^{k_l}, y^{k_l})} a_{k_l}(1/k_l - y_i^{k_l})$$
$$= a_{k_l}\left[k_l + \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} (1/k_l - y_i^{k_l})\right],$$

and that

$$\lim_{l \to \infty} \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} u_i^{k_l} \text{ exists},$$

$$\lim_{l \to \infty} \left[k_l + \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} (1/k_l - y_i^{k_l})\right] = +\infty,$$

we have that

$$\lim_{l \to \infty} a_{k_l} = 0. \tag{29}$$

Therefore, we obtain

$$\lim_{k \to \infty} a_k(1/k - y_i^k) = \lim_{l \to \infty} a_{k_l}(1/k_l - y_i^{k_l}) = 0, \qquad i \notin \mathcal{I}_F(\bar{x}, \bar{y}), \tag{30}$$

and

$$\lim_{k\to\infty} a_k F_j(x^k, y^k) = \lim_{l\to\infty} a_{k_l} F_j(x^{k_l}, y^{k_l}) = 0, \qquad j \notin \mathscr{I}_G(\bar{x}, \bar{y}). \tag{31}$$

On the other hand, by (15), (29), and (26)–(27), we have

$$\lim_{k\to\infty} u_i^k \geq 0, \qquad \lim_{k\to\infty} v_i^k \geq 0, \qquad i \in \mathscr{I}_G(\bar{x}, \bar{y}) \cap \mathscr{I}_F(\bar{x}, \bar{y}). \tag{32}$$

It then follows from (24) and (30)–(32) that conditions (17) and (18) hold. Therefore, $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1). This shows statement (c). The rest of the proof will be devoted to showing statement (a).

(ii) Suppose that $0 \in \mathscr{I}_{\phi^k}(x^k, y^k)$ for all sufficiently large k. Then, it follows from (5) that

$$\mathscr{I}_{\phi^k}(x^k, y^k) = \{0\} \cup \mathscr{I}_F(x^k, y^k). \tag{33}$$

(iia) If there exist a subsequence $\{k_l\}$ of $\{k\}$ and an index i_0 such that

$$i_0 \notin \mathscr{I}_G(\bar{x}, \bar{y}), \qquad i_0 \in \mathscr{I}_F(\bar{x}, \bar{y}) \backslash \mathscr{I}_F(x^{k_l}, y^{k_l}), \qquad \forall l,$$

or

$$i_0 \notin \mathscr{I}_F(\bar{x}, \bar{y}), \qquad i_0 \in \mathscr{I}_G(\bar{x}, \bar{y}) \backslash \mathscr{I}_G(x^{k_l}, y^{k_l}), \qquad \forall l,$$

then, by (26) and (27),

$$u_{i_0}^{k_l} = a_{k_l}(1/k_l - y_{i_0}^{k_l}), \qquad \forall l, \tag{34}$$

or

$$v_{i_0}^{k_l} = -a_{k_l} F_{i_0}(x^{k_l}, y^{k_l}), \qquad \forall l, \tag{35}$$

holds. Since

$$\lim_{l\to\infty} (1/k_l - y_{i_0}^{k_l}) = -\bar{y}_{i_0} < 0,$$

in the former case, or

$$\lim_{l\to\infty} F_{i_0}(x^{k_l}, y^{k_l}) = F_{i_0}(\bar{x}, \bar{y}) > 0,$$

in the latter case, it follows from (34) or (35) that $\{a_{k_l}\}$ converges. Then, we also have (30)–(32) and hence $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1).

(iib) Now, suppose that

$$\{1, 2, \ldots, m\} \backslash \mathscr{I}_F(\bar{x}, \bar{y}) \subseteq \mathscr{I}_G(x^k, y^k), \tag{36}$$

$$\{1, 2, \ldots, m\} \backslash \mathscr{I}_G(\bar{x}, \bar{y}) \subseteq \mathscr{I}_F(x^k, y^k), \tag{37}$$

for all sufficiently large k. Then, since $F_j(x^k, y^k) = 0$ for any $j \notin \mathcal{I}_G(\bar{x}, \bar{y})$ and $y_i^k = 0$ for any $i \notin \mathcal{I}_F(\bar{x}, \bar{y})$, (24) yields

$$\begin{aligned}\nabla f(x^k, y^k) = &\sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} u_i^k \nabla F_i(x^k, y^k) + \sum_{j \in \mathcal{I}_G(\bar{x}, \bar{y})} v_j^k \nabla G_j(x^k, y^k) \\ &+ \sum_{i \notin \mathcal{I}_F(\bar{x}, \bar{y})} a_k/k \nabla F_i(x^k, y^k) - \sum_{l \in \mathcal{I}_g(\bar{x}, \bar{y})} w_l^k \nabla g_l(x^k, y^k) \\ &- \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k), \end{aligned} \tag{38}$$

for all sufficiently large k. If $F(\bar{x}, \bar{y}) = 0$, i.e., $\mathcal{I}_F(\bar{x}, \bar{y}) = \{1, 2, \ldots, m\}$, then (38) implies that the limit $(\bar{x}, \bar{y})$ of $\{(x^k, y^k)\}$ satisfies the weak stationarity condition (17) for problem (1). If $F(\bar{x}, \bar{y}) \neq 0$, then there exists an index i such that $F_i(\bar{x}, \bar{y}) > 0$ and $\bar{y}_i = 0$, which implies

$$\mathcal{I}_G(\bar{x}, \bar{y}) \backslash \mathcal{I}_F(\bar{x}, \bar{y}) \neq \emptyset,$$

and

$$\sum_{i \in \mathcal{I}_G(\bar{x}, \bar{y}) \backslash \mathcal{I}_F(\bar{x}, \bar{y})} F_i(\bar{x}, \bar{y}) > 0. \tag{39}$$

By (36) and (37), for all sufficiently large k, we have

$$\begin{aligned} 0 &= \phi_0^k(x^k, y^k) \\ &= \sum_{i=1}^{m} (1/k - y_i^k) F_i(x^k, y^k) \\ &= \sum_{i \in \mathcal{I}_G(\bar{x}, \bar{y})} (1/k - y_i^k) F_i(x^k, y^k) \\ &= \sum_{i \in \mathcal{I}_G(\bar{x}, \bar{y}) \cap \mathcal{I}_F(\bar{x}, \bar{y})} (1/k - y_i^k) F_i(x^k, y^k) \\ &\quad + \sum_{i \in \mathcal{I}_G(\bar{x}, \bar{y}) \backslash \mathcal{I}_F(\bar{x}, \bar{y})} (1/k) F_i(x^k, y^k). \end{aligned} \tag{40}$$

