STUDIES ON

OPIMIZATION MODELS

OF FINANCIAL AND REAL OPTIONS

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by

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Preface

In this thesis, I study problems concerning both financial and real options in stochastic finance. As well-known, option pricing theory has originated from the Black-Scholes model in 1970s. In the model, Black, Scholes, and Merton proposed the concept of no-arbitrage pricing and derived theoretical prices of call and put options in closed forms. These studies gave deep impact to practitioners working around the Wall Street. Option pricing theory brought about the revolution in the field of financial engineering, since portfolio optimization proposed by Markowitz in 1950s. Option pricing theory still continues to develop for thirty years since the Black-Sholes model. While countless stochastic models which are much more complicated and sophisticated have been proposed up to now, this thesis contributes toward studies on option pricing from an opposite angle. In fact, this thesis conducts option pricing based on prices of other derivative securities without assuming any stochastic differential equation models.

In 1980s, the idea of option pricing began to be applied to evaluation of other things such as developing natural resources beyond pricing conventional derivative securities. After that, in 1990s, the word *real options studies*, which represent studies applying option pricing techniques to more general decision making, has become increasingly popular. At present, real options studies which are combined with other theory such as game theory, contract theory, and theory of optimal capital structure play an important role in corporate finance. Furthermore, in practice, many consulting firms utilize the idea of real options for resolving managerial matters.

Such active studies and rapid spread of real options will bring about the third revolution in the history of financial engineering. This third revolution will be shared among much more people than those in the previous revolutions. That is, real options methods are widely directed to all kinds of decision makers in all kinds of organizations, rather than restricted specialists in investment institutions. I hope that this thesis helps both researchers and practitioners to understand effectiveness of the real options approach.

Michi Nishihara

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Second, I would also like to express my gratitude to Professor Toshihide Ibaraki of Kwansei Gakuin University, Professor Mutsunori Yagiura of Nagoya University, and Professor Hiroshi Nagamochi of Kyoto University, who supervised me during my master course. Although I was one of typical students who were too confident to notice their ignorance at that time, they gave me opportunities to determine the research subject and conduct research freely. Without their tolerance, I would never start the studies on options in stochastic finance. During the guidance, they gave me many hints in obtaining some results on option pricing based only on the no-arbitrage assumption in this thesis.

I would also like to tender my acknowledgments to Professor Masaaki Kijima of Tokyo Metropolitan University. Since the second grade in my master course, I have learned a fundamental of financial engineering at his seminars. Whenever I had an opportunity to present my work at his seminar, he suggested many ideas that greatly improved the work. Some results on real options in this thesis were motivated and supported by the participants of his seminars as well as him.

Fortunately, I had valuable opportunities to discuss the related studies with many other researchers. The discussion always helped me to broaden my horizons and bring new insight about the studies. Above all, Professor Masamitsu Ohnishi of Osaka University, Professor Motoh Tsujimura of Ryukoku University, Professor Takashi Shibata of Tokyo Metropolitan University, and Atuyuki Ohyama, who is a doctoral course student of Kyoto University, gave me useful comments and suggestions. In addition, I would like to thank all the members of Professor Fukushima's Laboratory, Professor Kijima's Laboratory, Professor Nagamochi's Laboratory (formerly Professor Ibaraki's Laboratory) of Kyoto University, and the Center for the Study of Finance and Insurance, Osaka University for their generous support.

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Finally, I would like to dedicate this thesis to my parents with my appreciation for their warm-hearted support.

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Chapter 1

Introduction

1.1 Financial options

By the portfolio optimization theory, Markowitz [57] first showed effectiveness of an engineering method in finance. He regarded the variance of the return of a portfolio consisting of several stocks as a risk of the portfolio. Then, he formulated the portfolio optimization as the problem of finding a portfolio that minimizes the variance subject to the targeted return, and he reduced the problem to a quadratic programming problem. This is the well-known mean-variance model. The idea of optimizing a portfolio by taking account of both return and risk has developed into more sophisticated models such as multi-factor models (e.g., [70, 77]), and those models are now applied to investment business in financial institutions. The portfolio optimization theory by Markowitz was the first study that the spread fame of financial engineering. After that, Sharpe [76] extended the mean-variance model by Markowitz to a financial market that comprises a number of investors. By doing so, he built the CAPM (Capital Asset Pricing Model) that is the foundation of the modern asset pricing theory.

In the 1970s, the option pricing theory originating from the Black-Scholes model [10] made an enormous impact on the Wall Street. This was the second success in the field of financial engineering. Let us now make a brief introduction about options. Options (or derivatives) mean securities whose payoffs depend on the dynamics of underlying asset prices (e.g., stock prices, interest rates and exchange rates). Some firms trade derivative securities in order to hedge the risk of interest rates, exchange rates, etc. Although most traded derivatives are futures, call and put options, and swaps, numerous kinds of derivatives such as weather derivatives and credit derivatives are now commonly traded (cf., [7, 46]).

It is the famous Black-Scholes formula that first derived theoretical prices of call and put options. Black and Scholes [10] set up the model and somewhat intuitively derived the solution, which was mathematically proved by Merton [60] afterward. The Black-Scholes model is constructed on two major assumptions: one is that the price of the underlying stock follows one dimensional geometric Brownian motion; the other is the no-arbitrage assumption. The no-arbitrage assumption intuitively means that no portfolio generates a positive return without a risk of loss. By the first assumption, in the Black-Scholes model, for every option there exists a portfolio (which is called replicating portfolio) consisting of the underlying stock and the risk-free bond so that the payoff of the portfolio is the same as that of the option. Then, by the second assumption the price of the option must be equal to that of the replicating portfolio. This is how the theoretical price of call and put options are derived from the Black-Scholes formula.

Thereafter, in a more general setting, Harrison and Kreps [35] and Harrison and Pliska [36] showed the equivalence between the property that all derivative securities are replicable and the unique existence of the probability measures (called the risk-neutral measure or the equivalent martingale measure) under which every derivative price can be expressed as its expected discounted payoff. Thus, the fundamental theory of option pricing was established.

The Black-Scholes model still remains most popular and is frequently used as an important benchmark among both academic researchers and practitioners in financial engineering. Countless complicated models have been recently studied, such as incomplete market (this means that some derivatives can not be replicated in the market) models that include jump processes in the dynamics of underlying assets (e.g., [49, 64]). In the financial world, firms' demand for trading options continues to increase since options enable them to hedge various risks. In fact, many textbooks (e.g., [45]) for practitioners in financial institutions have been published and through those books the latest results about the option pricing theory are applied in actual trade of derivative securities. In such a situation, the option pricing theory is expected to further develop and to meet the needs of practitioners.

1.2 Real options

The study on the option pricing theory, as mentioned in the previous section, began in the 1970s. In the 1980s, the method of option pricing, beyond just pricing financial derivatives, began to be used for evaluating the rights (called real options) which have a similar property to financial options. The first study in this context was conducted by Brennan and Schwartz [15] who investigated the natural resource investment such as gold and copper mines and oil deposits.

Thereafter, Dixit [20], McDonald and Siegel [59] and Pindyck [71, 72] analyzed the investment decision problem in corporate finance by using the technique of option pricing. They regarded a firm as an option-holder who had a right to invest in a project and derived the optimal investment timing and the project value by utilizing the option pricing theory, in particular a method of evaluating American option. This kind of study on real options made a rapid spread into the field of corporate finance as a new theory that extended the investment timing theory by Jorgenson [47], Tobin's q theory [84] and the NPV (Net Present Value) in the project valuation. Actually, the real options study can capture both irreversibility (a firm can not easily withdraw a project once it makes investment) and uncertainty of future profit in investment. Above all, the result that higher uncertainty about future profit not only delays the firm's investment time but also increases the value of the investment project provided new insights which have never been gained in the previous works. See [22] for standard results from real options studies in the eighties and the early nineties. A large number of earlier literatures about real options are included in [74].

In the 1990s, the real options study about investment under uncertainty became a boom. Since this time, the mainstream of the real options study has shifted to more general studies about corporate decision makings, behaviors, and strategies under uncertainty. That is to say, the field of the real options study has greatly spread beyond the framework which was expected at the beginning.

One of the most growing studies about real options is the study on strategic real options (see [13] for an overview). This was started by Grenadier [29] who examined the strategic interactions between two firms by incorporating the timing game into a real options model. At present, more complex and realistic situations such as the case of allowing incomplete information between firms are actively conducted (e.g., [50, 65]). Furthermore, some literatures have focused on the agency conflicts between the owner and the manager in a single firm instead of competition among several firms, by combining the contract theory with the real options theory (e.g., [31, 66]).

The connection between the real options theory and the optimal capital structure theory that originated from Modigliani and Miller [62] has been gradually stronger. In fact, there have been several studies that incorporate the real investment problem into capital structure models proposed by [61, 52] (e.g., [58, 38, 81]). These studies clarify the interactions between how to finance for the investment and the investment timing.

Recently, the real options study has been applied to the business world. Several textbooks (e.g., [86, 80]) for business persons spread a real options approach among executive managers and consultants. Since the real options study is younger than the portfolio optimization and option pricing studies, it may hide a lot of potential and develop in both academic and practical aspects.

1.3 Overview of the thesis

This thesis makes several contributions toward the study on both financial and real options. Let us introduce each section of this thesis.

Sections 2 and 3 state contributions to the study on financial options. A conventional approach to the option pricing problem, as represented by Black-Scholes [10], assumes some stochastic differential equation model for the dynamics of asset prices to derive the no-arbitrage prices of derivatives. In contrast, we assume no particular models for the dynamics of asset prices. Instead, we examine a no-arbitrage price range of a derivative based only on the observed prices of other derivatives. This type of study is similar to the study of implied tree models proposed by [19, 23, 73] in the sense that both studies are based on the observed prices of derivatives. Using optimization techniques, Bertsimas and Popescu [6] investigated this type of option pricing problem in full detail. To put it more concretely, they showed that the problem of finding upper and lower bounds on derivative prices can be reduced to a semi-infinite programming problem and, in some special cases, a linear or semi-definite programming problem. In Sections 2 and 3, we extend their results toward the following two directions.

Section 2 clarifies financial meanings of duality of the semi-infinite programming problem, which has been used only from the computational profit in the previous studies such as [6, 34]. We show that the dual of the problem of finding the derivative price range from the observed prices of other derivatives is equivalent to the problem of finding the optimal buy-and-hold hedging portfolio consisting of the derivatives. The result shows another importance of this type of problem which was regarded as a problem of finding bounds on derivative prices.

Section 3 derives analytical bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call and put options. These bounds can be identified as bounds on risk neutral probabilities. We also investigate the characteristics and possible applications of the bounds by computing the bounds from Nikkei-225 option data in Japan.

On the other hand, Sections 4–6 states the results concerning with real options. As introduced in Section 1.2, one of the most important studies on real options is to analyze strategic real options, that is, competition among firms, conflicts between the owner and the manager, etc. We add new elements into the existing strategic real options models. While Sections 4 and 5 extend models where two firms compete in the same investment project, Section 6 extends a model which involves asymmetric information between the owner and the manager.

Section 4 extends the R&D competition model by [87] to a model where the firms can choose the target of the research from two alternative technologies of different standard. In the model, we can understand the simultaneous effects of the competition on the investment timing and the choice of the target. In particular, we show that in a de facto standard competition a lower-standard technology which is easy to invent could emerge than is developed in the monopoly. The results have also a theoretical contribution because little has been studied about the strategic real options involving both the investment timing and the choice of the project type.

Section 5 investigates a firm's loss due to incomplete information about its competitor's efficiency. We formulate a model where a start-up with a unique idea and technology pioneers a new market but will eventually be expelled from the market by a large firm's subsequent entry. We then evaluate the start-up's loss due to incomplete information about the large firm's behavior. There are several studies (e.g., [50, 40]) that focus the firms' equilibrium investment strategies under incomplete information. However, no study has tried to elucidate in which cases and how greatly the firm suffers the loss due to incomplete information, and therefore we obtain several new economic insights.

Section 6 mentions the results regarding asymmetric information in a decentralized firm where the owner delegates the investment decision to the manager with private information. The previous studies such as [31, 56] considered only the incentive mechanism as a measure to deal with asymmetric information. In practice, however, the owner conducts a costly audit to claim compensation and penalty against the owner's false and inefficient act. Taking this into account, we incorporate the auditing technology into a model of [31]. By doing this, we can make a realistic analysis of the decentralized firm in which the owner can resolve agency conflicts by means of both bonus-incentive and audit. The solution derived in this setting not only brings about economic implications, but also plays an important role of combining several existing studies.

Finally Section 7 summarizes the results obtained in this thesis, and then mentions important issues of future research relevant to each section.

Chapter 2

Option Pricing Based on Prices of Other Derivatives: Duality

2.1 Introduction

One of the most important issues in financial economics is to derive an appropriate price of a derivative security, which is called option pricing. Option pricing is based on the well-known fundamental assumption that the market is no-arbitrage, which intuitively means that we cannot increase a value of our portfolio without any risk. Under the no-arbitrage assumption, a derivative price must be the same as a value of a portfolio that replicates the derivative if such a hedging portfolio exists. In addition to the no-arbitrage assumption, many option pricing methods assume some stochastic differential equations for prices of risky assets. A typical approach, the Black-Scholes model introduced in [10] and [60] assumes a geometric Brownian motion for the risky stock price. By this assumption, every derivative can be replicated by a portfolio consisting of the risk-free bond and the underlying stock, and therefore has a unique price equal to the price of the hedging portfolio. However, it is well known that a stock price in the actual market does not obey the geometric Brownian motion. For example, a log-price of a stock displays a heavy tailed distribution different from a Gaussian distribution. It seems hard to find a stochastic differential equation that perfectly fits the dynamics of an asset price.

Thus, a natural question that arises is to derive a derivative price range based only on the no-arbitrage assumption and the observed prices of other derivatives without assuming any stochastic model for the dynamics of asset prices. This question has been studied in [14], [33] and [53]. They derived upper and lower bounds on option prices consistent with given mean and (co)variance of the underlying asset prices under a risk-neutral measure. Bertsimas and Popescu [6] showed that the question can be well treated in the framework of an SIP (semi-infinite programming problem). In particular, they showed that several problems are reducible to an SDP (semi-definite programming problem) by using duality in the SIP. By the same duality technique, Han et al. [34] investigated a case in which a derivative is written on multi-assets. While all studies mentioned above have treated the case of a single maturity, Bertsimas and Bushueva [4, 5] derived an option price range consistent with the prices of other derivatives with distinct maturities. This type of study is also related to a study of implied models proposed in [19], [23] and [73] in the sense that both studies use the observed prices of derivatives.

This chapter gives a financial interpretation of duality of the SIP, which has been used only from the computational profit in the previous studies [6] and [34]. We show that the dual problem is related to a hedging strategy called a buy-and-hold hedging portfolio. This financial interpretation also explains the relationship between the approach based only on the no-arbitrage assumption and the observed prices of derivatives and the usual stochastic approach such as the Black-Scholes model.

This chapter is organized as follows. Section 2.2 gives a brief review of the results which were obtained mainly in [6], after introducing two financial market models and notations. Section 2.3 describes the financial interpretation of duality of the SIP.

2.2 Preliminaries

This section introduces two financial market models, and then gives a brief explanation for the previous results obtained in [6]. We first introduce notations and two models which will be used throughout Chapters 2 and 3.

Notation Let $T > 0$ and let m be a positive integer. Let Φ^T and F_i^T denote simple claims written on m risky assets with exercise date T and payoff functions ϕ and $f_i : R_+^m \mapsto R_+$, respectively. The notation R_+ denotes $[0, \infty)$. Prices of Φ^T and F_i^T at time t are $\Phi^T(t)$ and $F_i^T(t)$, respectively. Let $\Lambda(R_+^m)$ denote the set of all probability measures on the measurable space $(R_+^m, \mathcal{B}(R_+^m))$, where $\mathcal{B}(R_+^m)$ is the Borel σ-algebra on R_+^m.

Model A Assume a no-arbitrage financial market which consists of m risky assets and one risk-free asset with constant risk-free rate $r(t) = 0$. The price process of m risky assets $S(t)$ is an m dimensional $\mathcal{F}_t$ adapted process with values in R_+^m defined on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$.

Model B In addition to the assumptions in Model A, $S(t)$ follows stochastic differential equations under P such that

$$P\left(S(t) \in \{x \in R_+ \mid |x-a| < \epsilon\}\right) > 0 \quad (t, \epsilon > 0,\ a \in R_+^m).$$

We can take any deterministic function for the risk-free rate $r(t)$, but we assume $r(t) = 0$ without loss of generality. Model A is a broad model based only on the no-arbitrage assumption, and several papers such as [6, 34] investigated option pricing in Model A. On the other hand, Model B is a more specific model including the Black-Scholes model which has been studied more frequently than Model A in option pricing.

Since the market is no-arbitrage in both models, there exists a risk-neutral measure $\tilde{P}$ on $(\Omega, \mathcal{F})$. By using $\tilde{P}$, the price of Φ^T at time t must be expressed as

$$\Phi^T(t) = E_{\tilde{P}}[\phi(S(T))|\mathcal{F}_t], \tag{2.2.1}$$

which follows from the Fundamental Theorem in option pricing (for instance, see p.133 – p.153 in [9]). Here, $E_{\tilde{P}}$ denotes the (conditional) mean under the probability measure $\tilde{P}$. In Model A, the problem of finding the supremum on prices of a simple claim Φ^T consistent with observed prices of F_i^T is described as follows:

$$\begin{array}{lll} \text{maximize }_{\tilde{P}(\sim P)} & E_{\tilde{P}}[\phi(S(T))] & \\ \text{subject to} & E_{\tilde{P}}[f_i(S(T))] = q_i & (i = 1, 2, \ldots, n), \end{array} \tag{2.2.2}$$

where $\tilde{P}$ moves over the set of probability measures on $(\Omega, \mathcal{F})$ such that $\tilde{P}$ is equivalent to the observed probability measure P ($\tilde{P} \sim P$ in problem (2.2.2) means that $\tilde{P}$ is a probability measure equivalent to P). We can also consider the problem of finding the infimum on $\Phi^T(0)$ consistent with $F_i^T(0)$ by replacing *maximize* with *minimize* in (2.2.2). Since no particular dynamics of $S(t)$ under P is given in Model A, the property of equivalence restricts nothing. By taking ξ as a distribution of $S(T)$ under $\tilde{P}$, problem (2.2.2) with respect to a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ can be reduced to the following SIP with respect to a probability measure ξ on $(R_+^m, \mathcal{B}(R_+^m))$.

$$\begin{array}{lll} \text{maximize }_{\xi \in \Lambda(R_+^m)} & \displaystyle\int_{R_+^m} \phi(x)\mathrm{d}\xi & \\ \text{subject to} & \displaystyle\int_{R_+^m} f_i(x)\mathrm{d}\xi = q_i & (i = 1, 2, \ldots, n). \end{array} \tag{2.2.3}$$

Note that this problem is a concrete problem compared with the abstract problem (2.2.2) on the abstract measurable space $(\Omega, \mathcal{F})$. Even in Model B, we can derive an upper bound on $\Phi^T(0)$ from the same formulation (2.2.3), but the upper bound could not be *tight* in Model B. In Model B, probability measure $\tilde{P}$ should be restricted to a smaller region by the additional assumption of stochastic differential equations. For example, $\tilde{P}$ is uniquely determined if Model B is a complete model such as the Black-Scholes model. In most cases, since problem (2.2.3) only finds too *loose* an upper bound in Model B, solving problem (2.2.3) in the framework of Model B is not helpful. Thus, in the remainder of chapter, problem (2.2.3) is considered only in Model A.