For any $i \in \mathcal{I}_F(\bar{x}, \bar{y}) \cap \mathcal{I}_G(\bar{x}, \bar{y})$, it follows from (26) and (33) that

$$\begin{aligned} &|a_k(1/k - y_i^k) F_i(x^k, y^k)| \\ &= \begin{cases} 0, & i \in \mathcal{I}_F(x^k, y^k), \\ |u_i^k F_i(x^k, y^k)|, & i \in \mathcal{I}_F(\bar{x}, \bar{y}) \backslash \mathcal{I}_F(x^k, y^k), \end{cases} \\ &\leq |u_i^k F_i(x^k, y^k)|, \end{aligned}$$

and so,

$$\lim_{k \to \infty} a_k (1/k - y_i^k) F_i(x^k, y^k) = 0. \tag{41}$$

Hence, by (39), (40), and (41), we have

$$\lim_{k\to\infty} a_k/k = -\lim_{k\to\infty} N_k/D_k = 0, \tag{42}$$

where

$$N_k = \sum_{i\in\mathcal{I}_G(\bar{x},\bar{y})\cap\mathcal{I}_F(\bar{x},\bar{y})} a_k(1/k - y_i^k)F_i(x^k, y^k),$$

$$D_k = \sum_{i\in\mathcal{I}_G(\bar{x},\bar{y})\setminus\mathcal{I}_F(\bar{x},\bar{y})} F_i(x^k, y^k).$$

Therefore, taking a limit in (38), we obtain (17) from (42). Now, we proceed to showing (19), i.e., $(\bar{x}, \bar{y})$ is C-stationary. Let $i\in\mathcal{I}_F(\bar{x},\bar{y})\cap\mathcal{I}_G(\bar{x},\bar{y})$. Note that, by the assumption of (ii),

$$\begin{aligned} kF_i(x^k, y^k) &= \phi_i^k(x^k, y^k) - \phi_0^k(x^k, y^k) \\ &= \phi_i^k(x^k, y^k) \geq 0, \qquad i = 1, 2, \ldots, m, \end{aligned}$$

i.e.,

$$F(x^k, y^k) \geq 0, \tag{43}$$

for all sufficiently large k. Suppose that there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$y_i^{k_l} F_i(x^{k_l}, y^{k_l}) \neq 0, \qquad \forall l.$$

It follows from (26) and (27) that

$$u_i^{k_l} = a_{k_l}(1/k_l - y_i^{k_l}), \qquad v_i^{k_l} = -a_{k_l}F_i(x^{k_l}, y^{k_l}).$$

By (42) and (43), we have

$$\begin{aligned} \lim_{k\to\infty} u_i^k v_i^k &= \lim_{l\to\infty} u_i^{k_l} v_i^{k_l} \\ &= \lim_{l\to\infty} a_{k_l}^2 y_i^{k_l} F_i(x^{k_l}, y^{k_l}) \geq 0. \end{aligned} \tag{44}$$

Next, we suppose that

$$y_i^k F_i(x^k, y^k) = 0, \tag{45}$$

for all sufficiently large k. First, consider the case where there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$y_i^{k_l} \neq 0, \qquad \forall l.$$

Then, by (45) and (27), $F_i(x^{k_l}, y^{k_l}) = 0$ and hence $v_i^{k_l} = 0$ for any sufficiently large l. So, we obtain

$$\lim_{k\to\infty} u_i^k v_i^k = \lim_{l\to\infty} u_i^{k_l} v_i^{k_l} = 0. \tag{46}$$

Next, consider the case where $y_i^k = 0$ for all sufficiently large k. If there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$F_i(x^{k_l}, y^{k_l}) \neq 0, \qquad \forall l,$$

then, by (26) and (42), we have

$$\lim_{l\to\infty} u_i^{k_l} = \lim_{l\to\infty} a_{k_l}/k_l = 0,$$

and so (46) also holds. If, for any sufficiently large k,

$$y_i^k = 0, \qquad F_i(x^k, y^k) = 0,$$

then, by (15), (26)–(27), and (42),

$$\begin{aligned}\lim_{k\to\infty} u_i^k v_i^k &= \lim_{k\to\infty} \mu_i^k (k\lambda_i^k + a_k/k)\\ &= \lim_{k\to\infty} k\lambda_i^k \mu_i^k \geq 0.\end{aligned}$$

Therefore, we always have

$$\lim_{k\to\infty} u_i^k v_i^k \geq 0,$$

i.e., $(\bar{x}, \bar{y})$ is a C-stationary point of problem (1). Moreover, if $(\bar{x}, \bar{y})$ is nondegenerate, then it follows readily from the definitions of the weak stationarity and nondegeneracy that $(\bar{x}, \bar{y})$ is B-stationary to problem (1).

(IIb) Consider the case where $\mathcal{I}_F(\bar{x}, \bar{y}) = \emptyset$. Then, $\mathcal{I}_G(\bar{x}, \bar{y}) = \{1, 2, \ldots, m\}$ and so $(\bar{x}, \bar{y})$ is nondegenerate. Moreover, (24) becomes

$$\begin{aligned}\nabla f(x^k, y^k) = &\sum_{i=1}^m a_k(1/k - y_i^k)\nabla F_i(x^k, y^k) + \sum_{j=1}^m v_j^k \nabla G_j(x^k, y^k)\\ &- \sum_{l\in\mathcal{I}_g(\bar{x},\bar{y})} w_l^k \nabla g_l(x^k, y^k) - \sum_{r=1}^q \gamma_r^k \nabla h_r(x^k, y^k).\end{aligned} \tag{47}$$

For any sufficiently large k, since $(x^k, y^k) \notin \mathcal{F}$, there exists an index $j \in \mathcal{I}_G(\bar{x}, \bar{y}) \backslash \mathcal{I}_G(x^k, y^k)$. Therefore, we can choose an index j_0 and a subsequence $\{k_l\}$ of $\{k\}$ such that

$$j_0 \in \mathcal{I}_G(\bar{x}, \bar{y}) \backslash \mathcal{I}_G(x^{k_l}, y^{k_l}), \qquad \forall l;$$

i.e., by (27),

$$v_{j_0}^{k_l} = -a_{k_l} F_{j_0}(x^{k_l}, y^{k_l}), \qquad \forall l.$$

Since $\{v_{j_0}^{k_l}\}$ converges and, by $\mathcal{I}_F(\bar{x}, \bar{y}) = \emptyset$,

$$\lim_{l\to\infty} F_{j_0}(x^{k_l}, y^{k_l}) > 0,$$

it follows that the sequence $\{a_{k_l}\}$ is convergent. Noticing that $\{y^k\}$ tends to $\bar{y}=0$ as $k\to\infty$, we have that, for each j,

$$\lim_{k\to\infty} a_k(1/k - y_j^k) = \lim_{l\to\infty} a_{k_l}(1/k_l - y_j^{k_l}) = 0.$$