Regardless of financial studies, it is known in the duality theory of SIP that the dual of problem (2.2.3) becomes

$$\begin{array}{lll} \text{minimize}_{\ z \in R^{n+1}} & z_0 + \sum_{i=1}^{n} q_i z_i & \\ \text{subject to} & z_0 + \sum_{i=1}^{n} z_i f_i(x) - \phi(x) \geq 0 & (x \in R_+^m) \end{array} \tag{2.2.4}$$

(see [12]) and furthermore the optimal values of (2.2.3) and (2.2.4) equalize under the Slater condition in problem (2.2.3), that is,

$$(1, q_1, \ldots, q_n) \in \text{int} \left\{ \left(\int_{R_+^m} 1 \mathrm{d}\xi, \int_{R_+^m} f_1(x) \mathrm{d}\xi, \ldots, \int_{R_+^m} f_n(x) \mathrm{d}\xi \right) \mid \xi \in \mathcal{A} \right\}. \tag{2.2.5}$$

Here, $\text{int}(\cdot)$ denotes the set of all interior points and $\mathcal{A}$ denotes the set of all measures (not necessarily *probability* measures) on $(R_+^m, \mathcal{B}(R_+^m))$ See Proposition 3.4 in [75]. Another condition for the strong duality to hold between (2.2.3) and (2.2.4) is that ϕ and f_i are continuous functions with compact support (see also Corollary 3.0.2 in [75]).

In Model A, several results have been obtained through the duality of the SIP. Using the duality, Bertsimas and Popescu [6] reduced the problem of finding the supremum and the infimum on $\Phi^T(0)$ consistent with the first n moments (i.e., $f_i(x) = x^i$ $(i = 1, 2, \ldots, n)$) to an SDP when $m = 1$ and ϕ is a piecewise polynomial. For the same but multi-dimensional (i.e., $m > 1$) problem studied in [6], Han et al. [34] constructed a sequence of SDP relaxations via the duality, where the approximation converges to the optimal solution as the dimension of the SDP relaxations increases.

However, the previous studies have employed the dual problem only from the computational advantage and lack a financial interpretation of the duality. The next section describes our results which reveal financial importance of the duality in terms of a buy-and-hold hedging portfolio. Viewed in this light, unlike problem (2.2.3), problem (2.2.4) is meaningful in Model B. The dual viewpoint gives another importance of the problem of finding a derivative price range based only on the no-arbitrage assumption and other derivative prices.

2.3 Financial interpretation of duality

This section clarifies the financial meaning of the duality between problems (2.2.3) and (2.2.4). We can actually show that problem (2.2.4) itself is a meaningful problem of finding the minimum investment cost of buy-and-hold super-hedging portfolios in Model B. We can also show that problem (2.2.4) finds an arbitrage buy-and-hold strategy if the observed prices of derivatives contradict the no-arbitrage assumption.

2.3.1 A buy-and-hold hedging portfolio

First, we explain a buy-and-hold portfolio before clarifying the meaning of the duality from the viewpoint of financial economics. We consider option pricing and hedging in Model B, which is a general approach. In Model B, buyers' price of a simple claim Φ^T and sellers' price of a simple claim Φ^T are usually defined as

$$q_{\text{buy}}(\Phi^T) = \sup \left\{ \Pi(0) \mid \Pi(t) : \begin{array}{c} \text{a value process of a self-financing} \\ \text{portfolio such that } \Pi(T) \leq \phi(S(T)) \end{array} \right\}$$

and

$$q_{\text{sell}}(\Phi^T) = \inf \left\{ \Pi(0) \mid \Pi(t) : \begin{array}{c} \text{a value process of a self-financing} \\ \text{portfolio such that } \Pi(T) \geq \phi(S(T)) \end{array} \right\}$$

respectively, where a value process $\Pi(t)$ is expressed as

$$\Pi(t) = H_0(t) + H_1(t) \cdot S(t) \tag{2.3.1}$$

for $\mathcal{F}_t$ adopted processes $H_0(t)$ and $H_1(t)$. Here, $H_0(t)$ and $H_1(t)$ mean the amounts of the risk-free asset and the risky assets included in a portfolio, respectively. Generally, the following relationship holds:

$$q_{\text{buy}}(\Phi^T) \leq \Phi^T(0) \leq q_{\text{sell}}(\Phi^T).$$

In a complete market both prices equalize, and we have

$$q_{\text{buy}}(\Phi^T) = q_{\text{sell}}(\Phi^T) = \Phi^T(0).$$

Notice that $H_0(t)$ and $H_1(t)$ are usually continuously re-balanced in portfolios which realize $q_{\text{buy}}(\Phi^T)$ and $q_{\text{sell}}(\Phi^T)$. In contrast to the usual buyers' and sellers' prices mentioned above, we define buyers' and sellers' buy-and-hold hedging prices by restricting a portfolio to a buy-and-hold portfolio, which means a constant portfolio with time t. For simple claims F_i^T $(i = 1, 2, \ldots, n)$, we define buyers' buy-and-hold hedging prices $\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T)$ and sellers' buy-and-hold hedging prices $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ as follows:

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T) = \sup \left\{ \Pi(0) \mid \Pi(t) : \begin{array}{l} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } \Pi(T) \leq \phi(S(T)) \end{array} \right\}, \tag{2.3.2}$$

$$\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T) = \inf \left\{ \Pi(0) \mid \Pi(t) : \begin{array}{l} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } \Pi(T) \geq \phi(S(T)) \end{array} \right\}, \tag{2.3.3}$$

where a value process $\Pi(t)$ is expressed as

$$\Pi(t) = z_0 + \sum_{i=1}^{n} z_i F_i^T(t), \tag{2.3.4}$$

for some constants z_i $(i = 0, 1, \ldots, n)$. In particular we can take F_i^T $(i = 1, 2, \ldots, m)$ as risky assets themselves, which means $F_i^T(t) = S_i(t)$. In this case, we have

$$\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T) \leq \mathrm{q}_{\mathrm{buy}}(\Phi^T) \leq \mathrm{q}_{\mathrm{sell}}(\Phi^T) \leq \mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T),$$

because we restrict the set of self-financing portfolios (2.3.1) to the set of buy-and-hold portfolios (2.3.4).

The sellers' price $\mathrm{q}_{\mathrm{sell}}(\Phi^T; F_i^T)$ means the minimum investment costs necessary to super-hedge the simple claim Φ^T with a buy-and-hold portfolio consisting of the risk-free asset and F_i^T, and hence is a favorable price for sellers of Φ^T. On the contrary, the buyers' price $\mathrm{q}_{\mathrm{buy}}(\Phi^T; F_i^T)$ is a favorable price for buyers. In the following subsection, we reveal the financial meaning of the duality in terms of buyers' and sellers' buy-and-hold hedging prices.

2.3.2 Financial interpretation of duality of the SIP

Now we give a financial interpretation of duality of problems (2.2.3) and (2.2.4), which arises as a problem of determining a derivative price range based only on the no-arbitrage assumption and the observed prices of other derivatives. The following proposition states the meaning of the dual problem (2.2.4).

Proposition 2.3.1 Let the derivative prices satisfy

$$F_i^T(0) = q_i \qquad (i = 1, 2, \ldots, n),$$

which are consistent with Model B. The optimal value in problem (2.2.4) is equivalent to $q_{\text{sell}}(\Phi^T; F_i^T)$ in Model B. An optimal solution $z^* \in R^{n+1}$ in problem (2.2.4) gives an optimal buy-and-hold super-hedging portfolio for Φ^T.

Proof By definition (2.3.3), we have

$$\begin{aligned} q_{\text{sell}}(\Phi^T; F_i^T) &= \inf \left\{ \Pi(0) \mid \{\Pi(t)\} : \begin{array}{c} \text{a value process of a buy-and-hold} \\ \text{portfolio such that } \Pi(T) \geq \phi(S(T)) \end{array} \right\} \\ &= \inf \left\{ z_0 + \sum_{i=1}^{n} z_i F_i^T(0) \mid z \in R^{n+1} \text{ such that } z_0 + \sum_{i=1}^{n} z_i F_i^T(T) \geq \phi(S(T)) \right\} \\ &= \inf \left\{ z_0 + \sum_{i=1}^{n} q_i z_i \mid z \in R^{n+1} \text{ such that } z_0 + \sum_{i=1}^{n} z_i f_i(x) \geq \phi(x) \ (x \in R_+^m) \right\}. \end{aligned}$$

The last equality holds because $S(T)$ could be all vectors in R_+^m by the assumptions of Model B. By the right-hand side of the last equality, the problem of finding $q_{\text{sell}}(\Phi^T; F_i^T)$ in Model B is equivalent to problem (2.2.4), and an optimal solution $z^* \in R^{n+1}$ in problem (2.2.4) gives an optimal buy-and-hold super-hedging portfolio for Φ^T if it exists. □

Remark 2.3.1 Problem (2.2.4) with minimizing and $\geq$ in the constraint replaced by maximizing and $\leq$, respectively, finds an optimal buy-and-hold under-hedging portfolio for Φ^T if it exists, and its optimal value becomes $q_{\text{buy}}(\Phi^T; F_i^T)$.

By the duality between problems (2.2.3) and (2.2.4), $q_{\text{sell}}(\Phi^T; F_i^T)$ in Model B is larger than the supremum on $\Phi^T(0)$ in Model A. Furthermore, if the Slater condition (2.2.5) is satisfied, then $q_{\text{sell}}(\Phi^T; F_i^T)$ in Model B is equal to the supremum on $\Phi^T(0)$ in Model A. This is the financial interpretation of the duality which emerges in the context of option pricing based only on the no-arbitrage assumption and prices of other derivatives.

Problem (2.2.4) gives an arbitrage buy-and-hold portfolio in the case where problem (2.2.3) is infeasible (i.e., observed prices $F_i^T(0) = q_i$ $(i = 1, 2, \ldots, n)$ contradict the no-arbitrage assumption).

Corollary 2.3.1 Let the derivative prices satisfy

$$F_i^T(0) = q_i \qquad (i = 1, 2, \ldots, n).$$

An optimal solution of the following problem gives an arbitrage buy-and-hold portfolio, if and only if the optimal value is less than 0 :

$$\begin{aligned} &\text{minimize}_{\,y\in R^{n+1}} && z_0 + \sum_{i=1}^{n} q_i z_i \\ &\text{subject to} && z_0 + \sum_{i=1}^{n} z_i f_i(x) \geq 0 \qquad (\forall x \in R^m_+) \\ & && z_i \in [-1,1] \qquad (i = 0,1,\dots,n). \end{aligned} \tag{2.3.5}$$

Remark 2.3.2 Problem (2.3.5) adds the extra constraints $z_i \in [-1,1]$ to problem (2.2.4) for $\phi(x) \equiv 0$, so that the optimal value is always bounded. For an investment in the actual market, we must take the range of z_i as a volume to which we can trade F_i^T at the observed prices q_i, and restrict z_i to be integral multiples of a minimum trade unit.

Proposition 2.3.1 shows that problem (2.2.4) itself is an important problem of finding the minimum investment costs of super-hedging buy-and-hold portfolios for Φ^T which consist of the risk-free asset and given derivatives F_i^T in Model B. This problem is meaningful especially for practical purpose, because in the actual market continuous hedging such as delta hedging has a problem of transaction costs. Since Corollary 2.3.1 enables us to make an arbitrage portfolio if it exists, it could be useful for a large investment company which can trade many kinds of European derivative securities with the same maturity.

Our interpretation from the financial viewpoint also unveils the relationship between results in Model A and Model B. For instance, it is shown in [4, 5] that function $\Psi_\xi(k) = \int_{R_+} \max\{x - a, 0\}\mathrm{d}\xi$ $(a \geq 0)$ determines a unique risk-neutral measure ξ. This has a dual relationship with the following proposition regarding buy-and-hold hedging in the Black-Scholes model on p.123 in [9].

Proposition 2.3.2 Assume the Black-Scholes model that consists of a risk-free asset and a risky asset S, and let $\phi : R_+ \mapsto R_+$ be a continuous function with compact support. Then, a simple claim with payoff function $\phi(S(T))$ can be replicated with arbitrary precision using a buy-and-hold portfolio consisting of the risk-free asset and several call options.

Figure 2.1 illustrates Proposition 2.3.2. Here, v_1 and v_2 represent the values of the super-hedging and under-hedging portfolios for Φ^T at T consisting of call options F_i^T with payoff $f_i = \max\{x - k_i, 0\}$, that is,

$$v_1(x) = \Pi_1(T) \geq \phi(x) \quad (x \in R_+),$$

$$v_2(x) = \Pi_2(T) \leq \phi(x) \quad (x \in R_+),$$

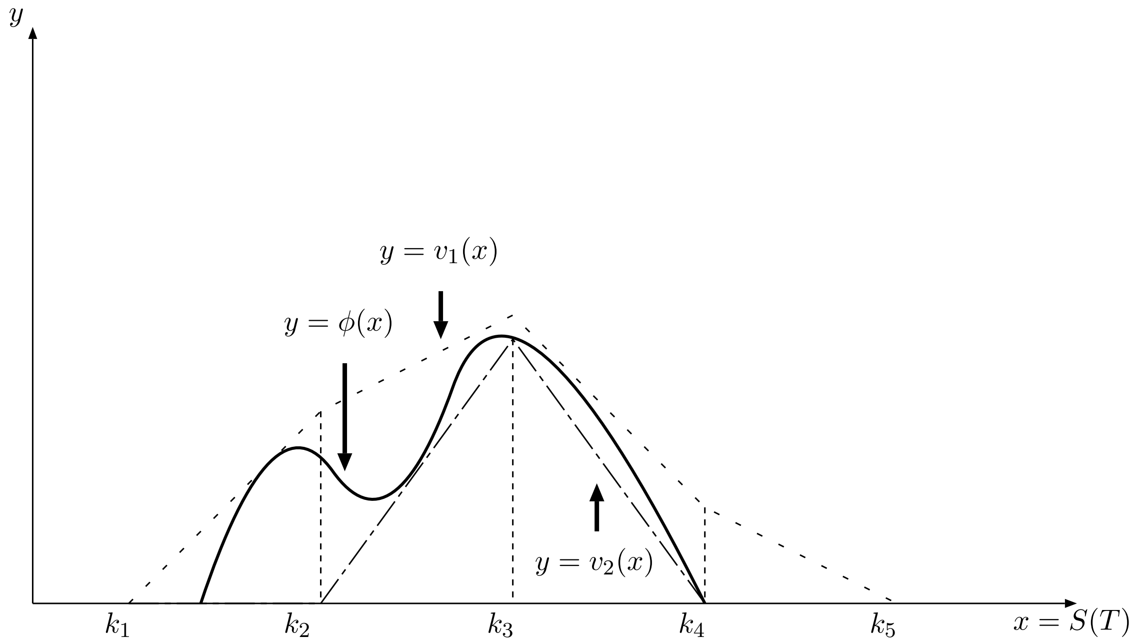

Figure 2.1: Buy-and-hold hedging portfolios.

where Π_j $(j = 1, 2)$ are of the form

$$\Pi_j(t) = \sum_{i=1}^{n} z_{i,j} F_i^T(t)$$

with certain constants $z_{i,j}$ $(i = 1, 2, \ldots, n,\ j = 1, 2)$. The relationship $\Pi_2(0) \leq \Phi^T(0) \leq \Pi_1(0)$ always holds, and Proposition 2.3.2 shows that $\Pi_j(0)$ can be made arbitrarily close to $\Phi^T(0)$ by letting $n \to +\infty$. Thus, the dual problem (2.2.4) could be more helpful to visualize the meaning than problem (2.2.3). As a special case of problem (2.2.3), the problem of determining a price range for a call option based on the observed prices of call options with other strikes has been fully investigated in [6]. From the dual viewpoint we can state that it is a problem of finding an optimal buy-and-hold hedging portfolio consisting of given call options.

2.4 Conclusion

This chapter has investigated the duality of the semi-infinite programming problem which arises in the context of determining a derivative price range based only on the observed prices of other derivatives and the no-arbitrage assumption (Model A). A contribution of this chapter is to give an interpretation of the duality from the viewpoint of financial economics and reveal another importance of studies in Model A. We have actually clarified that the dual of a problem of finding the supremum on derivative prices with the observed prices of other derivatives in Model A is equivalent to the problem of finding the minimum investment costs of buy-andhold super-hedging portfolios for the derivative in the usual financial market model (Model B). This problem is useful for investors because in the actual market rebalancing a hedging portfolio takes transaction costs. The interpretation links some previous studies in Model A to the results for Model B in terms of a buy-and-hold hedging portfolio.

Chapter 3

Option Pricing Based on Prices of Other Derivatives: Risk-Neutral Probabilities

3.1 Introduction

This chapter, as well as the previous chapter, investigates the option pricing based on the observed prices of other derivatives without assuming any stochastic model for the dynamics of asset prices. In particular, we investigate the problem of finding bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call options, based only on the no-arbitrage assumption. By considering this special case, we can analytically derive the bounds on risk-neutral measures, which saves us from computing the numerous corresponding LPs (linear programming problems) as discussed in [6]. We then compute the bounds from Nikkei-225 option data in Japan. To derive the risk-neutral measure implied from the real data is important, because the risk-neutral measure plays a decisive role in pricing financial securities, and it represents market's view of risk. Actually, several studies such as [26] and [37] have investigated this problem from other aspects.

This chapter is organized as follows. Section 3.2 explains the problem formulation and the results obtained in [4]. Section 3.3 describes our main result, that is, the bounds on risk-neutral measures in closed forms. Section 3.4 illustrates computational results obtained from Nikkei-225 option data in Japan.

3.2 Problem formulation

This section introduces the problem and describes some results obtained in [4], [5] and [6] for future use.

We consider the problem of finding bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of European call options with exercise date T in Model A which was introduced in Section 2.2. Throughout this chapter, we use the notations and models introduced in Section 2.2. Then, the problem which we consider in this section becomes problem (2.2.2) substituted $\phi(S(T)) = 1_{[0,a]}(S(T)) = \tilde{P}[S(T) \in [0,a]]$ and $f_i(S(T)) = \max\{S(T) - k_i, 0\}$, where $1_{[0,a]}(x)$ denotes the defining function of the set $[0,a]$. That is, for each $a \geq 0$,

$$\begin{array}{lll} \text{maximize}_{\tilde{P}(\sim P)} & \tilde{P}[S(T) \in [0,a]] & \\ \text{(or minimize)} & & \\ & & \\ \text{subject to} & E_{\tilde{P}}[\max\{S(T) - k_i, 0\}] = q_i & (i = 1, 2, \ldots, n), \end{array} \tag{3.2.1}$$

where $\tilde{P}$ moves over the set of probability measures on $(\Omega, \mathcal{F})$ such that $\tilde{P}$ is equivalent to the observed probability measure P. Recall that $\tilde{P} \sim P$ in problem (3.2.1) denotes that $\tilde{P}$ is a probability measure equivalent to P. Here, for $i = 1, 2, \ldots, n$, let q_i denote the observed prices of European call options with exercise date T and strikes k_i at time 0. Without loss of generality, we assume $0 \leq k_1 < k_2 < \cdots < k_n$ in the rest of this chapter. Note that the demension of $S(t)$, m, is always equal to 1 in the setting of chapter. Note that the payoff of the call option is defined by $\max\{S(T) - k_i, 0\}$, because the holder of the call option receives $S(T) - k_i$ by exercising the option on the exercise date T.

If we could derive the optimal values of problem (3.2.1) for all $a \geq 0$, the upper and lower bound functions can be obtained as functions of $a \geq 0$. Note that the obtained bounds may not be *tight* in the following sense: It is likely that no single risk-neutral probability measure $\tilde{P}$ gives the upper (or lower) bound function for all $a \geq 0$, though for any fixed $a \geq 0$ there exists a $\tilde{P}$ that attains the bound at a. To determine the bounds on risk-neutral cumulative distribution functions is a fundamental question, because every European option on $S(T)$ can be priced from the implied risk-neutral measure.

Since no particular dynamics of $S(t)$ under P is assumed, the equivalence $\tilde{P} \sim P$ in problem (3.2.1) does not add any restriction. Thus, by taking ξ as a distribution of $S(T)$ under $\tilde{P}$, we, like

(2.2.3), rewrite problem (3.2.1) for each $a \geq 0$,

$$\begin{aligned} &\text{maximize}_{\ \xi \in \Lambda(R_+)} \ \xi([0,a]) \\ &\text{(or minimize)} \\ &\text{subject to} \qquad \int_{R_+} \max\{x - k_i, 0\} \mathrm{d}\xi = q_i \qquad (i = 1, \ldots, n), \end{aligned} \tag{3.2.2}$$

Recall that $\Lambda(R_+)$ denotes the set of probability measures on the Borel space $(R_+, \mathcal{B}(R_+))$ (see Notaiton in Section 2.2). This problem is a concrete and solvable problem compared with the abstract problem (3.2.1) on the probability space $(\Omega, \mathcal{F})$.

Problem (3.2.2) is a special case of the problems investigated in [6], because $\xi([0, a])$ is the same as $\int_{R_+} 1_{[0,a]}(x)\mathrm{d}\xi$.

Although it involves the discontinuous payoff function $1_{[0,a]}(x)$, for a fixed $a \geq 0$, problem (3.2.2) can be reduced to an LP by using the same dual technique proposed in [6]. In this chapter, however, we derive the infimum and the supremum of problem (3.2.2) as functions of a (≥ 0) in closed forms. In other words, we can compute the upper and lower bounds without actually solving the numerous LPs. From the dual viewpoint revealed in the previous chapter (see also [67]), problem (3.2.2) is equivalent to finding the minimum costs necessary to super-hedge a binary option with payoff $1_{[0,a]}(S(T))$ with a buy-and-hold portfolio including the given call options in Model B.

In most cases, not only the prices of the call options but also the underlying asset price $S(0)$ itself is observed. In this case, we have only to put $k_1 = 0$ and take $S(0)$ as q_1, because the underlying asset price is equal to the price of the call option with strike 0. If $S(0)$ is observed the results in this chapter can be also applied to European put options, because the prices of the corresponding European call options can be derived from $S(0)$ and the prices of put options via the put-call parity (e.g., see p.123 in [9]), which is deduced only from the no-arbitrage assumption.

We note that the results in this chapter can also be applied to the modified problems, in which $S(T)$ in problem (3.2.1) are replaced with the maximum asset price $\max_{0 \leq t \leq T} S(t)$ and the average asset prices $1/T \int_0^T S(t)\mathrm{d}t$, by taking ξ as distributions of $\max_{0 \leq t \leq T} S(t)$ and $1/T \int_0^T S(t)\mathrm{d}t$, respectively.

Now, we describe the result derived in [4] before explaining our results. The following condition will be assumed in the subsequent analysis:

Condition A The observed prices of European call options q_i with strikes k_i (where $0 \leq k_1 <$

$k_2 < \cdots < k_n$) satisfy

$$q_1 \geq q_2 \geq \cdots \geq q_n \geq 0,$$
$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1},$$

where $\alpha_i = (q_i - q_{i-1})/(k_i - k_{i-1}) \quad (i = 2, \ldots, n)$, $\alpha_1 = -1$ and $\alpha_{n+1} = 0$. If there exists an l $(< n)$ such that $q_l = q_{l+1}$, then $q_l = q_{l+1} = \cdots = q_n = 0$.

Condition A tells that the piecewise linear price function obtained by connecting points (k_i, q_i) $(i = 1, 2, \ldots, n)$ is convex and monotonically decreasing (see Figure 3.1).

For $a \geq 0$, we define $\Psi_\xi(a)$ as the following function:

$$\Psi_\xi(a) = \int_{R_+} \max\{x - a, 0\} \mathrm{d}\xi. \tag{3.2.3}$$

This means the price of the call option with strike a under the assumption that the risk-neutral measure is ξ. The following proposition proved in [4] shows that Condition A is a necessary and sufficient condition for the existence of a risk-neutral measure ξ.

Proposition 3.2.1 At least one probability measure ξ on $(R_+, \mathcal{B}(R_+))$ exists such that

$$\Psi_\xi(k_i) = q_i \qquad (i = 1, \ldots, n)$$

if and only if Condition A holds, where Ψ_ξ is defined by (3.2.3).

Remark 3.2.1 Condition A is usually observed to hold on real data when the trade volume is large. We will discuss this in Section 3.4 (see Figure 3.3).

3.3 Bounds on risk-neutral measures

This section derives the optimal values of problem (3.2.2) in closed forms, for both versions of maximizing and minimizing the objective function. We then discuss potential applications of the results.

Let $f_{\max}(a)$ and $f_{\min}(a)$ denote the optimal values of problem (3.2.2) to maximize and to minimize, respectively. First, we introduce the following notations:

$$\gamma_i = q_i - \alpha_i k_i \qquad (i = 1, 2, \ldots, n),$$

$$\gamma_{n+1} = q_n,$$

$$l_i = \frac{\gamma_i - \gamma_{i+2}}{\alpha_{i+2} - \alpha_i} \qquad (i = 1, 2, \ldots, n-1),$$

$$l_n = -\frac{\gamma_n}{\alpha_n},$$

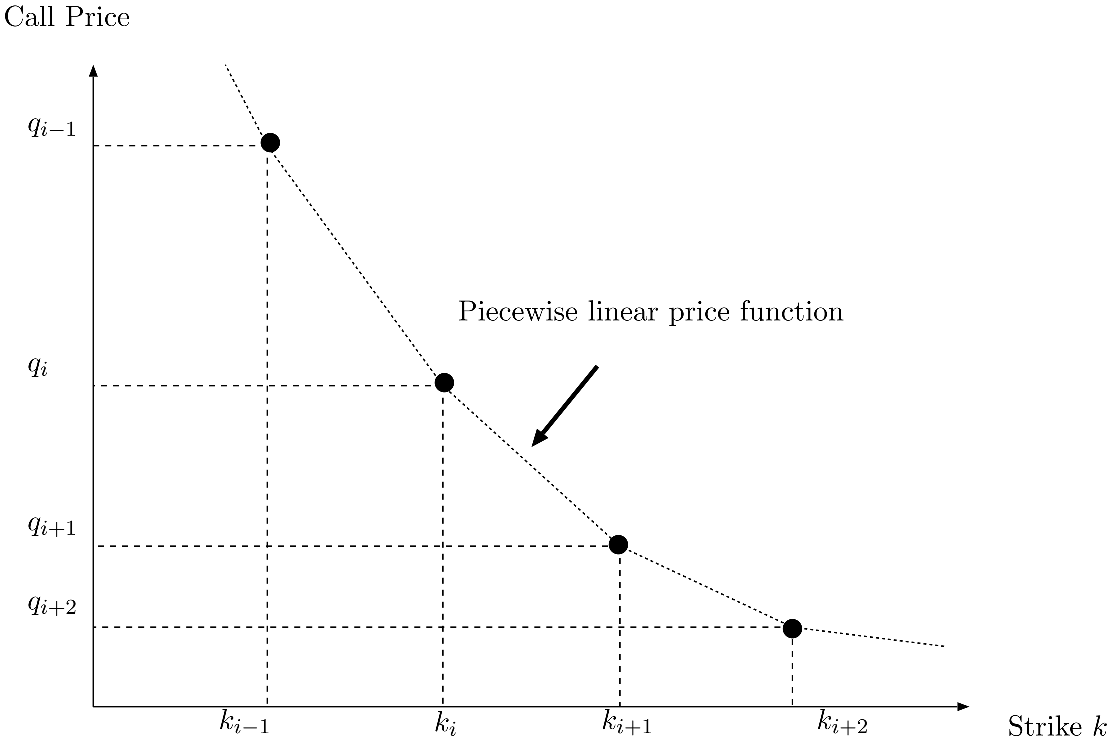

Figure 3.1: Convexity of call option prices.

where α_i $(i = 1, 2, \ldots, n + 1)$ are defined in Condition A. Figure 3.2 illustrates the meaning of these quantities. The following proposition, giving close forms of $f_{\max}(a)$ and $f_{\min}(a)$, is our main theoretical result. With this proposition, we no longer need to solve the corresponding LP for each a, as proposed in [6].

Proposition 3.3.1 For strikes k_i $(i = 1, 2, \ldots, n)$, let q_i (i.e., the prices of call options with payoff $\max\{S(T) - k_i, 0\}$) be given. If prices q_i satisfy Condition A and $q_n > 0$, then $f_{\max}(a)$ and $f_{\min}(a)$ are expressed as follows:

$$f_{\max}(a) = \begin{cases} 1 + \alpha_2 & (0 \le a < k_1) \\ 1 + \alpha_i + \dfrac{q_{i+1} - \alpha_i k_{i+1} - \gamma_i}{k_{i+1} - a} & (k_i \le a < l_i, i = 1, 2, \ldots, n-1) \\ 1 + \alpha_{i+2} & (l_i \le a < k_{i+1}, i = 1, 2, \ldots, n-1) \\ 1 & (k_n \le a), \end{cases} \tag{3.3.1}$$

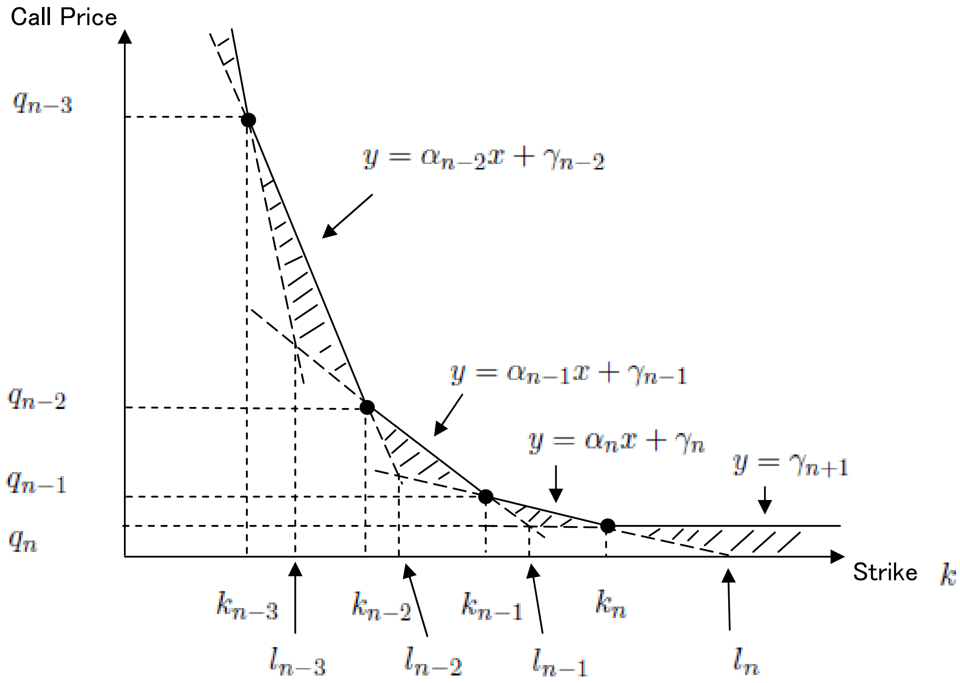

Figure 3.2: Meanings of α_i, γ_i, l_i.

$$f_{\min}(a) = \begin{cases} 0 & (0 \le a < k_1) \\ 1 + \alpha_{i+2} + \dfrac{\gamma_{i+2} + \alpha_{i+2}k_i - q_i}{a - k_i} & (l_i \le a < k_{i+1}, i = 1, 2, \dots, n-1) \\ 1 + \alpha_i & (k_i \le a < l_i, i = 1, \dots, n) \\ 1 - \dfrac{q_n}{a - k_n} & (l_n \le a). \end{cases} \tag{3.3.2}$$

Proof Assume that Condition A and $q_n > 0$ hold. Let $\Psi_\xi : R_+ \mapsto R_+$ be given by (3.2.3). The following equality was proved in [4]:

$$\begin{aligned} \Psi'_\xi(a+) &= -\xi((a, \infty]) \\ &= -1 + \xi([0, a]), \end{aligned} \tag{3.3.3}$$

where $\Psi'_\xi(a+)$ denotes the right derivative of Ψ_ξ at a. By (3.3.3) and the definition of $f_{\max}$ and

$f_{\min}$ (i.e., optimal values of problem (3.2.2)) we have

$$\begin{aligned} f_{\max}(a) &= \sup_{\xi \in \tilde{\Lambda}} \xi([0, a]) \\ &= 1 + \sup_{\xi \in \tilde{\Lambda}} \Psi'_{\xi}(a+). \end{aligned} \tag{3.3.4}$$

Here, $\tilde{\Lambda}$ is the set of probability measures that satisfy the constraints of problem (3.2.2). Similarly, we have

$$f_{\min}(a) = 1 + \inf_{\xi \in \tilde{\Lambda}} \Psi'_{\xi}(a+). \tag{3.3.5}$$

By Proposition 3.2.1, for a fixed a (≥ 0), there exists a probability measure $\xi \in \tilde{\Lambda}$ satisfying $q = \Psi_{\xi}(a)$ if and only if Condition A holds for the set of points consisting of (a, q) and (k_i, q_i) $(i = 1, 2, \ldots, n)$. Thus, we have

$$\sup_{\xi \in \tilde{\Lambda}} \Psi_{\xi}(a) = \begin{cases} \alpha_1 a + \gamma_1 & (0 \leq a < k_1) \\ \alpha_{i+1} a + \gamma_{i+1} & (k_i \leq a < k_{i+1}, i = 1, 2, \ldots, n-1) \\ q_n & (k_n \leq a) \end{cases} \tag{3.3.6}$$

and

$$\inf_{\xi \in \tilde{\Lambda}} \Psi_{\xi}(a) = \begin{cases} \alpha_2 a + \gamma_2 & (k < k_1) \\ \alpha_i a + \gamma_i & (k_i \leq a < l_i, i = 1, 2, \ldots, n) \\ \alpha_{i+2} a + \gamma_{i+2} & (l_i \leq a < k_{i+1}, i = 1, 2, \ldots, n-1) \\ 0 & (l_n \leq a), \end{cases} \tag{3.3.7}$$

from the fact that the piecewise linear function connecting (a, q) and (k_i, q_i) $(i = 1, 2, \ldots, n)$ is convex and decreasing. In Figure 3.2, the hatched regions between the upper dotted line and the lower dotted lines illustrate the area of points (a, q) between (3.3.6) and (3.3.7). Extending this results to all points a (≥ 0), we see that the price function $\Psi_{\xi}(a)$ must be a convex and decreasing function contained in the hatched regions. Conversely, we can show, by modifying Proposition 3.2.1 as in [4], that there exists a $\xi \in \tilde{\Lambda}$ such that $\Psi_{\xi}(a) = \psi(a)$ $(a \geq 0)$ for any convex and decreasing function $\psi(a)$ $(a \geq 0)$ in the hatched regions. Thus, by considering the right derivatives of all convex and decreasing functions in the hatched regions in Figure 3.2, for $k_i \leq a < l_i$ $(i \leq n-1)$, we have

$$\sup_{\xi\in\tilde{\Lambda}} \Psi'_\xi(a+) = \frac{q_{i+1} - (\alpha_i a + \gamma_i)}{k_{i+1} - a} \tag{3.3.8}$$
$$= \alpha_i + \frac{q_{i+1} - \alpha_i k_{i+1} - \gamma_i}{k_{i+1} - a}$$
$$\inf_{\xi\in\tilde{\Lambda}} \Psi'_\xi(a+) = \alpha_i. \tag{3.3.9}$$

Here, the right-hand side of (3.3.8) is the gradient of the line connecting two points (k_{i+1}, q_{i+1}) and $(a, \inf_{\xi\in\tilde{\Lambda}} \Psi_\xi(a))$, and the right-hand side of (3.3.9) is the gradient of the lower dotted line for $k_i \le k \le l_i$ in Figure 3.2. For $l_i \le a < k_{i+1}$ $(i \le n-1)$, we have

$$\sup_{\xi\in\tilde{\Lambda}} \Psi'_\xi(a+) = \alpha_{i+1} \tag{3.3.10}$$
$$\inf_{\xi\in\tilde{\Lambda}} \Psi'_\xi(a+) = \frac{\alpha_{i+2} a + \gamma_{i+2} - q_i}{a - k_i} \tag{3.3.11}$$
$$= \alpha_{i+2} + \frac{\gamma_{i+2} + \alpha_{i+2} k_i - q_i}{a - k_i},$$

where, the right-hand side of (3.3.10) is the gradient of the lower dotted line for $l_i \le k \le k_{i+1}$ in Figure 3.2, and the right-hand side of (3.3.11) is the gradient of the line connecting two points (k_i, q_i) and $(a, \inf_{\xi\in\tilde{\Lambda}} \Psi_\xi(a))$. For the cases of $a < k_1$ and $k_n \le a$ we can derive the supremum and the infimum on the right derivatives in (3.3.4) and (3.3.5) by a similar geometric consideration. The resulting functions are given as (3.3.1) and (3.3.2) in this proposition. □

Remark 3.3.1 In the above proposition, we assumed $q_n > 0$ for the practical reason that, in the actual market, no call option can be traded at price 0. However similar results can be obtained even if $q_n = 0$ is allowed.