Letting $k\to\infty$ in (47) and denoting

$$\bar{v}_j = \lim_{k\to\infty} v_j^k, \qquad \bar{w}_l = \lim_{k\to\infty} w_l^k, \qquad \bar{\gamma}_r = \lim_{k\to\infty} \gamma_r^k,$$

we obtain

$$\nabla f(\bar{x},\bar{y}) = \sum_{j=1}^{m} \bar{v}_j \nabla G_j(\bar{x},\bar{y}) - \sum_{l\in \mathcal{I}_g(\bar{x},\bar{y})} \bar{w}_l \nabla g_l(\bar{x},\bar{y}) - \sum_{r=1}^{q} \bar{\gamma}_r \nabla h_r(\bar{x},\bar{y}).$$

This, together with

$$\mathcal{I}_F(\bar{x},\bar{y}) \cap \mathcal{I}_G(\bar{x},\bar{y}) = \emptyset,$$

implies that $(\bar{x},\bar{y})$ is a B-stationary point of problem (1).

Combining Case IIa(ii) and Case IIb shows that statement (a) holds. This completes the proof. □

For a sequence $\{(x^k, y^k)\}$ of stationary points of problem (3), let us define

$$\mathcal{I}_1 = \{i : y_i^k > 0 \text{ for infinitely many } k\},$$

$$\mathcal{I}_2 = \{i : F_i(x^k, y^k) \neq 0 \text{ for infinitely many } k\}.$$

Then, we have

$$\{1,2,\ldots,m\}\setminus \mathcal{I}_G(\bar{x},\bar{y}) \subseteq \mathcal{I}_1 \cap \mathcal{I}_F(\bar{x},\bar{y}),$$

$$\{1,2,\ldots,m\}\setminus \mathcal{I}_F(\bar{x},\bar{y}) \subseteq \mathcal{I}_2 \cap \mathcal{I}_G(\bar{x},\bar{y}).$$

From the proof of Theorem 3.3, we have the next corollary immediately.

Corollary 3.1. Let the assumptions in Theorem 3.3 be satisfied. If $\mathcal{I}_1\setminus\mathcal{I}_F(\bar{x},\bar{y}) \neq \emptyset$ or $\mathcal{I}_2\setminus\mathcal{I}_G(\bar{x},\bar{y}) \neq \emptyset$, then $(\bar{x},\bar{y})$ is a B-stationary point of problem (1).

Next, we consider some other sufficient conditions on M-stationarity and B-stationarity for problem (1). We say $(x^k, y^k)\in\mathcal{F}_k$ satisfies the second-order necessary conditions if there exist multiplier vectors $\lambda^k \in R^{m+1}$,

$\mu^k \in R^m$, $\gamma^k \in R^q$, $\delta^k \in R^p$ such that

$$\lambda^k \geq 0, \qquad \mu^k \geq 0, \qquad \delta^k \geq 0, \tag{48}$$

$$(\lambda^k)^T \phi^k(x^k, y^k) = 0, \quad (\mu^k)^T G(x^k, y^k) = 0, \quad (\delta^k)^T g(x^k, y^k) = 0, \tag{49}$$

$$\nabla_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) = 0, \tag{50}$$

$$d^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d \geq 0, \qquad \forall d \in \mathcal{T}_k(x^k, y^k), \tag{51}$$

where

$$\begin{aligned} &L_k(x, y, \lambda, \mu, \delta, \gamma) \\ &= f(x, y) - \lambda^T \phi^k(x, y) - \mu^T G(x, y) + \delta^T g(x, y) + \gamma^T h(x, y) \end{aligned}$$

stands for the Lagrangian of problem (3) and, for $(x, y) \in \mathcal{F}_k$,

$$\begin{aligned} \mathcal{T}_k(x, y) = \{d \in R^{n+m}: \ & d^T \nabla \phi_i^k(x, y) = 0, i \in \mathcal{I}_{\phi^k}(x, y), \\ & d^T \nabla G_j(x, y) = 0, j \in \mathcal{I}_G(x, y), \\ & d^T \nabla g_l(x, y) = 0, l \in \mathcal{I}_g(x, y), \\ & d^T \nabla h_r(x, y) = 0, r = 1, 2, \ldots, q\}. \end{aligned}$$

We next introduce a new kind of conditions weaker than the second-order necessary conditions for problem (3). Suppose that α_k is a nonnegative number. We say that, at a stationary point (x^k, y^k) of problem (3), the matrix $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ is bounded below with constant α_k on the corresponding tangent space $\mathcal{T}_k(x^k, y^k)$ if

$$d^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d \geq -\alpha_k \|d\|^2, \qquad \forall d \in \mathcal{T}_k(x^k, y^k). \tag{52}$$

Condition (52) is clearly weaker than (51). In fact, for the matrix $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$, there must exist a number α_k such that (52) hold. For example, any nonnegative number α such that $-\alpha$ is less than the smallest eigenvalue of $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ must satisfy (52). However, condition (51) means that the matrix $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ should have some kind of semidefiniteness on the tangent space $\mathcal{T}_k(x^k, y^k)$. Note that, in (52), the constant $-\alpha_k$ may be larger than the smallest eigenvalue mentioned above.

Theorem 3.4. Let $(x^k, y^k) \in \mathcal{F}_k$ be a stationary point of problem (3) with multiplier vectors $\lambda^k, \mu^k, \delta^k, \gamma^k$ satisfying conditions (48)–(50) and, for each k, let $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ be bounded below with constant α_k on the corresponding tangent space $\mathcal{T}_k(x^k, y^k)$. Suppose that $(\bar{x}, \bar{y})$ is an accumulation point of the sequence $\{(x^k, y^k)\}$ with $F(\bar{x}, \bar{y}) \neq 0$, the sequence $\{\alpha_k\}$ is bounded, and the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$

$$i = 1, 2, \ldots, m, l \in \mathscr{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent. Then, $(\bar{x}, \bar{y})$ is an M-stationary point of problem (1). Furthermore, if $(\bar{x}, \bar{y})$ satisfies the upper level strict complementarity condition (20), it is B-stationary to problem (1).