Remark 3.3.2 Figure 3.4 illustrates the functions $f_{\max}(a)$ and $f_{\min}(a)$ for some given data (as will be discussed in Section 3.4).

3.4 Computational results

We compute the bounds of Proposition 3.3.1 from the data of Nikkei-225 options, which are most popular in the option market of Japan. Then, the underlying asset price $S(t)$ is the Nikkei-225 price at time t, and we took as q_i the closing prices of the options with strike k_i on the day 4 weeks before the exercise date (i.e., $t = 0$ on this day and $t = T$ on the exercise date). We set the risk-free rate as $r = 0$, as the maturity is only 4 weeks. For $k_1 = 0$, q_1 was taken as the closing price of Nikkei-225 on the day $t = 0$ (i.e., $q_1 = S(0)$), because $S(0)$ is identified as the price of the call option with strike 0. We chose the data according to the following rules to improve the data reliability:

(a) Use prices of all Nikkei-225 call and put options which have more than 500 trade volume.

(b) When both call and put options with the same strike and the same exercise date have more than 500 trade volumes, choose the one which has a larger trade volume. Then, if put option prices are chosen, determine the corresponding call option prices by applying the put-call parity (i.e., the relation between prices of a call option and a put option, see p.123 in [9]).

We confirmed that the call option prices obtained by the above rules mostly satisfy Condition A. An example is shown in Figure 3.3, which was computed from the data on March 11, 2004 (4 weeks before the exercise date April 8, 2004). In Figure 3.3, there is a large blank area between $k = 0$ and 8500, because we used not only prices of the call options with strikes $8500, 9000, \ldots, 13500$ but also the Nikkei-225 price $S(0) = 11297$ as the price of the call option with strike 0. For detailed data, refer to Tables 3.1 and 3.2. Nikkei-225 options are usually traded with 14 strikes, which are set at every 500 Japanese Yen around the present Nikkei-225 price.

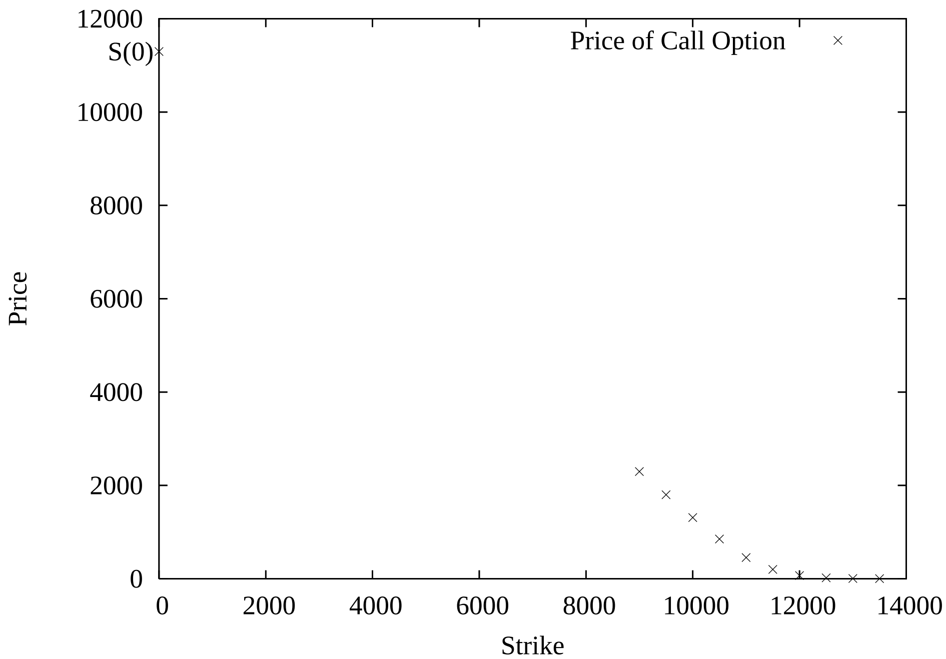

Figure 3.3: Prices of Nikkei-225 call options on March 11, 2004, with the exercise date of April 8, 2004.

Table 3.1: Data of Nikkei-225 options on March 11, 2004, with the exercise date April 8, 2004.

Strike	Call Option Price	Trade Volume	Put Option Price	Trade Volume	Choose
7500	N/A	0	N/A	0	N/A
8000	N/A	0	N/A	0	N/A
8500	2780	30	N/A	0	N/A
9000	N/A	0	1	2840	Put
9500	N/A	0	4	3409	Put
10000	N/A	0	15	4128	Put
10500	810	32	55	3940	Put
11000	455	1188	180	866	Call
11500	200	1249	415	163	Call
12000	70	3044	775	94	Call
12500	20	1908	N/A	0	Call
13000	7	2328	N/A	0	Call
13500	2	1127	N/A	0	Call

Table 3.2: propositionices of call options on March 11, 2004, with the exercise date April 8, 2004.

Strike k_i	Price q_i
0	11297
9000	2298
9500	1801
10000	1312
10500	852
11000	455
11500	200
12000	70
12500	20
13000	7
13500	2

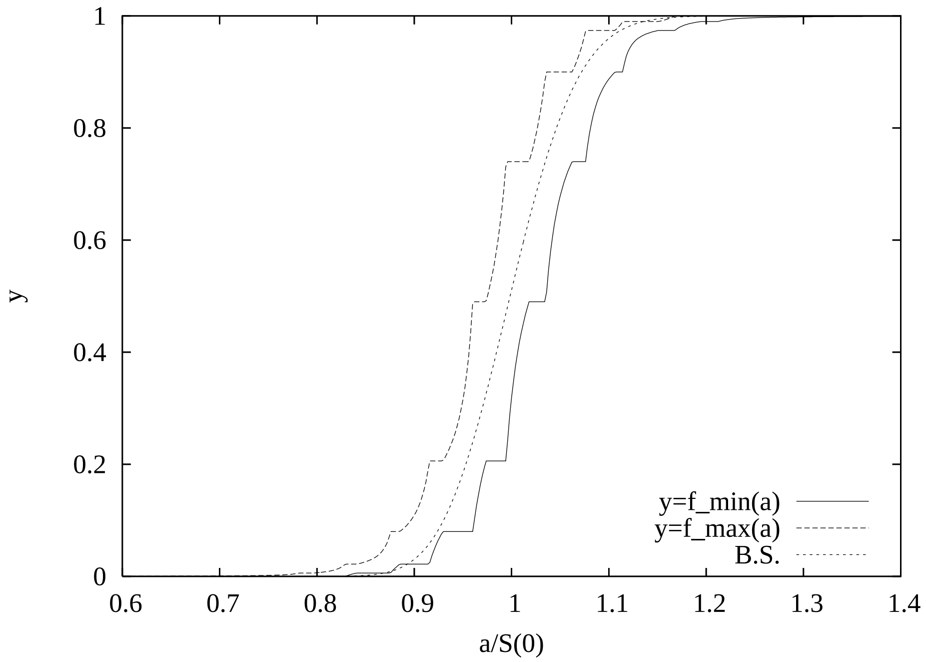

Figure 3.4: $f_{\max}(a)$ and $f_{\min}(a)$ on April 8, 2004.

Then, we compute $f_{\max}(a)$ and $f_{\min}(a)$ by Proposition 3.3.1 from the data in Figure 3.3, and illustrate them in Figure 3.4, where the scale of the x-axis is normalized by the present Nikkei-225 price $S(0)$. For comparison, we also show the risk-neutral measure obtained from the Black-Scholes model [10] with volatility $\sigma = 0.2$ (see B.S. in Figure 3.4); i.e., $\Phi((1/(\sigma\sqrt{T}))(\log(a/S(0))+\sigma^2 T/2))$, where $\Phi(y) = (1/\sqrt{2\pi})\int_{-\infty}^{y} \mathrm{e}^{-x^2/2}\mathrm{d}x$ denotes the standard normal cumulative distribution. Since the risk-neutral measure in the Black-Scholes model does not exactly satisfy the constraints in (3.2.2), the B.s. curve in Figure 3.4 slightly violates the boundaries of $f_{\max}(a)$ and $f_{\min}(a)$.

The results show that the difference between the upper and lower bounds is large in the region close to the present Nikkei-225 price $S(0)$ (i.e., $a/S(0) \approx 1$) but it is small in the region far from $S(0)$. We also computed the bounds for 32 different exercise dates from January, 2002 to August, 2004, and confirmed that a similar trend always held for all exercise dates. For example, see Figure 3.5 showing the results for 3 different exercise dates in 2004.

In closing this section we suggest a few potential applications of our results. A first application of Proposition 3.3.1 is of course to use the upper and lower bounds on ξ for the purpose of estimating the price of European options.

Another use may be to utilize the above trend of the gap between the upper and lower bounds. It tells that adding extra strikes in the region close to $S(0)$ will reduce the difference between the upper and lower bounds more efficiently than adding them in far regions. Smaller the difference, the easier it becomes to hedge other European options with the same exercise date. As an extreme case, let us assume that call options with all nonnegative strikes are actually traded. In this case, the gap between the upper and lower bounds obtained from the observed prices becomes 0, and therefore all European options with the same exercise date can be replicated by a buy-and-hold portfolio consisting of several call options, meaning that the market is *complete* (for details see the previous section [67]). Since one of the important roles of the option market is to change the market closer to being *complete*, it is more meaningful to set the strikes, not equally spaced but less spaced in the region near the present Nikkei-225 price $S(0)$. In this way, we could use the bounds of Proposition 3.3.1 to set the strikes with which the call options are traded. This suggestion will also be supported by the observation that trade volume of the options became smaller for the strikes set farther from the present Nikkei-225 price $S(0)$ (e.g., see Table 3.1).

In general, the risk-neutral cumulative distribution function ξ tells us how investors view the uncertain risk of $S(T)$. If the ξ implied by the computed bounds is similar to the cumulative distribution function obtained from the historical data of the underlying asset price, we can expect that investors in the market are risk-neutral. This kind of observation will help us when we make investment in the market.

In analyzing Nikkei-225 data, we observed that the $f_{\max}$ and $f_{\min}$ computed by Proposition 3.3.1 showed different behaviors depending on whether $S(T)$ has actually become smaller or larger than the $S(0)$ of 4 weeks ago. This may suggest the possibility of using $f_{\max}$ and $f_{\min}$ to forecast the future price of an asset, which would be one of our future topics.

3.5 Conclusion

This chapter investigated the problem of deriving the upper and lower bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call options, based only on the no-arbitrage assumption. The main contribution of this chapter is to provide the bounds in closed forms, without solving the corresponding LPs. The bounds are easy to compute. Based on the bounds computed from the real data of the Nikkei-225 options, we made several observations and discussed possible applications, which could be used by investors. Finding more applications of the computed bounds remains as an important and interesting issue.

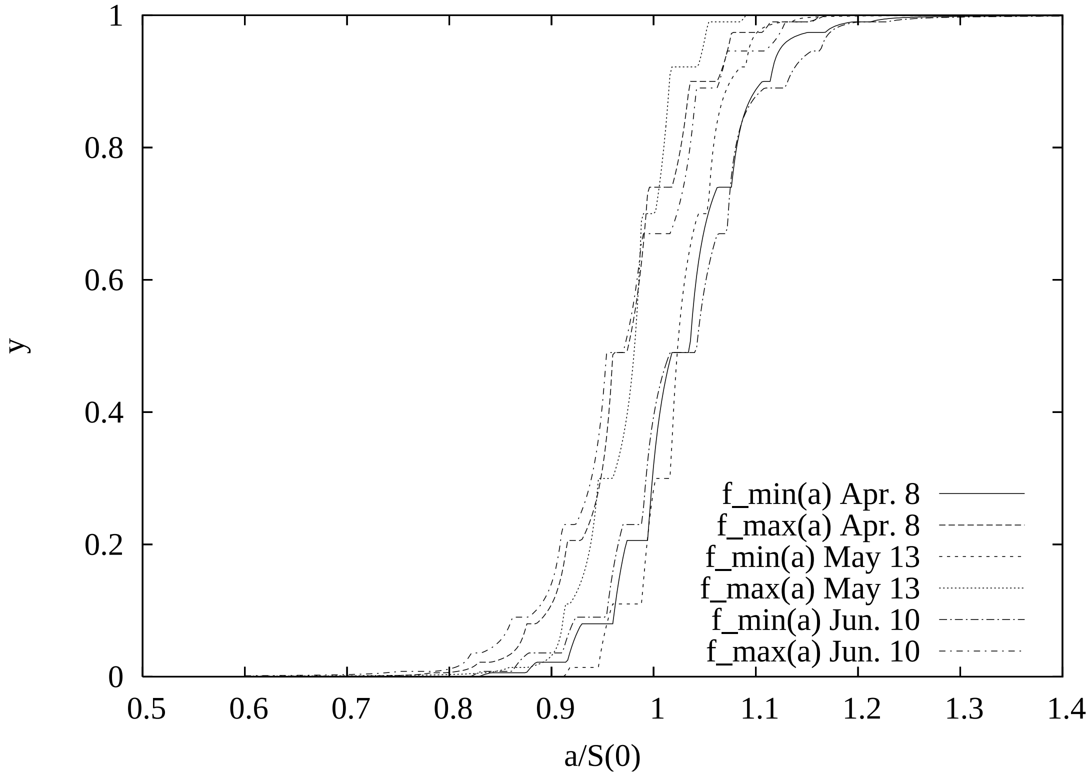

Figure 3.5: $f_{max}(a)$ and $f_{min}(a)$ on different dates in 2004.

Chapter 4

Real Options Involving Alternative Investment Projects

4.1 Introduction

Real options approaches have become a useful tool for evaluating irreversible investment under uncertainty such as R&D investment (see [22]). Although the early literature on real options (e.g., [20, 59]) treated the investment decision of a single firm, more recent studies provoked by [29] have investigated the problem of several firms competing in the same market from a game theoretic approach (see [13] for an overview). Grenadier [30] derived the equilibrium investment strategies of the firms in the Cournot–Nash framework and Weeds [87] provided the asymmetric outcome (called *preemption equilibrium*) in R&D competition between the two firms using the equilibrium in a timing game studied in [27]. In [42, 83], a possibility of mistaken simultaneous investment resulting from an absence of rent equalization that was assumed in [87] was investigated.

On the other hand, there are several studies on the decision of a single firm with an option to choose both the type and the timing of the investment projects. In this literature, [21] was the first study to pay attention to the problem and Décamps et al. [18] investigated the problem in more detail. In [25], a similar model is applied to the problem of constructing small wind power units.

Despite such active studies on real options, to our knowledge few studies have tried to elucidate how competition between two firms affects their investment decisions in the case where the firms have the option to choose both the type and the timing of the projects. This chapter investigates the above problem by extending the R&D model in [87] to a model where the firms can choose the target of the research from two alternative technologies of different standards with the same uncertainty about the market demand[1], where the technology standard is to be defined in some appropriate sense. As in [87], technological uncertainty is taken into account, in addition to the product market uncertainty. We assume that the time between project initiation and project discovery (henceforth the research term) follows the Poisson distribution[2] with its hazard rate determined by the standard of the technology. This assumption is realistically intuitive since a higher-standard technology is likely to require a longer research term and is expected to generate higher profits at its completion.

In the model, we show that the competition between the two firms affects not only the firms' investment time, but also their choice of the technology targeted in the project. In fact, we observe that the effect on the choice of the standard consists of two components. The presence of the other firm straightforwardly changes the value of the technologies. We call this the *direct* effect on the choice of the project type. In addition to the direct effect, the timing game caused by the competition affects the firms' choice of the targeted technology. This is due to the hastened timing through the strategic interaction with the competitor; accordingly we call this the *indirect* effect, distinguishedly from the direct effect.

We highlight two typical cases that are often observed in a market and, at the same time, reveal interesting implications. The first case is that a firm that completes a technology first can monopolize the profit flow regardless of the standard of the technology. *De facto standardization struggles* such as VHS vs Betamax for video recorders are true for this case (henceforth called the de facto standard case). In such cases, a firm can impose its technology as a de facto standard by introducing it before its competitors. Once one technology becomes the de facto standard for the market, the winner may well enjoy a monopolistic cash flow from the patent of the de facto standard technology for a long term. It is then quite difficult for other firms to replace it with other technologies even if those technologies are superior to the de facto standard one. Indeed, it has been often observed in de facto standardization races that the existing technology drives out a newer (superior) technology, which can be regarded as a sort of *Gresham's law*[3]. In conclusion, what is important in the de facto standard case is introducing the completed technology into the market before the opponents.

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[1]We assume that the two technologies are applied to homogeneous products.

[2]Most studies, such as [16, 44, 55, 87], model technical innovation as a Poisson arrival; we also follow this convention.

[3]Gresham's law is the economic principle that in the circulation of money "bad money drives out good," i.e., when depreciated, mutilated, or debased coinage (or currency) is in concurrent circulation with money of high value in terms of precious metals, the good money is withdrawn from circulation by hoarders.

The other case is where a firm with higher-standard technology can deprive a firm with lower-standard technology of the cash flow by completing the higher-standard technology. This case applies to technologies of the innovative type (henceforth called the innovative case). As observed in evolution from cassette-based Walkmans to CD- and MD-based Walkmans, and further to flash memory- and hard drive-based digital audio players (e.g., iPod), the appearance of a newer technology drives out the existing technology. In such cases, a firm often attempts to develop a higher-standard technology because it fears the invention of superior technologies by its competitors. As a result, in the innovative case, a higher-standard technology tends to appear in a market.

The analysis in the two cases gives a good account of the characteristics mentioned above. In the de facto standard case, the competition increases the incentive to develop the lower-standard technology, which is easy to complete, while in the innovative case, the competition increases the incentive to develop the higher-standard technology, which is difficult to complete. The increase comes from both the direct and indirect effect of the completion. In particular, we show that in the de facto standard case the competition is likely to lead the firms to invest in the lower-standard technology, which is never chosen in the single firm situation. This result explains a real problem caused by too bitter R&D competition. It is a possibility that the competition spoils the higher-standard technology that consumers would prefer[4], while the development hastened by the competition increases consumers' profits compared with that of the monopoly. That is, the result accounts for both positive and negative sides of the R&D competition for consumers. Of course, as described in [69], practical R&D management is often much more flexible and complex (e.g., growth and sequential options studied in [54, 51]) than the simple model in this chapter. However, it is likely that the essence of the results remains unchanged in more practical setups.

In addition to the implications about the R&D competition given above, we also discuss our theoretical contribution in relation to existing streams of the studies on real options with strategic interactions. In fact, there are enormous number of papers that analyze strategic real options models between two firms. While there is a stream of literature concentrating on incomplete and asymmetric information[5], our model is built on complete information. In literature under complete information, Grenadier [29] proposed the basic model, and it has been extended to several directions (e.g., involving the research term in [87], the exit decision in [63], the entry and exit decision in [28]). Among those studies, a distinctive feature of our model is that the firms have the option to choose

[4] It is reasonable to suppose that consumers benefit from the invention of higher-standard technologies, though, strictly speaking, we need to incorporate consumers' value functions into the model.

[5] See, for example, [31, 40, 50, 65, 66].

the project type, which is properly defined in connection with the research term. Technically, we combine the model by [87] with that of [18]. By doing so, we capture the simultaneous changes of the investment timing and the choice of the project type due to the competition. In particular, there is an interaction between the timing and project type choices (recall the indirect effect).

In terms of treating high and low standard technologies, this chapter is related to [43, 44]. Their models give the technological innovation exogenously and assume that the firms can receive revenue flows immediately after the investment using the available technology. Then, they show how the possibility that a higher-standard technology will emerge in the future influences the investment decision. In our model, on the other hand, the endogenous factor (i.e., the firm 's choice of the type and the timing of the investment projects) in addition to the exogenous one (i.e., randomness in the research term) causes the technological innovation.

That is, the firm itself can trigger the technological innovation generating the patent value. Thus, this chapter, unlike [43, 44], investigates the investment decision of R&D which will provoke future technological innovation.

The chapter is organized as follows. After Section 4.2 derives the optimal investment timing for the monopolist, Section 4.3 formulates the problem of the R&D competition between two firms. Section 4.4 derives the equilibrium strategies in the two typical cases, namely, the de facto standard case and the innovative case. Section 4.5 gives numerical examples, and finally Section 4.6 concludes the chapter.

4.2 Single firm situation

Throughout Chapters 4–6, we assume all stochastic processes and random variables are defined on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. This chapter is based on the model in [87]. This section considers the investment decision of the single firm without fear of preemption. The firm can set up a research project for developing a new technology i (we denote technologies 1 and 2 for the lower-standard and higher-standard technologies, respectively) by paying an indivisible investment cost K_i. As in [87], for analytical advantage we assume that the firm has neither option to suspend nor option to switch the projects, though practical R&D investment often allows more managerial flexibility, such as to abandon, expand and switch (see [69]).

In developing technology i, from the time of the investment the invention takes place randomly according to a Poisson distribution with constant hazard rate h_i. The firm must pay the research expense $\tilde{K}_i$ per unit of time during the research term and can receive the profit flow $D_iY(t)$ from the discovery. Here, $Y(t)$ represents a market demand of the technologies at time t and influence cash flows which the technologies generate. It must be noted that the firm's R&D investment is affected by two different types of uncertainty (i.e., technological uncertainty and product market uncertainty). For simplicity, $Y(t)$ obeys the following geometric Brownian motion, which is independent of the Poisson processes representing technological uncertainty.

$$\mathrm{d}Y(t) = \mu Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}B(t) \quad (t > 0),\ Y(0) = y, \tag{4.2.1}$$

where the voratility $\sigma > 0$ and the initial value $y > 0$ are given constants and $B(t)$ denotes the one-dimensional $\mathcal{F}_t$ standard Brownian motion. Quantities K_i, h_i, D_i and $\tilde{K}_i$ are given constants satisfying

$$0 \leq K_1 \leq K_2,\ 0 < h_2 < h_1,\ 0 < D_1 < D_2,\ 0 < \tilde{K}_1 \leq \tilde{K}_2, \tag{4.2.2}$$

so that technology 2 is more difficult to develop and generates a higher profit flow from its completion than technology 1.

Let us now comment upon the model. For analysis in later sections, we modified the original setup by [87] at the two following points, but there are no essential difference. In [87], the completed technology generates not a profit flow but a momentary profit as the value of the patent at its completion, and there is no research expense during the research term (i.e, $\tilde{K}_i = 0$). In [44] the Poisson process determining technological innovation is exogenous to the firms as in [32], but we assume that a firm's investment initiates the Poisson process determining the completion of the technology. This is the main difference from the model studied in [44] that also treats two technologies.

The firm that monitors the state of the market can set up development of either technologies 1 or 2 at the optimal timing maximizing the expected payoff under discount rate $r(> \mu)$. Then, the firm's problem is expressed as the following optimal stopping problem:

$$M(y) = \sup_{\tau \in \mathcal{T}} E\left[\max_{i=1,2} E\left[\int_{\tau+t_i}^{\infty} \mathrm{e}^{-rt} D_i Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau} K_i - \int_{\tau}^{\tau+t_i} \mathrm{e}^{-rt}\tilde{K}_i \mathrm{d}t \mid \mathcal{F}_\tau\right]\right], \tag{4.2.3}$$

where $\mathcal{T}$ is a set of all $\mathcal{F}_t$ stopping times and t_i denotes a random variable following the exponential distribution with hazard rate h_i. In problem (4.2.3), $\max_{i=1,2} E[\cdot \mid \mathcal{F}_\tau]$ means that the firm can choose the optimal technology at the investment time τ.

By the calculation we obtain

$$E\left[\int_{\tau+t_i}^{\infty} \mathrm{e}^{-rt} D_i Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau} K_i - \int_{\tau}^{\tau+t_i} \mathrm{e}^{-rt}\tilde{K}_i \mathrm{d}t \mid \mathcal{F}_\tau\right] \tag{4.2.4}$$

$$= \mathrm{e}^{-r\tau} E^{Y(\tau)}\left[\int_{t_i}^{\infty} \mathrm{e}^{-rt} D_i Y(t)\mathrm{d}t - K_i - \int_0^{t_i} \mathrm{e}^{-rt}\tilde{K}_i \mathrm{d}t\right] \tag{4.2.5}$$

$$= \mathrm{e}^{-r\tau} \int_0^{\infty}\left(\int_s^{\infty} \mathrm{e}^{-rt} D_i E^{Y(\tau)}[Y(t)]\mathrm{d}t - K_i - \int_0^{s} \mathrm{e}^{-rt}\tilde{K}_i \mathrm{d}t\right) h_i \mathrm{e}^{-h_i s}\mathrm{d}s \tag{4.2.6}$$

$$= \mathrm{e}^{-r\tau}(\rho_{i0} Y(\tau) - I_i), \tag{4.2.7}$$

where we use the strong Markov property of $Y(t)$ in (4.2.5) and independence between t_i and $Y(t)$ in (4.2.6). Here, we need to explain the notation $E^{Y(\tau)}[\cdot]$ in (4.2.5) and (4.2.6). For a real number x, the notation $E^x[\cdot]$ denotes the expectation operator given that $Y(0) = x$, which can be changed from the original initial value y. When the initial value is unchanged from the original value y, we omit the superscript y, that is, $E[\cdot] = E^y[\cdot]$. The notation $E^{Y(\tau)}[\cdot]$ represents the random variable $\psi(Y(\tau))$, where $\psi(x) = E^x[\cdot]$. For example, $E^x[Y(t)]$ is $x\mathrm{e}^{\mu t}$, and therefore $E^{Y(\tau)}[Y(t)]$ in (4.2.6) becomes $Y(\tau)\mathrm{e}^{\mu t}$. Thus, problem (4.2.3) can be reduced to

$$M(y) = \sup_{\tau\in\mathcal{T}} E[\mathrm{e}^{-r\tau} \max_{i=1,2}(\rho_{i0} Y(\tau) - I_i)], \tag{4.2.8}$$

where ρ_{i0} and I_i are defined by

$$\rho_{i0} = \frac{D_i h_i}{(r-\mu)(r+h_i-\mu)} \tag{4.2.9}$$

$$I_i = K_i + \frac{\tilde{K}_i}{r+h_i}. \tag{4.2.10}$$

Here, $\rho_{i0} Y(\tau)$ represents the expected discounted value of the future profit generated by technology i at the investment time τ, and I_i represents its total expected discounted cost at time τ.[6] Eq. (4.2.2) and (4.2.10) imply $I_1 < I_2$, but the inequality $\rho_{10} < \rho_{20}$ does not necessarily hold depending upon a trade-off between h_i and D_i. Note that (4.2.8) is essentially the same as the problem examined in [18].

We make a brief explanation as to the difference between (4.2.8) and

$$\tilde{M}(y) = \max_{i=1,2}\left\{\sup_{\tau\in\mathcal{T}} E[\mathrm{e}^{-r\tau}(\rho_{i0} Y(\tau) - I_i)]\right\}. \tag{4.2.11}$$

(4.2.11) is a problem in which at time 0 the firm must decide which technology it develops. That is, in problem (4.2.11), the firm cannot switch the technology even before the investment time τ

[6]Our setup is essentially the same as the setup by [87] that assumes $\tilde{K}_i = 0$, because we do not allow suspension in the research term. That is, I_i defined by (4.2.10) can be regarded as a sunk investment cost. However, in this chapter we consider $\tilde{K}_i$ plus K_i in order to relate the cost I_i with the hazard rate h_i.

once the firm choose the technology at initial time. Since fewer cases of R&D investment applies to the setting (4.2.11), we consider the setting (4.2.8) in which the firm can determine the technology standard at the investment time τ. In addition, it holds that $M(y) \geq \tilde{M}(y)$ because the firm has more managerial flexibility in problem (4.2.8) than in problem (4.2.11).

Let $M(y)$ and τ_M^* denote the value function and the optimal stopping time of problem (4.2.8), respectively. The letter "M" means the case of monopoly. Note that τ_M^* is expressed in a form independent of the initial value y. As in most real options literature (e.g., [22]), we define

$$\beta_{10} = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{4.2.12}$$

$$\beta_{20} = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0. \tag{4.2.13}$$

They are usual characteristic roots in an optimal stopping problem with discount rate r and state process $Y(t)$ following (4.2.1), and we can easily check $\beta_{10} > 1$ and $\beta_{20} < 0$.

Proposition 4.2.1 The value function $M(y)$ and the optimal stopping time τ_M^* of the monopolist (4.2.8) are given as follows:

Case 1: $0 < \rho_{20}/\rho_{10} \leq 1$

$$M(y) = \begin{cases} A_0 y^{\beta_{10}} & (0 < y < y_{10}^*) \\ \rho_{10} y - I_1 & (y \geq y_{10}^*), \end{cases} \tag{4.2.14}$$

$$\tau_M^* = \inf\{t \geq 0 \mid Y(t) \geq y_{10}^*\}. \tag{4.2.15}$$

Case 2: $1 < (\rho_{20}/\rho_{10})^{\beta_{10}/(\beta_{10}-1)} < I_2/I_1$

$$M(y) = \begin{cases} A_0 y^{\beta_{10}} & (0 < y < y_{10}^*) \\ \rho_{10} y - I_1 & (y_{10}^* \leq y \leq y_{20}^*) \\ B_0 y^{\beta_{10}} + C_0 y^{\beta_{20}} & (y_{20}^* < y < y_{30}^*) \\ \rho_{20} y - I_2 & (y \geq y_{30}^*), \end{cases} \tag{4.2.16}$$

$$\tau_M^* = \inf\{t \geq 0 \mid Y(t) \in [y_{10}^*, y_{20}^*] \cup [y_{30}^*, +\infty)\}. \tag{4.2.17}$$

Case 3: $(\rho_{20}/\rho_{10})^{\beta_{10}/(\beta_{10}-1)} \geq I_2/I_1$

$$M(y) = \begin{cases} B_0 y^{\beta_{10}} & (0 < y < y_{30}^*) \\ \rho_{20} y - I_2 & (y \geq y_{30}^*), \end{cases} \tag{4.2.18}$$

$$\tau_M^* = \inf\{t \geq 0 \mid Y(t) \geq y_{30}^*\}. \tag{4.2.19}$$

Here, constants A_0, B_0, C_0 and thresholds $y_{10}^*, y_{20}^*, y_{30}^*$ are determined by imposing value matching and smooth pasting conditions (see [22]). Note that $I_1 < I_2$ and $\beta_{10} > 1$.

Proof In Case 2, (4.2.16) and (4.2.17) immediately follow from the discussion in [18]. In Case 1, using relationships $\rho_{10} \geq \rho_{20}, I_1 < I_2$ and $Y(t) > 0$, we have

$$\sup_{\tau \in \mathcal{T}} E[\mathrm{e}^{-r\tau} \max_{i=1,2}(\rho_{i0} Y(\tau) - I_i)] = \sup_{\tau \in \mathcal{T}} E[\mathrm{e}^{-r\tau}(\rho_{10} Y(\tau) - I_1)],$$

which implies (4.2.14) and (4.2.15). In Case 3, by taking into consideration that the right-hand side of (4.2.18) dominates $\rho_{10} y - I_1$, we can show (4.2.18) and (4.2.19) by a standard technique to solve an optimal stopping problem (see [68]). □

In Proposition 4.2.1, $A_0 y^{\beta_{10}}, B_0 y^{\beta_{10}}$ and $C_0 y^{\beta_{20}}$ correspond to the values of the option to invest in technology 1 at the trigger y^*_{10}, the option to invest in technology 2 at the trigger y^*_{30} and the option to invest in technology 1 at the trigger y^*_{20}, respectively. In Case 1, where the expected discounted profit of technology 1 is higher than that of technology 2, the firm initiates development of technology 1 at time (4.2.15) independently of the initial market demand y. In Case 3, where technology 2 is much superior to technology 1, on the contrary, the firm invests in technology 2 at time (4.2.19) regardless of y. In Case 2, where both projects has similar values by the trade-off between the profitability and the research term and cost, the firm's optimal investment strategy has three thresholds y^*_{10}, y^*_{20} and y^*_{30}, and therefore the project chosen by the firm depends on the initial value y. Above all, if $y \in (y^*_{20}, y^*_{30})$, the firm defers not only investment, but also choice among the two projects (i.e, whether the firm invests in technology 2 when the market demand $Y(t)$ increases to the upper trigger y^*_{30} or invests in technology 1 when $Y(t)$ decreases to the lower trigger y^*_{20}). Hence, $M(y) > \tilde{M}(y)$ holds only for $y \in (y^*_{20}, y^*_{30})$ in Case 2, while $M(y)$ equals to $\tilde{M}(y)$ in other regions in Case 2 and other cases.

By letting volatility $\sigma \to +\infty$ with other parameters fixed, we have $\beta_{10} \to 1$ by definition (4.2.12) and therefore $(\rho_{20}/\rho_{10})^{\beta_{10}/(\beta_{10}-1)} \to +\infty$ if $\rho_{10} < \rho_{20}$. As a result, with high product market uncertainty, instead of Case 2, Case 3 holds and the firm chooses the higher-standard technology 2 rather than the lower-standard technology 1, unless the expected discounted profit generated by technology 1 exceeds that of technology 2. The similar result has also been mentioned in [18].

4.3 Situation of two noncooperative firms

We turn now to a problem of two symmetric firms. This chapter considers a symmetric setting to avoid unnecessary confusion, but the results in this chapter could remain true to some extent in an asymmetric case. For a standard discussion of an asymmetric situation, see [42]. We assume that two Poisson processes modeling the two firms' innovation are independent of each other, which means that the progress of the research project by one of the firms does not affect that of its rival. The scenarios of the cash flows into the firms can be classified into four cases. Figure 4.1 illustrates the cash flows into the firm that has completed a technology first (denoted Firm 1) and the other (denoted Firm 2).

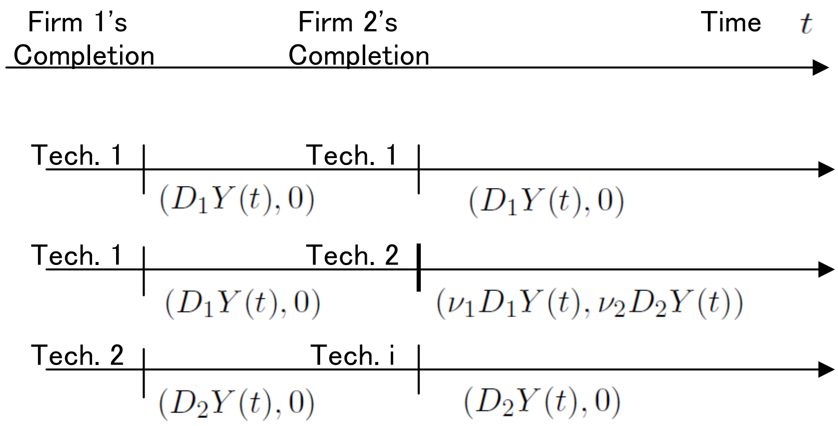

Figure 4.1: (Firm 1's cash flow, Firm 2's cash flow).

In the period when a single firm has succeeded in the development of technology i, the firm obtains the monopoly cash flow $D_iY(t)$. If both firms develop the same technology i, the one that has completed first receives the profit flow $D_iY(t)$ resulting from the patent perpetually and the other obtains nothing, according to the setup by [87]. Of course, the firm that has completed the lower-standard technology 1 after the competitor's completion of the higher-standard technology 2 obtains no cash flow. When the firm has completed technology 2 after the competitor's completion of technology 1, from the point technology 2 generates the profit flow $\nu_2D_2Y(t)$, and technology 1 generates $\nu_1D_1Y(t)$, where ν_i is a constant satisfying $0 \leq \nu_1, \nu_2 \leq 1$. It is considered that the technology's share in the product market determines ν_1 and ν_2.

As usual (see the books [22, 42]), we solve the game between two firms backwards. We begin by supposing that one of the firms has already invested, and find the optimal decision of the other. In the remainder of this chapter, we call the one who has already invested *leader* and call the other *follower*, though we consider two symmetrical firms. Thereafter, in the next section, we look at the situation where neither firms has invested, and consider the decision of either as it contemplates whether to go first, knowing that the other will react in the way just calculated as the follower's optimal response. The main difference from the existing literature such as [22, 29, 42, 63, 87] is that the follower's optimal response depends on the technology i chosen by the leader. Let $F_i(Y)$ and $\tau^*_{F_i}$ denote the expected discounted payoff (at time t) and the investment time of the follower responding optimally to the leader who has invested in technology i at time t satisfying $Y(t) = Y$. We denote by $L_i(Y)$ the expected discounted payoff (at time t) of the leader who has invested in technology i at $Y(t) = Y$.