Proof. Since (48)–(50) are equivalent to (14) and (15), it follows from Theorem 3.3 (a) that $(\bar{x}, \bar{y})$ is a C-stationary point of problem (1). By the proof of Theorem 3.3, $(\bar{x}, \bar{y})$ is not B-stationary only in the case IIa(iib); i.e., for all sufficiently large k,

$$(x^k, y^k) \notin \mathscr{F}, \qquad \mathscr{I}_F(\bar{x}, \bar{y}) \neq \emptyset, \tag{53}$$

$$\mathscr{I}_{\phi^k}(x^k, y^k) = \{0\} \cup \mathscr{I}_F(x^k, y^k), \tag{54}$$

$$\{1, 2, \ldots, m\} \backslash \mathscr{I}_F(\bar{x}, \bar{y}) \subseteq \mathscr{I}_G(x^k, y^k), \tag{55}$$

$$\{1, 2, \ldots, m\} \backslash \mathscr{I}_G(\bar{x}, \bar{y}) \subseteq \mathscr{I}_F(x^k, y^k). \tag{56}$$

In the rest of the proof, we therefore assume (53)–(56) and use the same setting as in the proof of Theorem 3.3. Then, (24) holds with (25)–(27). Suppose that $(\bar{x}, \bar{y})$ is not an M-stationary point of problem (1). Then, by the definitions of C-stationarity and M-stationarity, there exists an $i_0 \in \mathscr{I}_F(\bar{x}, \bar{y}) \cap \mathscr{I}_G(\bar{x}, \bar{y})$ such that

$$\bar{u}_{i_0} = \lim_{k \to \infty} u_{i_0}^k < 0, \qquad \bar{v}_{i_0} = \lim_{k \to \infty} v_{i_0}^k < 0, \tag{57}$$

where we use the fact that both the sequences $\{u_{i_0}^k\}$ and $\{v_{i_0}^k\}$ are convergent.

We claim that

$$y_{i_0}^k F_{i_0}(x^k, y^k) \neq 0, \tag{58}$$

for all sufficiently large k. In fact, if there exists a subsequence $\{k_l\}$ of $\{k\}$ such that

$$y_{i_0}^{k_l} F_{i_0}(x^{k_l}, y^{k_l}) = 0, \qquad \forall l,$$

namely,

$$y_{i_0}^{k_l} = 0 \quad \text{or} \quad F_{i_0}(x^{k_l}, y^{k_l}) = 0, \qquad \forall l, \tag{59}$$

then we have from (26)–(27) and (59) that

$$u_{i_0}^{k_l} \geq 0 \quad \text{or} \quad v_{i_0}^{k_l} \geq 0, \qquad \forall l.$$

This contradicts (57), and so (58) holds for all sufficiently large k. Then, (57) becomes

$$\bar{u}_{i_0} = \lim_{k\to\infty} u_{i_0}^k = \lim_{k\to\infty} a_k(1/k - y_{i_0}^k) < 0, \tag{60}$$

$$\bar{v}_{i_0} = \lim_{k\to\infty} v_{i_0}^k = -\lim_{k\to\infty} a_k F_{i_0}(x^k, y^k) < 0, \tag{61}$$

by (26) and (27). By Theorem 3.1 (ii), we may suppose that k is sufficiently large so that, for any k, the set of vectors

$$\{\nabla F_i(x^k, y^k), \nabla G_i(x^k, y^k), \nabla g_l(x^k, y^k), \nabla h_r(x^k, y^k): i = 1, 2, \ldots, m, l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent. Note that

$$\lim_{k\to\infty} (1/k - y_{i_0}^k)/F_{i_0}(x^k, y^k) = -\bar{u}_{i_0}/\bar{v}_{i_0} < 0, \tag{62}$$

by (60) and (61). Therefore, we can choose a bounded sequence $\{d^k\} \subseteq R^{n+m}$ such that, for all sufficiently large k,

$$(d^k)^T \nabla F_i(x^k, y^k) = 0, \qquad i = 1, \ldots, m, \qquad i \neq i_0, \tag{63}$$

$$(d^k)^T \nabla G_j(x^k, y^k) = 0, \qquad j = 1, \ldots, m, \qquad j \neq i_0, \tag{64}$$

$$(d^k)^T \nabla F_{i_0}(x^k, y^k) = 1, \tag{65}$$

$$(d^k)^T \nabla G_{i_0}(x^k, y^k) = (1/k - y_{i_0}^k)/F_{i_0}(x^k, y^k), \tag{66}$$

$$(d^k)^T \nabla g_l(x^k, y^k) = 0, \qquad l \in \mathcal{I}_g(\bar{x}, \bar{y}), \tag{67}$$

$$(d^k)^T \nabla h_r(x^k, y^k) = 0, \qquad r = 1, 2, \ldots, q. \tag{68}$$

Since

$$\nabla \phi_0^k(x^k, y^k) = \sum_{i=1}^{m} (1/k - y_i^k) \nabla F_i(x^k, y^k) - \sum_{j=1}^{m} F_j(x^k, y^k) \nabla G_j(x^k, y^k), \tag{69}$$

we have from (63)–(66) that

$$(d^k)^T \nabla \phi_0^k(x^k, y^k) = \sum_{i=1}^{m} (1/k - y_i^k)(d^k)^T \nabla F_i(x^k, y^k) - \sum_{j=1}^{m} F_j(x^k, y^k)(d^k)^T \nabla G_j(x^k, y^k) = 0. \tag{70}$$

On the other hand, noting that $i_0 \notin \mathscr{I}_F(x^k, y^k) \cap \mathscr{I}_G(x^k, y^k)$ for all sufficiently large k by (58), we have from (5), (63), and (70) that

$$(d^k)^T \nabla \phi_i^k(x^k, y^k) = (d^k)^T \nabla \phi_0^k(x^k, y^k) + (d^k)^T \nabla F_i(x^k, y^k) = 0, \qquad 0 \neq i \in \mathscr{I}_{\phi^k}(x^k, y^k). \tag{71}$$