4.3.1 Case where the leader has invested in technology 2

This subsection derives $F_2(Y), \tau^*_{F_2}$ and $L_2(Y)$. Given that the leader has invested in technology 2 at $Y(t) = Y$, the follower solves the following optimal stopping problem:

$$\begin{aligned}
&F_2(Y) = \mathrm{e}^{rt} \sup_{\tau\in\mathcal{T},\tau\geq t} E\left[\mathrm{e}^{-h_2\tau} \max\left\{E\left[1_{\{t_1<s_2\}}\left(\int_{\tau+t_1}^{\tau+s_2} \mathrm{e}^{-rs}D_1Y(s)\mathrm{d}s\right.\right.\right.\right.\\
&+\left.\int_{\tau+s_2}^{\infty} \mathrm{e}^{-rs}\nu_1 D_1 Y(s)\mathrm{d}s\right) - \mathrm{e}^{-r\tau}K_1 - \left.\int_{\tau}^{\tau+t_1} \mathrm{e}^{-rs}\tilde{K}_1\mathrm{d}s \mid \mathcal{F}_\tau\right], \qquad (4.3.1)\\
&\left.\left.E\left[1_{\{t_2<s_2\}}\int_{\tau+t_2}^{+\infty} \mathrm{e}^{-rs}D_2Y(s)\mathrm{d}s - \mathrm{e}^{-r\tau}K_2 - \int_{\tau}^{\tau+t_2} \mathrm{e}^{-rs}\tilde{K}_2\mathrm{d}s \mid \mathcal{F}_\tau\right]\right\} \mid Y(t) = Y\right],
\end{aligned}$$

where $E[\cdot \mid Y(t) = Y]$ represents the expectation conditioned that $Y(t) = Y$. Recall t_i represents a random variable following the exponential distribution with hazard rate h_i. The random variable s_i is independent of t_i and also follows the exponential distribution with hazard rate h_i. Note that the research term of the follower choosing technology i is expressed as t_i in (4.3.1). The interval between τ and the discovery time of the leader follows the exponential distribution with hazard rate h_2 (hence, it is expressed as s_2 in (4.3.1)) *under the condition that the leader has yet to complete technology* 2 *at time* τ. The reason is that the discovery occurs according to the Poisson process which is Markovian. What has to be noticed is that the follower's problem (4.3.2) is discounted by $\mathrm{e}^{-h_2\tau}$ differently from the single firm's problem (4.2.3). This is because the leader's completion of technology 2 deprives the follower of the future option to invest. As in the single firm's problem (4.2.3), $\max_{i=1,2} E[\cdot \mid \mathcal{F}_\tau]$ means that the follower chooses the better project at the investment time τ. Furthermore, $1_{\{t_i<s_2\}}$ denotes a defining function and means that the follower's payoff becomes nothing if the leader completes technology 2 first. In order to derive $F_2(Y)$ and $\tau^*_{F_2}$, we rewrite problem (4.3.1) as the following problem with initial value $Y(0) = Y$, using the Markov property,

$$\begin{aligned}
&F_2(Y) = \sup_{\tau\in\mathcal{T}} E^Y\left[\mathrm{e}^{-h_2\tau} \max\left\{E^Y\left[1_{\{t_1<s_2\}}\left(\int_{\tau+t_1}^{\tau+s_2} \mathrm{e}^{-rs}D_1Y(s)\mathrm{d}s\right.\right.\right.\right.\\
&+\left.\int_{\tau+s_2}^{\infty} \mathrm{e}^{-rs}\nu_1 D_1 Y(s)\mathrm{d}s\right) - \mathrm{e}^{-r\tau}K_1 - \left.\int_{\tau}^{\tau+t_1} \mathrm{e}^{-rs}\tilde{K}_1\mathrm{d}s \mid \mathcal{F}_\tau\right], \qquad (4.3.2)\\
&\left.\left.E^Y\left[1_{\{t_2<s_2\}}\int_{\tau+t_2}^{+\infty} \mathrm{e}^{-rs}D_2Y(s)\mathrm{d}s - \mathrm{e}^{-r\tau}K_2 - \int_{\tau}^{\tau+t_2} \mathrm{e}^{-rs}\tilde{K}_2\mathrm{d}s \mid \mathcal{F}_\tau\right]\right\}\right],
\end{aligned}$$

where $E^Y[\cdot]$ means the (conditional) expectation operator given that the initial value $Y(0)$ is Y instead of y, as explained in page 36. Then, τ and s in problem (4.3.2), unlike those in (4.3.1), represents how long it has passed since the leader's investment time t. Strictly speaking, the optimal stopping time in problem (4.3.2) is different from that in problem (4.3.1), $\tau^*_{F_2}$, since the initial time in problem (4.3.2) corresponds to time t in problem (4.3.1). However, it is easy to derive $\tau^*_{F_2}$ from the solution in problem (4.3.2), and hence we hereafter identify problem (4.3.2) with problem (4.3.1).

Via the similar calculation to (4.2.4)–(4.2.7) we can rewrite problem (4.3.2) as

$$F_2(Y) = \sup_{\tau \in \mathcal{T}} E^Y[\mathrm{e}^{-(r+h_2)\tau} \max_{i=1,2}(\rho_{i2}Y(\tau) - I_i)], \tag{4.3.3}$$

where ρ_{ij} are defined by

$$\rho_{11} = \frac{D_1 h_1}{(r-\mu)(r+2h_1-\mu)}, \tag{4.3.4}$$

$$\rho_{12} = \frac{D_1 h_1}{(r+h_1+h_2-\mu)(r+h_2-\mu)}\left(1+\frac{\nu_1 h_2}{r-\mu}\right), \tag{4.3.5}$$

$$\rho_{21} = \frac{D_2 h_2}{(r-\mu)(r+h_1+h_2-\mu)}\left(1+\frac{\nu_2 h_1}{r+h_2-\mu}\right), \tag{4.3.6}$$

$$\rho_{22} = \frac{D_2 h_2}{(r-\mu)(r+2h_2-\mu)}. \tag{4.3.7}$$

Quantity $\rho_{ij}Y(\tau)$ represents the expected discounted value of the future cash flow of the firm that invests in technology i at time τ when its opponent is on the way to development of technology j. From the expression (4.3.3), we can show the following proposition.

Proposition 4.3.1 The follower's payoff $F_2(Y)$, investment time $\tau^*_{F_2}$ and the leader's payoff $L_2(Y)$ are given as follows:

Case 1: $0 < \rho_{22}/\rho_{12} \leq 1$

$$F_2(Y) = \begin{cases} A_2 Y^{\beta_{12}} & (0 < Y < y^*_{12}) \\ \rho_{12}Y - I_1 & (Y \geq y^*_{12}), \end{cases}$$

$$\tau^*_{F_2} = \inf\{s \geq t \mid Y(s) \geq y^*_{12}\},$$

$$L_2(Y) = \begin{cases} \rho_{20}Y - I_2 - \tilde{A}_2 Y^{\beta_{12}} & (0 < Y < y^*_{12}) \\ \rho_{21}Y - I_2 & (Y \geq y^*_{12}). \end{cases}$$

Case 2: $1 < (\rho_{22}/\rho_{12})^{\beta_{12}/(\beta_{12}-1)} < I_2/I_1$

$$
\begin{aligned}
F_2(Y) &= \begin{cases} A_2Y^{\beta_{12}} & (0 < Y < y^*_{12}) \\ \rho_{12}Y - I_1 & (y^*_{12} \le Y \le y^*_{22}) \\ B_2Y^{\beta_{12}} + C_2Y^{\beta_{22}} & (y^*_{22} < Y < y^*_{32}) \\ \rho_{22}Y - I_2 & (Y \ge y^*_{32}), \end{cases} \\
\tau^*_{F_2} &= \inf\{s \ge t \mid Y(s) \in [y^*_{12}, y^*_{22}] \cup [y^*_{32}, +\infty)\}, \\
L_2(Y) &= \begin{cases} \rho_{20}Y - I_2 - \tilde{A}_2Y^{\beta_{12}} & (0 < Y < y^*_{12}) \\ \rho_{21}Y - I_2 & (y^*_{12} \le Y \le y^*_{22}) \\ \rho_{20}Y - I_2 - \tilde{B}_2Y^{\beta_{12}} - \tilde{C}_2Y^{\beta_{22}} & (y^*_{22} < Y < y^*_{32}) \\ \rho_{22}Y - I_2 & (Y \ge y^*_{32}). \end{cases}
\end{aligned}
$$

Case 3: $(\rho_{22}/\rho_{12})^{\beta_{12}/(\beta_{12}-1)} \ge I_2/I_1$

$$
\begin{aligned}
F_2(Y) &= \begin{cases} B_2Y^{\beta_{12}} & (0 < Y < y^*_{32}) \\ \rho_{22}Y - I_2 & (Y \ge y^*_{32}), \end{cases} \\
\tau^*_{F_2} &= \inf\{s \ge t \mid Y(s) \ge y^*_{32}\}, \\
L_2(Y) &= \begin{cases} \rho_{20}Y - I_2 - \tilde{B}_2Y^{\beta_{12}} & (0 < Y < y^*_{32}) \\ \rho_{22}Y - I_2 & (Y \ge y^*_{32}). \end{cases}
\end{aligned}
$$

Here, β_{12} and β_{22} denote (4.2.12) and (4.2.13) replaced r by $r + h_2$, respectively. Here, $r + h_2$ is the discount factor taking account of the possibility that the option is vanished with intensity h_2. After constants A_2, B_2, C_2 and thresholds $y^*_{12}, y^*_{22}, y^*_{32}$ are determined by both value matching and smooth pasting conditions in the follower's value function $F_2(Y)$, constants $\tilde{A}_2, \tilde{B}_2$ and $\tilde{C}_2$ are determined by the value matching condition alone in the leader's payoff function $L_2(Y)$. Note that $I_1 < I_2$ and $\beta_{12} > 1$.

Proof Problem (4.3.3) coincides with problem (4.2.8) replaced r and ρ_{i0} by $r + h_2$ and ρ_{i2}, respectively. Thus, we easily obtain the follower's payoff $F_2(Y)$ and investment time $\tau^*_{F_2}$ in the same way as Proposition 4.2.1. We next consider the leader's payoff $L_2(Y)$. In Case 1 and 3, we readily have the same expression as that of [87] since the follower's trigger is single. In Case 2, we obtain the similar expression, though the calculation becomes more complicated because of the three triggers. □.

Constants A_2, B_2, C_2 and thresholds $y^*_{12}, y^*_{22}, y^*_{32}$ in Proposition 4.3.1 correspond to constants A_0, B_0, C_0 and thresholds $y^*_{10}, y^*_{20}, y^*_{30}$ in Proposition 4.2.1, respectively. Let us explain the leader's payoff briefly. Constants $\tilde{A}_2, \tilde{B}_2$ and $\tilde{C}_2$ value the possibility that Y rises above y^*_{12} prior to the leader's completion, the possibility that Y rises above y^*_{32} prior to the leader's completion, and the possibility that Y falls bellow y^*_{22} prior to the leader's completion, respectively. Since these situations cause the follower's investment, the leader's payoff is reduced from the monopoly profit $\rho_{20}Y - I_2$ (see $Y \in (0, y^*_{12})$ in Case 1, $Y \in (0, y^*_{12}) \cup (y^*_{22}, y^*_{32})$ in Case 2, and $Y \in (0, y^*_{32})$ in Case 3).

4.3.2 Case where the leader has invested in technology 1

We now consider $F_1(Y), \tau^*_{F_1}$ and $L_1(Y)$. In the previous subsection, i.e., in the case where the leader has chosen technology 2, the follower's opportunity to invest is completely lost at the leader's completion of technology 2. However, in the case where the leader has invested in technology 1, there remains the inactive follower's option after the leader's invention of technology 1. That is, the follower can invest in technology 2 even after the leader's discovery if the follower has not invested in any technology yet. Due to this option value, we need more complicated discussion in this subsection.

Let $\tilde{f}_1(Y)$ and $\tau^*_{\tilde{f}_1}$ be the expected discounted payoff and the optimal stopping time of the follower responding optimally to the leader who has already succeeded in development of technology 1 at $Y(t) = Y$. In other words, $\tilde{f}_1(Y)$ represents the remaining option value to invest in technology 2 after the leader's completion of technology 1. We need to derive $\tilde{f}_1(Y)$ and $\tau^*_{\tilde{f}_1}$ before analyzing $F_1(Y)$ and $\tau^*_{F_1}$. Given that the leader has already completed technology 1 at $Y(t) = Y$, the follower's problem becomes

$$\tilde{f}_1(Y) = \sup_{\tau \in \mathcal{T}} E^Y \left[\int_{\tau+t_2}^{\infty} \mathrm{e}^{-rt} \nu_2 D_2 Y(t) \mathrm{d}t - \mathrm{e}^{-r\tau} K_2 - \int_{\tau}^{\tau+t_2} \mathrm{e}^{-rt} \tilde{K}_2 \mathrm{d}t \right], \tag{4.3.8}$$

which is equal to a problem of a firm that can develop only technology 2. In this subsection, we omit a description of a problem which corresponds to (4.3.1), and describe only a problem (which corresponds to (4.3.1)) with initial value $Y(0) = Y$. In the same way as calculation (4.2.4)–(4.2.7), we can rewrite problem (4.3.8) as

$$\tilde{f}_1(Y) = \sup_{\tau \in \mathcal{T}} E^Y [\mathrm{e}^{-r\tau} (\nu_2 \rho_{20} Y(t) - I_2)]. \tag{4.3.9}$$

It is easy to obtain the value function $\tilde{f}_1(Y)$ and the optimal stopping time $\tau^*_{\tilde{f}_1}$ of the follower. If $\nu_2 > 0$, then

$$\tilde{f}_1(Y) = \begin{cases} B' Y^{\beta_{10}} & (0 < Y < y') \\ \nu_2 \rho_{20} Y - I_2 & (Y \geq y'), \end{cases} \tag{4.3.10}$$

$$\tau^*_{\tilde{f}_1} = \inf\{s \geq t \mid Y(s) \geq y'\}, \tag{4.3.11}$$

where B' and y' are constants determined by the value matching and smooth pasting conditions (we omit the explicit solutions to avoid cluttering). If $\nu_2 = 0$, we have $\tilde{f}_1(Y) = 0$ and $\tau^*_{\tilde{f}_1} = +\infty$.

Assuming that the leader has begun developing technology 1 at $Y(t) = Y$, the follower's problem is expressed as follows:

$$F_1(Y) = \sup_{\tau\in\mathcal{T}} E^Y \Bigg[\mathrm{e}^{-h_1\tau} \max\Bigg\{ E^Y \bigg[1_{\{t_1<s_1\}} \int_{\tau+t_1}^{+\infty} \mathrm{e}^{-rs} D_1 Y(s)\mathrm{d}s - \mathrm{e}^{-r\tau} K_1 - \int_{\tau}^{\tau+t_1} \mathrm{e}^{-rs}\tilde{K}_1 \mathrm{d}s \mid \mathcal{F}_\tau \bigg], \\ E^Y \bigg[1_{\{t_2<s_1\}} \int_{\tau+t_2}^{+\infty} \mathrm{e}^{-rs} D_2 Y(s)\mathrm{d}s + 1_{\{t_2\geq s_1\}} \int_{\tau+t_2}^{+\infty} \mathrm{e}^{-rs}\nu_2 D_2 Y(s)\mathrm{d}s - \mathrm{e}^{-r\tau} K_2 \\ - \int_{\tau}^{\tau+t_2} \mathrm{e}^{-rs}\tilde{K}_2 \mathrm{d}s \mid \mathcal{F}_\tau \bigg] \Bigg\} + 1_{\{\tau\geq s_1'\}} \mathrm{e}^{-rs_1'} \tilde{f}_1(Y(s_1')) \Bigg], \tag{4.3.12}$$

where s_1' represents another random variable following the exponential distribution with hazard rate h_2. In (4.3.12), the interval between the leader's investment time t and completion time is expressed as s_1'. By the Markov property, the interval between t and the completion time of the non-conditional leader has the same distribution as the interval between τ and the completion time of the leader *who is conditioned to be yet to complete technology* 1 *at* τ. Compared with the follower's problem (4.3.2) in the previous subsection, problem (4.3.12) has the additional term $E^Y[1_{\{\tau\geq s_1'\}}\mathrm{e}^{-rs_1'}\tilde{f}_1(Y(s_1'))]$. This term corresponds to the remaining option value of the inactive follower. As in (4.2.4)–(4.2.7), problem (4.3.12) can be reduced to

$$F_1(Y) = \sup_{\tau\in\mathcal{T}} E^Y[\mathrm{e}^{-(r+h_1)\tau} \max_{i=1,2}(\rho_{i1} Y(\tau) - I_i) + 1_{\{\tau\geq s_1'\}}\mathrm{e}^{-rs_1'}\tilde{f}_1(Y(s_1'))], \tag{4.3.13}$$

where ρ_{11} and ρ_{21} are defined by (4.3.4) and (4.3.6), respectively. Generally, problem (4.3.13), unlike (4.3.3), is difficult to solve analytically because of the additional term. In the next section, we overcome the difficulty by focusing on two typical cases, namely, the de fact standard case, where $(\nu_1, \nu_2) = (1, 0)$, and the innovative case, where $(\nu_1, \nu_2) = (0, 1)$.

4.4 Analysis in two typical cases

This section examines the firms' behaviour in the de fact standard case, where $(\nu_1, \nu_2) = (1, 0)$, and the innovative case, where $(\nu_1, \nu_2) = (0, 1)$. In the real world both $\nu_1 > 0$ and $\nu_2 > 0$ are usually hold and the two cases are extreme. However, such a real case approximates to one of the two cases or has an intermediate property, depending on the relationship between ν_1 and ν_2, and therefore analysis in the two cases helps us to understand the essence of the problem. In order to exclude a situation where both firms mistakenly invest simultaneously[7], we assume that the initial value y is small enough, that is,

Assumption A

$$\max_{i=1,2}(\rho_{i0}y - I_i) < 0,$$

as in [87] when we discuss the firms' equilibrium strategies. Assumption A is likely to hold in the context of R&D. A firm tends to delay its investment decision of R&D (rarely invest immediately), because the R&D investment decision is carefully made taking account of the distant future.

We moreover restrict our attention to the case where the firm always chooses the higher-standard technology 2 in the single firm situation, for the purpose of contrasting the competitive situation with the single firm situation. To put it more concretely, we assume

Assumption B

$$\left(\frac{\rho_{20}}{\rho_{10}}\right)^{\frac{\beta_{10}}{\beta_{10}-1}} \geq \frac{I_2}{I_1},$$

so that Case 3 follows in Proposition 4.2.1.

In the first place, we analytically derive the follower's payoff $F_1(Y)$ and the leader's payoff $L_1(Y)$ in both de fact standard and innovative cases. Note that the results on $F_2(Y)$ and $L_2(Y)$ in Proposition 4.3.1 hold true by substituting $(\nu_1, \nu_2) = (1, 0)$ and $(\nu_1, \nu_2) = (0, 1)$ into (4.3.5) and (4.3.6). Then,we compare the leader's payoff $L(Y)$ with the follower's payoff $F(Y)$, where $L(Y)$ and $F(Y)$ are defined by

$$\begin{aligned} L(Y) &= \max_{i=1,2} L_i(Y), \\ F(Y) &= \begin{cases} F_1(Y) \ (L_1(Y) > L_2(Y)) \\ F_2(Y) \ (L_1(Y) \leq L_2(Y)). \end{cases} \end{aligned}$$

By the comparison, we see the situation where both firms try to preempt each other.

4.4.1 De facto standard case

Since $\nu_2 = 0$ holds in this case, the follower's option value $\tilde{f}_1(Y(s'_1))$ vanishes just like in Subsection 3.2. Thus, we can solve the follower's problem (4.3.13) in the same way as problem (4.3.3). Indeed, $F_1(Y)$ and $\tau^*_{F_1}$ agree with $F_2(Y)$ and $\tau^*_{F_2}$ replaced ρ_{i2}, β_{i2} with ρ_{i1}, β_{i1}, respectively in Proposition

[7] We must distinguish between mistaken simultaneous investment and joint investment which is examined in Subsection 4.3. For the details of the stopping time game, see Appendix A.

4.3.1, where β_{11} (>1) and β_{21} (<0) denote (4.2.12) and (4.2.13) replaced discount rate r with $r+h_1$, respectively. Recall that ρ_{11} and ρ_{21} were defined by (4.3.4) and (4.3.6). In this case, we denote three thresholds corresponding to y_{12}^*, y_{22}^* and y_{32}^* in Proposition 4.3.1 by y_{11}^*, y_{21}^* and y_{31}^*, respectively. Then, the payoff $L_1(Y)$ of the leader who has invested in technology 1 at $Y(t)=Y$ coincides with $L_2(Y)$ replaced $\rho_{2i}, I_2, \beta_{i2}$ and y_{i2}^* by $\rho_{1i}, I_1, \beta_{i1}$ and y_{i1}^*, respectively in Proposition 4.3.1.

Let us compare the follower's decision in the de facto standard case with the single firm's decision derived in Section 4.2. Using

$$\begin{aligned}\frac{\rho_{20}}{\rho_{10}} &= \frac{D_2h_2(r+h_1-\mu)}{D_1h_1(r+h_2-\mu)}\\ &> \frac{D_2h_2(r+h_1+h_2-\mu)}{D_1h_1(r+h_2+h_2-\mu)} = \frac{\rho_{22}}{\rho_{12}}\\ &> \frac{D_2h_2(r+h_1+h_1-\mu)}{D_1h_1(r+h_2+h_1-\mu)} = \frac{\rho_{21}}{\rho_{11}},\end{aligned}$$

which result from $r-\mu>0$ and $h_1>h_2>0$, we have

$$\frac{\rho_{21}}{\rho_{11}} < \frac{\rho_{22}}{\rho_{12}} < \frac{\rho_{20}}{\rho_{10}}. \tag{4.4.1}$$

Eq. (4.4.1) states that the relative expected profit of technology 2 to technology 1 is smaller than that of the single firm case. Using $1<\beta_{10}<\beta_{12}<\beta_{11}$, we also obtain

$$1 < \frac{\beta_{11}}{\beta_{11}-1} < \frac{\beta_{12}}{\beta_{12}-1} < \frac{\beta_{10}}{\beta_{10}-1}. \tag{4.4.2}$$

Eqs. (4.4.1) and (4.4.2) suggest a possibility that $(\rho_{2i}/\rho_{1i})^{\beta_{1i}/(\beta_{1i}-1)}$ is smaller than I_2/I_1 and 1 even under Assumption B, and then the follower's optimal choice could be technology 1. In consequence, the presence of the leader increases the follower's incentive to choose the lower-standard technology 1, which is easy to complete, compared with in the single firm situation (the direct effect).

From $\rho_{i1}<\rho_{i2}$, $r+h_2<r+h_1$, problem formulations (4.3.3) and (4.3.13) (note that $\tilde{f}_1=0$ in the de facto standard case), it follows that

$$F_1(Y) < F_2(Y) \quad (Y>0).$$

That is, from the follower's viewpoint, the case where the leader has chosen technology 2 is preferable to the case where the leader has chosen technology 1. This is due to that the leader who has invested in technology 1 is more likely to preempt the follower because of its short research term.

Finally, we take a look at the situation where neither firm has invested. Let us see that there exists a possibility that technology 1 can be developed owing to the competition even if technology

2 generates much more profit than technology 1 at its completion. Although, as has been pointed out, $(\rho_{2i}/\rho_{1i})^{\beta_{1i}/(\beta_{1i}-1)}$ could be smaller than I_2/I_1 and 1 under Assumption B, we now consider the case where

$$\left(\frac{\rho_{2i}}{\rho_{1i}}\right)^{\frac{\beta_{1i}}{\beta_{1i}-1}} \geq \frac{I_2}{I_1} \tag{4.4.3}$$

holds, which means that a cash flow resulting from technology 2 is expected to be much greater than that of technology 1.

Since the initial value $Y(0) = y$ is small enough (Assumption A), in the single firm situation the firm invests in technology 2 (Assumption B) as soon as the market demand $Y(t)$ rises to the level y^*_{30} (Figure 4.2). Development of technology 1 is meaningless because the firm without fear of preemption can defer the investment sufficiently. However, the firm with the fear of preemption by its rival will try to obtain the leader's payoff by investing a slight bit earlier than its rival when the leader's payoff $L(Y)$ is larger than the follower's payoff $F(Y)$. Repeating this process causes the investment trigger to fall to the point where $L(Y)$ is equal to $F(Y)$ (y_P in Figure 4.3). At the point the firms are indifferent between the two roles, and then one of the firms invests at time $\inf\{t \geq 0 \mid Y(t) \geq y_P\}$ as leader, while the other invests at time $\tau^*_{F_i}$ (if there remains the option to invest) as follower. This phenomenon is *rent equalization* explained in [27, 87]. This asymmetric outcome where one of the firms becomes a leader and the other becomes a follower is called *preemption equilibrium*. For further details of the stopping time game and the equilibrium, see Appendix A. If the fear of preemption hastens the investment time sufficiently (e.g., threshold y_P becomes smaller than $\tilde{y}$ in Figure 4.2), then threshold y_P becomes the intersection of $L_1(Y)$ and $F_1(Y)$ rather than the intersection of $L_2(Y)$ and $F_2(Y)$ (Figure 4.3). It suggests a possibility that in the preemption equilibrium the leader invests in technology 1 (the indirect effect). Needless to say, the leader is more likely to choose technology 1 if (4.4.3) is not satisfied. The above discussion gives a good account of the phenomenon observed frequently in de facto standard wars.

4.4.2 Innovative case

This subsection examines the innovative case, where $(\nu_1, \nu_2) = (0, 1)$ is satisfied. We now consider the follower's optimal response assuming that the leader has invested in technology 1 at $Y(t) = Y$. Let $\tilde{F}_1(Y)$ denote the payoff (strictly speaking, the expected discounted payoff at time t) of the follower who initiate developing technology 2 at time $\tau^*_{\tilde{f}_1}$ defined by (4.3.11). We can show that in the innovative case the follower's best response $\tau^*_{F_1}$ coincides with $\tau^*_{\tilde{f}_1}$ and also show $\tilde{F}_1(Y) = \tilde{f}_1(Y) = F_1(Y) = M(Y)$ as follows.

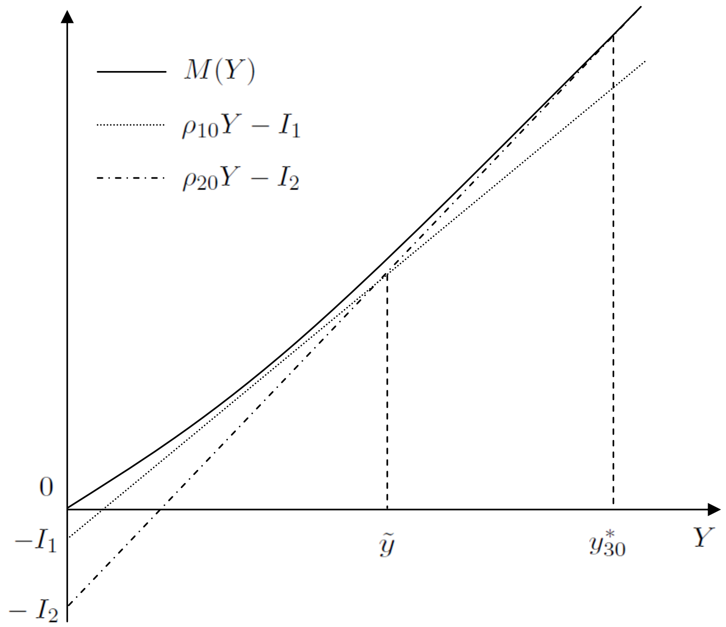

Figure 4.2: The value function $M(Y)$ in the single firm case.

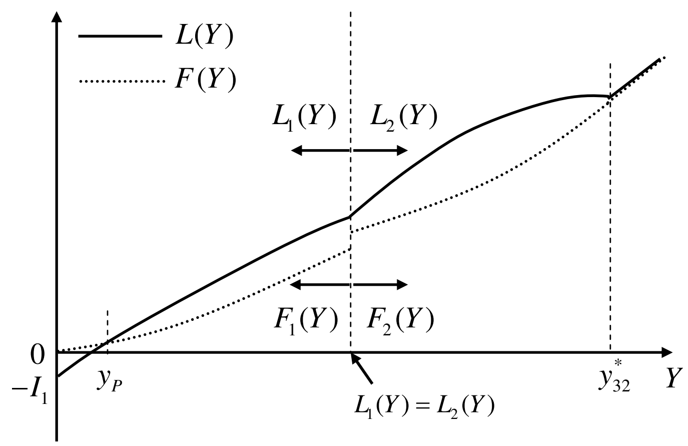

Figure 4.3: The leader's payoff $L(Y)$ and the follower's payoff $F(Y)$.

By $\nu_2 = 1$, the payoff of the follower who invests in technology 2 at time s ($\geq t$) is $\rho_{20}Y(s) - I_2$, whether the leader has completed technology 1 or not. Then we have $\tilde{F}_1(Y) = \tilde{f}_1(Y)$. Under Assumption B the single firm's value function $M(Y)$ is expressed as that of Case 3 in Proposition 4.2.1. Using $\nu_2 = 1$, we have

$$M(Y) = \tilde{f}_1(Y) = \tilde{F}_1(Y). \tag{4.4.4}$$

On the other hand, by definition of the follower's problem (4.3.13), it can readily be seen that the relationship

$$F_1(Y) \leq M(Y) \tag{4.4.5}$$

holds between $F_1(Y)$ and $M(Y)$. Note that the follower's option value to invest in technology 2 is the same as that of the single firm case. In contrast, the follower's option value to invest in technology 1 is lower than that of the single firm case. The reason is that the follower's option value to invest in technology 1 vanishes completely at the leader's invention of technology 1. Eqs. (4.4.4) and (4.4.5) suggest $F_1(Y) \leq \tilde{F}_1(Y)$. Thus, we have $\tilde{F}_1(Y) = F_1(Y)$, taking account of $F_1(Y) \geq \tilde{F}_1(Y)$ resulting from the optimality of $F_1(Y)$. Consequently, the follower's optimal response $\tau^*_{F_1}$ coincides with $\tau^*_{\tilde{f}_1}$ and $\tilde{F}_1(Y) = \tilde{f}_1(Y) = F_1(Y) = M(Y)$ holds. We should notice that the follower behaves as if there were no leader.

Using the follower's investment time $\tau^*_{F_2} = \tau^*_{\tilde{f}_1}$ derived above (note that $y' = y^*_{30}$ in (4.3.11) by $\nu_2 = 1$), we have the leader's payoff $L_1(Y)$ as $L_2(Y)$ replaced $\rho_{2i}, I_2, \beta_{12}$ and y^*_{32} by $\rho_{1i}, I_1, \beta_{11}$ and y^*_{30}, respectively in Case 3 in Proposition 4.3.1.

Next, we compare the follower's decision in the innovative case with the single firm's decision. Using

$$\begin{aligned}\frac{\rho_{20}}{\rho_{10}} &= \frac{\rho_{22}}{\rho_{12}} \times \frac{r + h_1 - \mu}{r + h_1 + h_2 - \mu} \times \frac{(r-\mu)(r + 2h_2 - \mu)}{(r + h_2 - \mu)^2} \\ &< \frac{\rho_{22}}{\rho_{12}},\end{aligned}$$

$\rho_{21} = \rho_{20}$ and $\rho_{11} < \rho_{10}$, we have

$$1 < \frac{\rho_{20}}{\rho_{10}} < \frac{\rho_{2i}}{\rho_{1i}} \quad (i = 1, 2). \tag{4.4.6}$$

Eq. (4.4.6) means that the relative expected profit of technology 2 to technology 1 is greater than that of the single firm case, contrary to (4.4.1) in the de facto standard case. Since (4.4.2) remains true, the relationship between $(\rho_{22}/\rho_{12})^{\beta_{12}/(\beta_{12}-1)}$ and I_2/I_1 depends on the parameters even under Assumption B. This suggests a slight possibility that the follower chooses technology 1 in the case where the leader has chosen technology 2, while as we showed in the beginning of this subsection the follower's best response to the leader who has invested in technology 1 is choosing technology 2 regardless of Y. However, in most cases the effect of (4.4.6) dominates the effect of (4.4.2), that is,

$$\frac{I_2}{I_1} < \left(\frac{\rho_{20}}{\rho_{10}}\right)^{\frac{\beta_{10}}{\beta_{10}-1}} < \left(\frac{\rho_{22}}{\rho_{12}}\right)^{\frac{\beta_{12}}{\beta_{12}-1}}$$

hold. To sum up, the presence of the leader, unlike in the de facto standard case, tends to decreases the incentive of the lower-standard technology 1, which is easy to complete (the direct effect).

By definition of the follower's problem (4.3.3) we can immediately show

$$F_2(Y) < M(Y) = F_1(Y) \quad (Y > 0).$$

In other words, contrary to the de facto standard case, the follower prefers the leader developing technology 1 to the leader developing technology 2. This is because the follower can deprive the leader who has chosen technology 1 of the profit by completing technology 2.

Finally, let us examine the situation where neither firm has taken action. We obtain the following proposition with respect to the preemption equilibrium.

Proposition 4.4.1 The inequality

$$L_1(Y) < F_1(Y) \quad (Y > 0) \tag{4.4.7}$$

holds, and therefore in the preemption equilibrium the leader always chooses technology 2. Furthermore, if

$$\left(\frac{\rho_{22}}{\rho_{12}}\right)^{\frac{\beta_{12}}{\beta_{12}-1}} > \frac{I_2}{I_1} \tag{4.4.8}$$

(Eq. (4.4.8) is satisfied for reasonable parameter values as mentioned earlier), then in the preemption equilibrium the follower, also, always chooses technology 2.

Proof The leader's payoff $L_1(Y)$ is equal to

$$L_1(Y) = \begin{cases} \rho_{10}Y - I_1 - \tilde{B}_1 Y^{\beta_{12}} & (0 < Y < y_{30}^*) \\ \rho_{12}Y - I_1 & (Y \geq y_{30}^*), \end{cases}$$

where the constant $\tilde{B}_1 > 0$ is determined by the value matching condition at the trigger y_{30}^*. Using $\rho_{10} > \rho_{12}$ and $\tilde{B}_1 > 0$, we have

$$L_1(Y) < \rho_{10}Y - I_1 \leq M(Y) = F_1(Y) \quad (Y > 0),$$

which implies that there is no incentive to invest in technology 1 earlier than the competitor. Therefore, there arises no preemption equilibrium where the leader invests in technology 1. Next, we assume (4.4.8). In this case, the follower's decision corresponds to that of Case 3 in Proposition 4.2.1. In consequence, in the preemption equilibrium, the follower, also, always chooses technology 2. □

Table 4.1 summarizes the comparison results between the de facto standard and innovative cases.

4.4.3 Case of joint investment

The joint investment equilibria, which are, unlike the preemption equilibria, symmetric outcomes, may also occur even if the two firms are noncooperative. The results on the joint investment equilibria in our setup is similar to that in [87] and therefore they are briefly described below.

Assuming that the two firms are constrained to invest in the same technology at the same timing, the firm's problem can be reduced to

$$J(y) = \sup_{\tau \in \mathcal{T}} E[\mathrm{e}^{-r\tau} \max_{i=1,2}(\rho_{ii} Y(\tau) - I_i)], \tag{4.4.9}$$

in the same procedure as (4.2.4)–(4.2.7). Recall that ρ_{11} and ρ_{22} were defined by (4.3.4) and (4.3.7), respectively. It is worth noting that the expression (4.4.9) does not depend on whether the de facto standard case or the innovative case. Using

$$\begin{aligned} \frac{\rho_{20}}{\rho_{10}} &= \frac{D_2 h_2 (r + h_1 - \mu)}{D_1 h_1 (r + h_2 - \mu)} \\ &< \frac{D_2 h_2 (r + 2h_1 - \mu)}{D_1 h_1 (r + 2h_2 - \mu)} = \frac{\rho_{22}}{\rho_{11}} \end{aligned}$$

and Assumption B, we have

$$\frac{I_2}{I_1} < \left(\frac{\rho_{20}}{\rho_{10}}\right)^{\frac{\beta_{10}}{\beta_{10}-1}} < \left(\frac{\rho_{22}}{\rho_{11}}\right)^{\frac{\beta_{10}}{\beta_{10}-1}}.$$

Thus, the value function (denoted by $J(y)$) and the optimal stopping time (denoted by τ_J^*) of problem (4.4.9) coincide with $M(Y)$ and τ_0^* replaced ρ_{20} with ρ_{22} in Case 3 in Proposition 4.2.1, that is, the two firms set up the development of technology 2 at the same time

$$\tau_J^* = \inf\{t \geq 0 \mid Y(t) \geq y_{33}^*\}, \tag{4.4.10}$$

where y_{33} denotes the joint investment trigger corresponding to y_{30} in Proposition 4.2.1. The letter "J" refers to the case of joint investment. As in the single firm case, in joint investment both firms always choose technology 2.

If there exists any Y satisfying $L(Y) > J(Y)$, then the only preemption equilibria (not necessarily unique), which are asymmetric outcomes, occur. Otherwise, there arises the joint investment equilibria (not necessarily unique) in addition to the preemption equilibria. In this case, the joint investment equilibrium attained by the optimal joint investment rule (4.4.10) Pareto-dominates the other equilibria. For further details of the joint investment equilibria, see [42, 87].

4.5 Numerical examples

This section presents some examples in which the single firm's payoff $M(Y)$, the leader's payoff $F(Y)$, the joint investment payoff $J(Y)$ and the equilibrium strategies are numerically computed. We set the parameter values as Table 4.2 in order that Assumption B is satisfied and the single firm case corresponds a standard example in [22] (note $\rho_{20} = I_2 = 1$). Table 4.3 shows β_{ij}, and Table 4.4 and 4.5 indicate ρ_{ij}, I_i and y^*_{ij}. To begin with, we compute the single firm's problem. Figure 4.4 illustrates its value function $M(Y)$ corresponding to Case 3 in Proposition 4.2.1, where the investment time τ^*_M is

$$\tau^*_M = \inf\{t \geq 0 \mid Y(t) \geq y^*_{30} = 2\}. \tag{4.5.1}$$

Second, let us turn to the de facto standard case. Because the inequalities

$$1 < \left(\frac{\rho_{2i}}{\rho_{1i}}\right)^{\frac{\beta_{1i}}{\beta_{1i}-1}} < \frac{I_2}{I_1} \quad (i = 1, 2)$$

hold, the follower's optimal response $\tau^*_{F_i}$ has three triggers (see Table 4.5), that is, which technology the follower chooses depends on the initial value Y. Figure 4.5 illustrates the leader's payoff $L_i(Y)$ and the follower's payoff $F_i(Y)$. In Figure 4.5, $F_i(Y)$ is smooth while $L_i(Y)$ changes drastically at the follower's triggers y^*_{1i}, y^*_{2i} and y^*_{3i}. This means that the leader is greatly affected by the technology chosen by the follower. Particularly, a sharp rise of $L_i(Y)$ in the interval $[y^*_{2i},\ y^*_{3i}]$ in Figure 4.5 states that the leader prefers the follower choosing technology 2 to the follower choosing technology 1.