It follows from (64), (66)–(68), and (70)–(71) that

$$d^k \in \mathscr{T}_k(x^k, y^k), \tag{72}$$

for all sufficiently large k. By (69), we have

$$\nabla^2 \phi_0^k(x^k, y^k) = \sum_{i=1}^{m} (1/k - y_i^k) \nabla^2 F_i(x^k, y^k) - \sum_{i=1}^{m} \nabla F_i(x^k, y^k) G_i(x^k, y^k)^T - \sum_{j=1}^{m} \nabla G_j(x^k, y^k) \nabla F_j(x^k, y^k)^T, \tag{73}$$

where we use the fact that $\nabla G_j(x^k, y^k)$, $j = 1, \cdots, m$, are constant vectors. On the other hand, we can write

$$\begin{aligned}
&\nabla_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) \\
&= \nabla f(x^k, y^k) - \sum_{i=0}^{m} \lambda_i^k \nabla \phi_i^k(x^k, y^k) - \sum_{j=1}^{m} \mu_j^k \nabla G_j(x^k, y^k) \\
&\quad + \sum_{l=1}^{p} \delta_l^k \nabla g_l(x^k, y^k) + \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k) \\
&= \nabla f(x^k, y^k) - a_k \nabla \phi_0^k(x^k, y^k) \\
&\quad - \sum_{i=1}^{m} k\lambda_i^k \nabla F_i(x^k, y^k) - \sum_{j=1}^{m} \mu_j^k \nabla G_j(x^k, y^k) \\
&\quad + \sum_{l=1}^{p} \delta_l^k \nabla g_l(x^k, y^k) + \sum_{r=1}^{q} \gamma_r^k \nabla h_r(x^k, y^k),
\end{aligned}$$

where

$$a_k = \sum_{i=0}^{m} \lambda_i^k$$

is the same as that in the proof of Theorem 3.3, and so we have from (73) that

$$\begin{aligned}
&\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) \\
&= \nabla^2 f(x^k, y^k) - a_k \nabla^2 \phi_0^k(x^k, y^k) - \sum_{i=1}^{m} k\lambda_i^k \nabla^2 F_i(x^k, y^k) \\
&\quad + \sum_{l=1}^{p} \delta_l^k \nabla^2 g_l(x^k, y^k) + \sum_{r=1}^{q} \gamma_r^k \nabla^2 h_r(x^k, y^k) \\
&= \nabla^2 f(x^k, y^k) + a_k \sum_{i=1}^{m} \nabla F_i(x^k, y^k)\, G_i(x^k, y^k)^T \\
&\quad + a_k \sum_{j=1}^{m} \nabla G_j(x^k, y^k)\, \nabla F_j(x^k, y^k)^T \\
&\quad - \sum_{i=1}^{m} (k\lambda_i^k + a_k(1/k - y_i^k)) \nabla^2 F_i(x^k, y^k) \\
&\quad + \sum_{l=1}^{p} \delta_l^k \nabla^2 g_l(x^k, y^k) + \sum_{r=1}^{q} \gamma_r^k \nabla^2 h_r(x^k, y^k).
\end{aligned}$$

Since $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$ is bounded below with constant α_k on the corresponding tangent space $\mathcal{T}_k(x^k, y^k)$, we have from (52) and (72) that there exists a constant C such that

$$(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \geq -\alpha_k \|d^k\|^2 \geq C, \tag{74}$$

where the last inequality follows from the boundedness of the sequences $\{\alpha_k\}$ and $\{d^k\}$. Note that

$$\begin{aligned}
&(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \\
&= (d^k)^T \nabla^2 f(x^k, y^k) d^k + a_k \sum_{i=1}^{m} (d^k)^T \nabla F_i(x^k, y^k) G_i(x^k, y^k)^T d^k \\
&\quad + a_k \sum_{j=1}^{m} (d^k)^T \nabla G_j(x^k, y^k) \nabla F_j(x^k, y^k)^T d^k \\
&\quad - \sum_{i=1}^{m} [k\lambda_i^k + a_k(1/k - y_i^k)] (d^k)^T \nabla^2 F_i(x^k, y^k) d^k \\
&\quad + \sum_{l=1}^{p} \delta_l^k (d^k)^T \nabla^2 g_l(x^k, y^k) d^k + \sum_{r=1}^{q} \gamma_r^k (d^k)^T \nabla^2 h_r(x^k, y^k)\, d^k
\end{aligned}$$

$$= (d^k)^T \nabla^2 f(x^k, y^k) d^k + 2a_k(1/k - y^k_{i_0})/F_{i_0}(x^k, y^k)$$
$$- \sum_{i=1}^{m} [k\lambda_i^k + a_k(1/k - y_i^k)](d^k)^T \nabla^2 F_i(x^k, y^k) d^k$$
$$+ \sum_{l=1}^{p} \delta_l^k (d^k)^T \nabla^2 g_l(x^k, y^k) d^k + \sum_{r=1}^{q} \gamma_r^k (d^k)^T \nabla^2 h_r(x^k, y^k) d^k. \qquad (75)$$

By the twice continuous differentiability of the functions involved, the boundedness of the sequence $\{d^k\}$, and the convergence of the sequences $\{(x^k, y^k)\}$, $\{\delta_l^k\}$, and $\{\gamma_r^k\}$, the terms

$$(d^k)^T \nabla^2 f(x^k, y^k) d^k, \qquad \sum_{l=1}^{p} \delta_l^k (d^k)^T \nabla^2 g_l(x^k, y^k) d^k,$$
$$\sum_{r=1}^{q} \gamma_r^k (d^k)^T \nabla^2 h_r(x^k, y^k) d^k$$

are all bounded. Noticing that, for all sufficiently large k, $i \notin \mathscr{I}_F(x^k, y^k)$ implies $i \notin \mathscr{I}_{\phi^k}(x^k, y^k)$ by (54) and so $\lambda_i^k = 0$ by (48) and (49), we have from the convergence of the sequences in (28) and the definition (26) of u_i^k that the sequence $\{k\lambda_i^k + a_k(1/k - y_i^k)\}$ is bounded for any $i = 1, 2, \ldots, m$. Hence, the term

$$\sum_{i=1}^{m} [k\lambda_i^k + a_k(1/k - y_i^k)](d^k)^T \nabla^2 F_i(x^k, y^k) d^k$$

is also bounded. However, since

$$\lim_{k \to \infty} y^k_{i_0} = \bar{y}_{i_0} = 0,$$

we have $a_k \to +\infty$ by (60) and so

$$2a_k(1/k - y^k_{i_0})/F_{i_0}(x^k, y^k) \to -\infty,$$

as $k \to \infty$ by (62). Therefore, it follows from (75) that

$$(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \to -\infty,$$

as $k \to \infty$. This contradicts (74) and hence $(\bar{x}, \bar{y})$ is M-stationary to problem (1). This completes the proof of the first part of the theorem. The second part of the theorem follows from the definitions of M-stationarity and the upper level strict complementarity immediately. □

Corollary 3.2. Let $\{(x^k, y^k)\}$ and $(\bar{x}, \bar{y})$ be the same as in Theorem 3.4. If (x^k, y^k) together with the corresponding multiplier vectors $\lambda^k, \mu^k, \delta^k, \gamma^k$

satisfies the second-order necessary conditions (48)–(51) and the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$

$$i = 1, 2, \ldots, m, l \in \mathscr{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent, then the conclusion of Theorem 3.4 remains true.

Corollary 3.2 establishes convergence to a B-stationary point under the second-order necessary conditions and the upper level strict complementarity. These or similar conditions have also been assumed in Refs. 2–3, but they are somewhat restrictive and may be difficult to verify in practice. The next theorem provides a new condition for convergence to a B-stationary point, which can be dealt with more easily. We note that, unlike Refs. 2–3, it relies on neither upper level strict complementarity nor asymptotic weak nondegeneracy.

Theorem 3.5. Let $\{(x^k, y^k)\}$ and $(\bar{x}, \bar{y})$ be the same as in Theorem 3.4 and let $\lambda^k, \mu^k, \delta^k, \gamma^k$ be the multiplier vectors corresponding to (x^k, y^k) with (48)–(51). Let β_k be the smallest eigenvalue of the matrix $\nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k)$. If the sequence $\{\beta_k\}$ is bounded below and if the set of vectors

$$\{\nabla F_i(\bar{x}, \bar{y}), \nabla G_i(\bar{x}, \bar{y}), \nabla g_l(\bar{x}, \bar{y}), \nabla h_r(\bar{x}, \bar{y}):$$

$$i = 1, 2, \ldots, m, l \in \mathscr{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent, then $(\bar{x}, \bar{y})$ is a B-stationary point of problem (1).

Proof. It is easy to see that the assumptions of Theorem 3.4 are satisfied with $\alpha_k = \max\{-\beta_k, 0\}$ and so $(\bar{x}, \bar{y})$ is an M-stationary point of problem (1). Suppose that $(\bar{x}, \bar{y})$ is not B-stationary to problem (1). As mentioned at the beginning of the proof of Theorem 3.4, this occurs only in the case where (53)–(56) hold for all sufficiently large k. By the definitions of B-stationarity and M-stationarity, there exists an $i_0 \in \mathscr{I}_F(\bar{x}, \bar{y}) \cap \mathscr{I}_G(\bar{x}, \bar{y})$ such that

$$\bar{u}_{i_0} = \lim_{k \to \infty} u^k_{i_0} < 0, \qquad \bar{v}_{i_0} = \lim_{k \to \infty} v^k_{i_0} = 0, \tag{76}$$

or

$$\bar{u}_{i_0} = \lim_{k \to \infty} u^k_{i_0} = 0, \qquad \bar{v}_{i_0} = \lim_{k \to \infty} v^k_{i_0} < 0. \tag{77}$$

From (25)–(27) and (48), we know that either of (76) and (77) implies

$$\lim_{k \to \infty} a_k = +\infty. \tag{78}$$

By Theorem 3.1, we may suppose that k is large enough so that (53)–(56) hold,

$$\mathcal{I}_F(x^k, y^k) \subseteq \mathcal{I}_F(\bar{x}, \bar{y}), \qquad \mathcal{I}_G(x^k, y^k) \subseteq \mathcal{I}_G(\bar{x}, \bar{y}),$$
$$\mathcal{I}_g(x^k, y^k) \subseteq \mathcal{I}_g(\bar{x}, \bar{y}),$$

and the set of vectors

$$\{\nabla F_i(x^k, y^k), \nabla G_i(x^k, y^k), \nabla g_l(x^k, y^k), \nabla h_r(x^k, y^k):$$
$$i = 1, 2, \ldots, m, l \in \mathcal{I}_g(\bar{x}, \bar{y}), r = 1, 2, \ldots, q\}$$

is linearly independent. Therefore, we can choose a vector $d^k \in R^{n+m}$ such that (63)–(64) and (67)–(68) hold and

$$(d^k)^T \nabla F_{i_0}(x^k, y^k) = 1, \qquad (d^k)^T \nabla G_{i_0}(x^k, y^k) = -1.$$

Furthermore, we can choose the sequence $\{d^k\}$ to be bounded. By the assumptions of the theorem, there exists a constant C such that

$$(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \geq \beta_k \|d^k\|^2 \geq C \tag{79}$$

holds for all sufficiently large k. Note that, by the definition of d^k and (75),

$$\begin{aligned}
&(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \\
&= (d^k)^T \nabla^2 f(x^k, y^k) d^k - 2a_k \\
&\quad - \sum_{i=1}^{m} [k\lambda_i^k + a_k(1/k - y_i^k)](d^k)^T \nabla^2 F_i(x^k, y^k) d^k \\
&\quad + \sum_{l=1}^{p} \delta_l^k (d^k)^T \nabla^2 g_l(x^k, y^k) d^k \\
&\quad + \sum_{r=1}^{q} \gamma_r^k (d^k)^T \nabla^2 h_r(x^k, y^k) d^k.
\end{aligned} \tag{80}$$

In a way similar to Theorem 3.4, we can show that all the terms on the right-hand side of (80) except the term $-2a_k$ are bounded. This, together with (78), implies that

$$(d^k)^T \nabla^2_{(x,y)} L_k(x^k, y^k, \lambda^k, \mu^k, \delta^k, \gamma^k) d^k \to -\infty,$$

as $k \to \infty$. This contradicts (79) and hence $(\bar{x}, \bar{y})$ is B-stationary to problem (1). This completes the proof. □

4. Computational Results

We have tested the method on various small scale examples of MPECs, which have been used to test other methods in the literature. We applied the MATLAB 6.0 built-in solver function fmincon to problem (3) with various values of k. The computational results are summarized in Tables 1–4, which indicate that the proposed method produces good approximate solutions of (1) in a small number of iterations. In the tables, (x^k, y^k) is the (approximate) solution of (1) produced by solving (3), Ite stands for the number of iterations spent by fmincon, and $r(x^k, y^k)$ denotes the residual for the constraints in problem (3) at (x^k, y^k), i.e.,

$$r(x^k, y^k) = \sum_{l=1}^{p} (g_l(x^k, y^k))_+ + \sum_{r=1}^{q} |h_r(x^k, y^k)| + \sum_{j=1}^{m} (-y_j^k)_+ + \sum_{i=1}^{m} (-F_i(x^k, y^k))_+ + |(y^k)^T F(x^k, y^k)|,$$

where $(u)_+ = \max\{0, u\}$ for a scalar u.

Problem 4.1. This problem is given in Ref. 11, which has two upper-level variables $(x_1, x_2) \in R^2$ and one lower-level variable $y \in R$:

$$\begin{aligned} \min \quad & x_1^2 + 10(x_2 - 1)^2 + (y+1)^2, \\ \text{s.t.} \quad & x_2 \geq 0, \qquad x_1 - e^{x_2} - e^y \geq 0, \\ & y \geq 0, \qquad y(x_1 - e^{x_2} - e^y) = 0. \end{aligned}$$

Table 1. Computational results for Problem 4.1 ($k = 10, 10^2$).

Size (p, m, n)	(1, 1, 2)
Initial point	(3, 0, 0)
(x^k, y^k)	(2.7101, 0.5365, 0)
Ite	7
$f(x^k, y^k)$	10.4925
$r(x^k, y^k)$	0

Table 2. Computational results for Problem 4.2 ($k = 10, 10^2, 10^4$).

Size (m, n, p, q)	(2, 4, 4, 2)
Initial point	(1, 1, 1, 1, 0, 0)
(x^k, y^k)	(0.5000, 0.5000, 0.5000, 0.5000, 0, 0)
Ite	2
$f(x^k, y^k)$	–1.0000
$r(x^k, y^k)$	0

Table 3. Computational results for Problem 4.3.

Size (p, m, n)		(3, 6, 2)
Initial point		(0, 0, 1.60, 0.20, 0.44, 1.36, 0, 0)
$k = 10^2$	(x^k, y^k)	(0, 2, 1.9034, 0.9276, 0, 1.2689, 0, 0)
	Ite	7
	$f(x^k, y^k)$	−12.7533
	$r(x^k, y^k)$	0.0703
$k = 10^4$	(x^k, y^k)	(0, 2, 1.8753, 0.9065, 0, 1.2502, 0, 0)
	Ite	6
	$f(x^k, y^k)$	−12.6795
	$r(x^k, y^k)$	0.0007
$k = 10^6, 10^8$	(x^k, y^k)	(0, 2, 1.8750, 0.9063, 0, 1.2500, 0, 0)
	Ite	6
	$f(x^k, y^k)$	−12.6787
	$r(x^k, y^k)$	0.0004

Table 4. Computational results for Problem 4.4.

Size (m, n, p, q)		(12,8,9,4)
x^0		(0, 0, 0, 0, 0, 0, 0, 0)
$k = 10^2$	x^k	(6.9858, 2.9766, 12.0064, 18.0312, −0.0173, 10.0143, 30.0896, −0.0173)
	Ite	19
	$f(x^k, y^k)$	−6.6097e + 003
	$r(x^k, y^k)$	0.5500
$k = 10^4$	x^k	(7.0369, 3.0553, 11.9632, 17.9447, 0.0921, 10.0000, 29.9079, 0)
	Ite	20
	$f(x^k, y^k)$	−6.6000e + 003
	$r(x^k, y^k)$	0.1953e − 004
$k = 10^6$	x^k	(6.4449, 2.7621, 12.3172, 18.4758, 0, 9.2069, 30.7930, 0.0001)
	Ite	7
	$f(x^k, y^k)$	−6.5987e + 003
	$r(x^k, y^k)$	0.1200e − 003
$k = 10^8$	x^k	(6.4447, 2.7620, 12.3173, 18.4758, 0, 9.2068, 30.7932, 0)
	Ite	7
	$f(x^k, y^k)$	−6.5987e + 003
	$r(x^k, y^k)$	0.1200e − 005

Problem 4.2. This is equivalent to Problem 5 in Ref. 7 and goes back to Ref. 12:

$$\begin{aligned}
\min \quad & x_1^2 - 2x_1 + x_2^2 - 2x_2 + x_3^2 + x_4^2, \\
\text{s.t.} \quad & 0 \le x_1 \le 2, \qquad 0 \le x_2 \le 2, \\
& x_3 - x_1 + x_3 y_1 - y_1 = 0, \\
& x_4 - x_2 + x_4 y_2 - y_2 = 0,
\end{aligned}$$

$$y_1 \geq 0, \qquad y_2 \geq 0,$$
$$F(x, y) \geq 0, \qquad y^T F(x, y) = 0,$$

where

$$F(x, y) = \begin{bmatrix} 0.25 - (x_3 - 1)^2 \\ 0.25 - (x_4 - 1)^2 \end{bmatrix}.$$

Problem 4.3. This is Problem 11 in Ref. 7, which is equivalent to the following MPEC:

$$\begin{aligned} \min \quad & -x_1^2 - 3x_2 - 4y_1 + y_2^2, \\ \text{s.t.} \quad & y \geq 0, \qquad F(x, y) \geq 0, \qquad y^T F(x, y) = 0, \\ & x_1^2 + 2x_2 \leq 4, \qquad x_1 \geq 0, \qquad x_2 \geq 0, \end{aligned}$$

where

$$F(x, y) = \begin{bmatrix} 2y_1 + 2y_3 - 3y_4 - y_5 \\ -5 - y_3 + 4y_4 - y_6 \\ x_1^2 - 2x_1 + x_2^2 - 2y_1 + y_2 + 3 \\ x_2 + 3y_1 - 4y_2 - 4 \\ y_1 \\ y_2 \end{bmatrix}.$$

Problem 4.4. This is equivalent to Problem 10 in Ref. 7:

$$\begin{aligned} \min \quad & (x_5 + x_7 - 200)(x_5 + x_7) + (x_6 + x_8 - 160)(x_6 + x_8), \\ \text{s.t.} \quad & 0 \leq x_1 \leq 10, \qquad 0 \leq x_2 \leq 5, \\ & 0 \leq x_3 \leq 15, \qquad 0 \leq x_4 \leq 20, \\ & x_1 + x_2 + x_3 + x_4 \leq 40, \\ & x_5 - 4 + 0.4y_1 + 0.6y_2 - y_3 + y_4 = 0, \\ & x_6 - 13 + 0.7y_1 + 0.3y_2 - y_5 + y_6 = 0, \\ & x_7 - 35 + 0.4y_7 + 0.6y_8 - y_9 + y_{10} = 0, \\ & x_8 - 2 + 0.7y_7 + 0.3y_8 - y_{11} + y_{12} = 0, \\ & y \geq 0, \qquad F(x, y) \geq 0, \qquad y^T F(x, y) = 0, \end{aligned}$$

where

$$F(x, y) = \begin{bmatrix} x_1 - 0.4x_5 - 0.7x_6 \\ x_2 - 0.6x_5 - 0.3x_6 \\ x_5 \\ -x_5 + 20 \\ x_6 \\ -x_6 + 20 \\ x_3 - 0.4x_7 - 0.7x_8 \\ x_4 - 0.6x_7 - 0.3x_8 \\ x_7 \\ -x_7 + 40 \\ x_8 \\ -x_8 + 40 \end{bmatrix}.$$

Since y stands for the Lagrangian multiplier vector in the original problem (Ref. 7), we list only the values of x in Table 4.

5. Concluding Remarks

Suppose that the condition

$$\sum_{j=1}^{m} y_j \leq m/k + k$$

is retained in the constraints of problem (3), i.e., problem (3) is replaced by the problem

$$\min \quad f(x, y), \tag{81a}$$

$$\text{s.t.} \quad g(x, y) \leq 0, \qquad h(x, y) = 0, \tag{81b}$$

$$y \geq 0, \qquad \sum_{j=1}^{m} y_j \leq m/k + k, \tag{81c}$$

$$(e_i^k - y)^T F(x, y) \geq 0, \qquad i = 0, 1, \ldots, m. \tag{81d}$$

Then, since the constraint

$$\sum_{j=1}^{m} y_j \leq m/k + k$$

eventually becomes inactive at any fixed point as k tends to ∞, all the results established in the previous sections remain true except that the results in

Theorem 2.1 are replaced by

$$\mathcal{F} = \lim_{k\to\infty} \mathcal{F}_k.$$

When the set

$$Z = \{z \in R^{n+m}: g(z) \le 0, h(z) = 0\}$$

is bounded, problem (81) has a compact feasible region and so it is solvable for any k as long as it is feasible.

In addition, we remark that the term $(1/k)e$ in (2) is necessary for problem (3) to have desirable properties. In fact, the problem

$$\min \quad f(x, y), \tag{82a}$$

$$\text{s.t.} \quad g(x, y) \le 0, \qquad h(x, y) = 0, \qquad y \ge 0, \tag{82b}$$

$$(k\, e_i - y)^T F(x, y) \ge 0, \qquad i = 0, 1, \ldots, m, \tag{82c}$$

is difficult to handle because problem (82) does not satisfy the MFCQ at any point $(\bar{x}, \bar{y}) \in \mathcal{F}$ for all sufficiently large k. For simplicity, we assume that the constraints $g(x, y) \le 0$ and $h(x, y) = 0$ are absent and let

$$G(x, y) = y, \qquad \psi_i^k(x, y) = (ke_i - y)^T F(x, y), \qquad i = 0, 1, \ldots, m. \tag{83}$$

Note that

$$\psi_i^k(x, y) = kF_i(x, y) + \psi_0^k(x, y), \qquad i = 1, 2, \ldots, m. \tag{84}$$

At $(\bar{x}, \bar{y}) \in \mathcal{F}$, the set of active constraints is

$$\{\psi_0^k, \psi_i^k, G_j: i \in \mathcal{I}_F(\bar{x}, \bar{y}), j \in \mathcal{I}_G(\bar{x}, \bar{y})\}.$$

Suppose the MFCQ holds at $(\bar{x}, \bar{y})$ for problem (82). Then, there exists a vector $(x, y) \in R^{n+m}$ such that

$$\nabla \psi_0^k(\bar{x}, \bar{y})^T \begin{bmatrix} x \\ y \end{bmatrix} > 0, \tag{85}$$

$$\nabla \psi_i^k(\bar{x}, \bar{y})^T \begin{bmatrix} x \\ y \end{bmatrix} > 0, \qquad i \in \mathcal{I}_F(\bar{x}, \bar{y}), \tag{86}$$

$$y_j = \nabla G_j(\bar{x}, \bar{y})^T \begin{bmatrix} x \\ y \end{bmatrix} > 0, \qquad j \in \mathcal{I}_G(\bar{x}, \bar{y}). \tag{87}$$

(i) Assume that $\mathcal{I}_F(\bar{x}, \bar{y}) \neq \emptyset$ and k is large enough to satisfy

$$1 - (1/k) \sum_{i \in \mathcal{I}_F(\bar{x}, \bar{y})} \bar{y}_i > 0. \tag{88}$$

By (84) and (86), we have

$$-\nabla F_i(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}<(1/k)\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix},\qquad i\in\mathscr{I}_F(\bar{x},\bar{y}).\tag{89}$$

It then follows from (83), (87), and (89) that

$$\begin{aligned}\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}&=-\sum_{i=1}^m\bar{y}_i\nabla F_i(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}\\&\quad-\sum_{j=1}^m F_j(\bar{x},\bar{y})y_j\\&=-\sum_{i\in\mathscr{I}_F(\bar{x},\bar{y})}\bar{y}_i\nabla F_i(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}\\&\quad-\sum_{j\in\mathscr{I}_G(\bar{x},\bar{y})}F_j(\bar{x},\bar{y})y_j\\&\leq\Big((1/k)\sum_{i\in\mathscr{I}_F(\bar{x},\bar{y})}\bar{y}_i\Big)\,\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix},\end{aligned}$$

i.e.,

$$\Big(1-(1/k)\sum_{i\in\mathscr{I}_F(\bar{x},\bar{y})}\bar{y}_i\Big)\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}\leq 0.$$

By (88), we have

$$\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}\leq 0.$$

This contradicts (85) and hence the MFCQ does not hold at $(\bar{x},\bar{y})$ for problem (82) when k is sufficiently large.

(ii) Suppose that $\mathscr{I}_F(\bar{x},\bar{y})=\emptyset$. Then, we have

$$\bar{y}=0,\qquad F(\bar{x},\bar{y})>0,$$

and by (87), $y>0$. It follows that

$$\begin{aligned}\nabla\psi_0^k(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}&=-\sum_{i=1}^m\bar{y}_i\nabla F_i(\bar{x},\bar{y})^T\begin{bmatrix}x\\y\end{bmatrix}\\&\quad-\sum_{j=1}^m F_j(\bar{x},\bar{y})y_j\\&=-\sum_{j=1}^m F_j(\bar{x},\bar{y})y_j<0,\end{aligned}$$

which also contradicts (85) and then the MFCQ does not hold at $(\bar{x}, \bar{y})$ for problem (82).

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