The payoffs $L(Y)$, $F(Y)$, and $J(Y)$ appear in Figure 4.6. Let us consider the firms' equilibrium strategies under Assumption A, i.e., the condition that the initial market demand y is small enough. Note that as mentioned in Section 4.4.3 the optimal joint investment strategy has the unique trigger y^*_{33} and both firms always choose technology 2. We see from Figure 4.6 that the preemption equilibrium is a unique outcome in the completion between the two firms, since there exists Y satisfying $J(Y) < L(Y)$. By assumption A, in the preemption equilibrium one of the firms becomes

Table 4.1: Comparison between the de facto standard and innovative cases.

	De facto standard	Innovative
Relative expected profit	$\rho_{2i}/\rho_{1i} < \rho_{20}/\rho_{10}$	$\rho_{2i}/\rho_{1i} > \rho_{20}/\rho_{10}$
Follower's value function	$F_1(Y) < F_2(Y)$	$F_1(Y) > F_2(Y)$
Preemption equilibrium	Both firms: likely to choose Tech. 1	Leader: Tech. 2, Follower: Tech. 2 (in most cases)

Table 4.2: Parameter setting.

r	μ	σ	D_1	D_2	h_1	h_2	K_1	K_2	$\tilde{K}_1$	$\tilde{K}_2$
0.04	0	0.2	0.025	0.05	0.32	0.16	0	0	0.18	0.2

Table 4.3: β_{ij}.

β_{10}	β_{20}	β_{11}	β_{21}	β_{12}	β_{22}
2	1	4.77	-3.77	3.7	-2.7

Table 4.4: Values common to both cases.

ρ_{10}	ρ_{20}	ρ_{11}	ρ_{22}	I_1	I_2	y_{30}^*	y_{33}^*
0.56	1	0.29	0.56	0.5	1	2	3.6

Table 4.5: Values dependent on the cases.

	ρ_{12}	ρ_{21}	y_{11}^*	y_{21}^*	y_{31}^*	y_{12}^*	y_{22}^*	y_{32}^*
De facto standard	0.38	0.38	2.15	5.46	5.59	1.78	2.81	3.04
Innovative	0.08	1	N/A	N/A	2	N/A	N/A	2.47

a leader investing in technology 1 at

$$\inf\{t \geq 0 \mid Y(t) \geq y_P = 0.93\} \tag{4.5.2}$$

(y_P is the intersection of $L(Y)$ and $F(Y)$ in Figure 4.6) and the other invests in technology 1 as follower at

$$\tau_{F_1}^* = \inf\{t \geq 0 \mid Y(t) \geq y_{11}^* = 2.15\}$$

if the leader has not succeeded in the development until this point. We observe that the leader's investment time (4.5.2) becomes earlier than the single firm's investment time (4.5.1). Furthermore, we see that the preemption trigger y_P in Figure 4.6 is the intersection of $L_1(Y)$ and $F_1(Y)$ instead of that of $L_2(Y)$ and $F_2(Y)$ and see that both firms switch the target from technology 2 chosen in the single firm situation to technology 1. Thus, consumers could suffer disadvantage that the only lower-standard technology emerges due to the competition.

It is obvious from Figure 4.6 that in the case where the roles of the firms are exogenously given, i.e., in the leader-follower game

$$\sup_{\tau \in \mathcal{T}} E[\mathrm{e}^{-r\tau} L(Y(\tau))],$$

the leader invests in technology 1. Therefore, in this instance, rather than the fear of preemption by the competitor, the presence of the competitor causes development of the lower-standard technology 1, which is never developed in the single firm situation. That is, the direct effect is strong enough to change the technology standard chosen by the firms.

Let us now replace $\sigma = 0.2$ by $\sigma = 0.8$ with other parameters fixed in Table 4.2 and consider the firms' strategic behavior under Assumption A. Notice that the higher product market uncertainty σ becomes the greater the advantage of technology 2 over technology 1 becomes. Figure 4.7 illustrates $L(Y), F(Y)$ and $J(Y)$. Since $J(Y) > L(Y)$ in Figure 4.7, the joint investment equilibria arise together with the preemption equilibria. There are two preemption equilibria corresponding the two leader's triggers y_{P_1} and y_{P_2}. It is reasonable to suppose that which type of equilibria occurs depends on the firms' inclination to the preemption behavior. In this instance, it can be readily seen from Figure 4.7 that in the corresponding leader-follower game the leader invests in technology 2 at the joint investment trigger y_{33}^*. This suggests that relative to the case in Figure 4.6, the fear of preemption by the competitor could drive the leader to develop the lower-standard technology 1, which never emerges in the noncompetitive situation, at the trigger y_{P_1}. That is, the direct effect is not strong enough to change the technology standard chosen by the firms, and the indirect effect together with the direct effect changes the firms' investment strategies.

Finally, we examine the innovative case. It can be deduced from the inequality

$$\left(\frac{\rho_{22}}{\rho_{12}}\right)^{\frac{\beta_{12}}{\beta_{12}-1}} > \frac{I_2}{I_1}$$

that the follower always chooses technology 1 (Table 4.5). The leader's payoff $L_i(Y)$ and the follower's payoff $F_i(Y)$ appear in Figure 4.8. The payoff $F_1(Y)$ dominates the others since it is equal to $M(Y)$ as shown in Section 4.4.2. Figure 4.9 illustrates $L(Y), F(Y)$ and $J(Y)$. We examine the firms' strategic behaviour under Assumption A. There occurs no joint investment outcome as there exists Y satisfying $J(Y) < L(Y)$. In the preemption equilibrium, as shown in Proposition 4.4.1, both firms invest in the same technology 2 but the different timings. Indeed, in equilibrium one of the firms invests in technology 2 at

$$\inf\{t \geq 0 \mid Y(t) \geq y_P = 1.06\} \tag{4.5.3}$$

(y_P denotes the intersection of $L(Y)$ and $F(Y)$ in Figure 4.9) as leader, while the other invests in the same technology at

$$\tau^*_{F_2} = \inf\{t \geq 0 \mid Y(t) \geq y^*_{32} = 2.47\}$$

as follower if the leader has yet to complete the technology at this point. We see that the leader's investment time (4.5.2) is earlier than the single firm's investment time (4.5.1) but is later than (4.5.2) in the de facto standard case. The preemption trigger y_P is the intersection of $L_2(Y)$ and $F_2(Y)$, and therefore the technology developed by firms remains unchanged by the competition. It is worth noting that y_P agrees with the preemption trigger in the case where the firms has no option to choose technology 1, that is, the preemption trigger derived in [87].

We make an additional comment on Assumption A. As assumed in the beginning of Section 4.4, this chapter have investigated the equilibrium strategy under Assumption A. However, Figures 4.6, 4.7, and 4.9 also show $L(Y), F(Y)$, and $J(Y)$ for Y larger than $\max_{i=1,2}\{I_i/\rho_{i0}\}$. Thus, from the figures, we could examine the firm's equilibrium strategy in cases where the initial value is too large to satisfy Assumption A. It must be noted that the results in those cases may depend on the parameter values; for this reason, we have limited the discussion to the case where Assumption A holds.

4.6 Conclusion

This chapter extended the R&D model in [87] to the case where a firm has the freedom to choose the timing and the standard of the research project, where the higher-standard technology is difficult

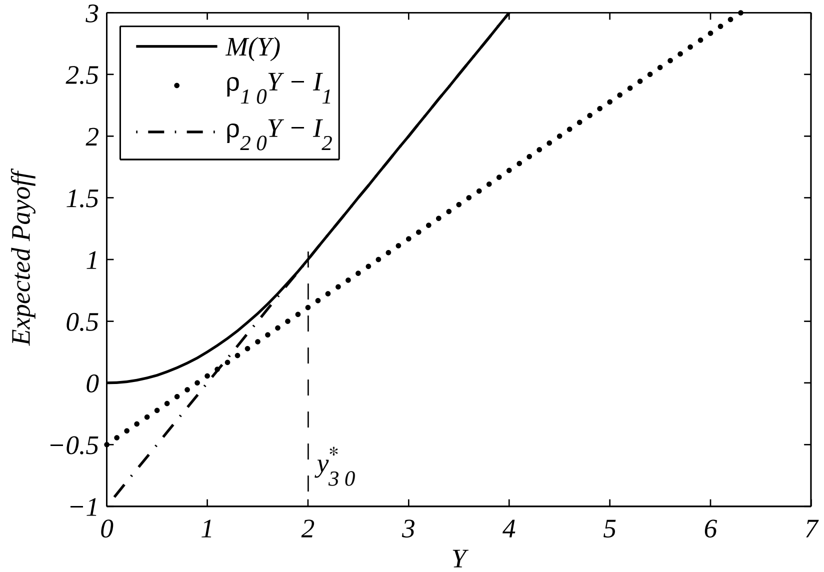

Figure 4.4: The monopolist's value function $M(Y)$.

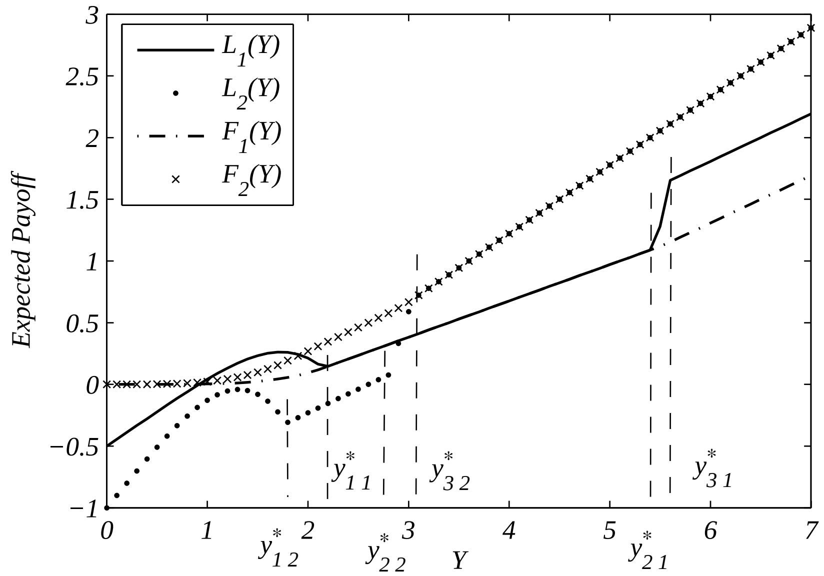

Figure 4.5: $L_i(Y)$ and $F_i(Y)$ in the de facto standard case.

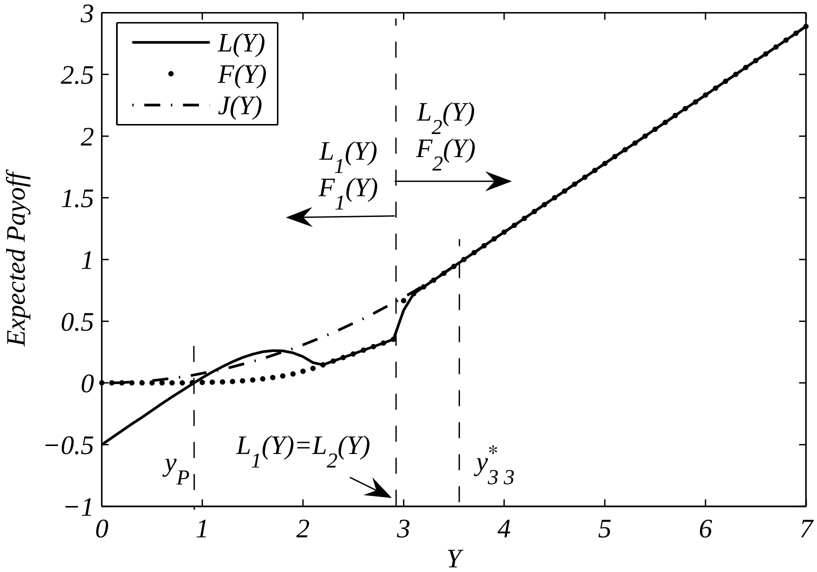

Figure 4.6: $L(Y), F(Y)$ and $J(Y)$ in the de facto standard case.

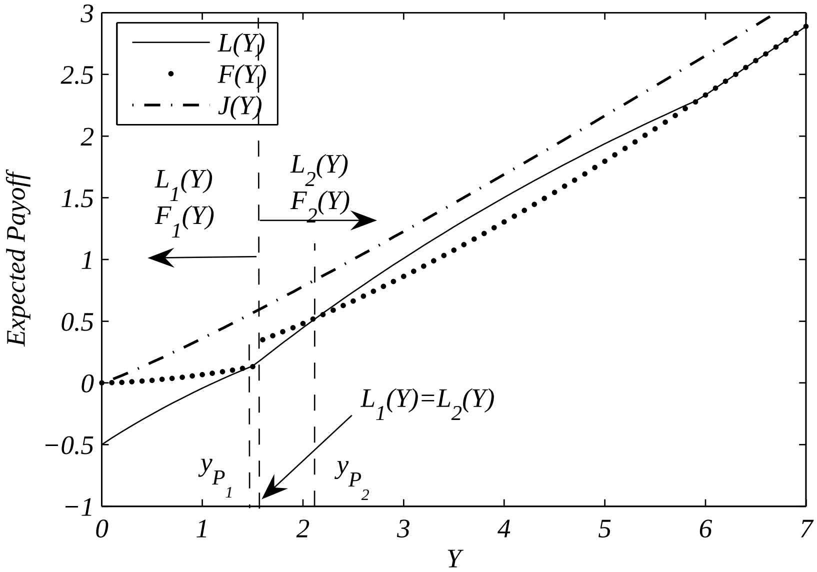

Figure 4.7: $L(Y), F(Y)$ and $J(Y)$ for $\sigma = 0.8$ in the de facto standard case.

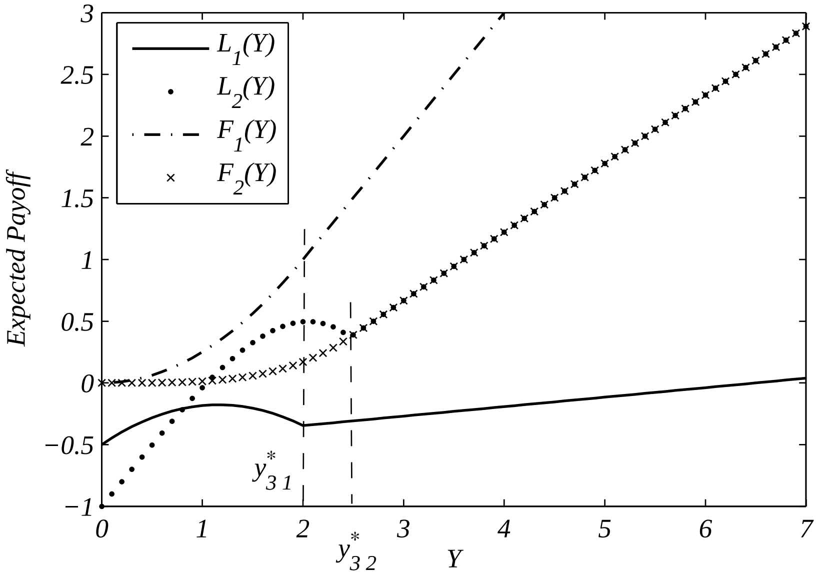

Figure 4.8: $L_i(Y)$ and $F_i(Y)$ in the innovative case.

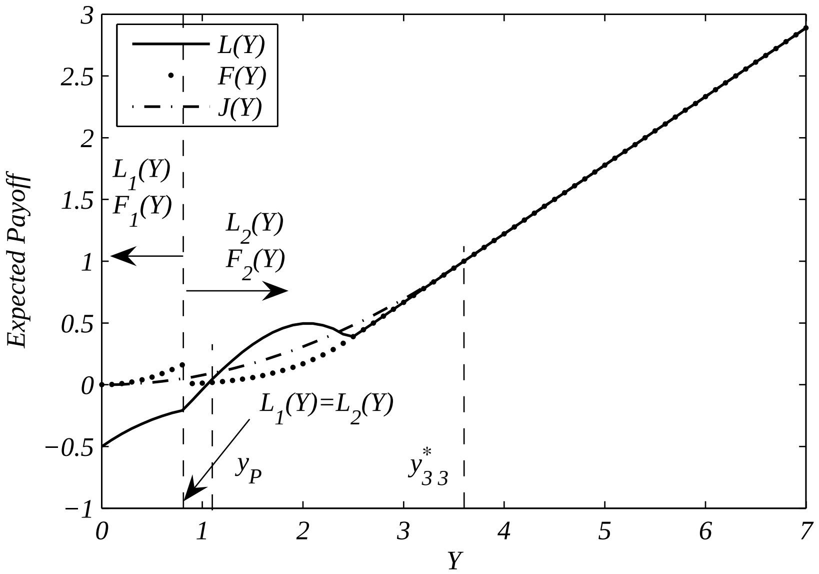

Figure 4.9: $L(Y), F(Y)$ and $J(Y)$ in the innovative case.

to complete and generates a greater cash flow. First, we derived the firm's optimal decision in the single firm situation. We thereafter extended the model to the situation of two firms and examined in full detail two typical cases, i.e., the de facto standard case and the innovative case. The results obtained in this chapter can be summarized as follows.

The competition between the two firms affects not only the firms' investment timing decision, but also their choice of the technology standard directly and indirectly. The choice of the project standard is indirectly affected by the hastened investment timing in the stopping time game between the two firms, as well as by the direct change of the project value by the presence of the competitor. In the de facto standard case, the competition increases the incentive to choose the lower-standard technology, which is easy to complete; in the innovative case, on the contrary, the competition increases the incentive to choose the higher-standard technology, which is difficult to complete. The main contribution of this chapter is showing that in the de facto standard case a lower-standard technology could emerge than is developed in the single firm situation. This implies the possibility that too bitter competition among firms adversely affects not only the firms but also consumers.

Finally, we mention potential extensions of this research. One of the remaining problems is to find a system in which noncooperative firms conduct more efficient R&D investment from the viewpoint of social welfare including consumers. A tax and a subsidy investigated in [39, 48] could provide viable solutions to the problem. Although this chapter considers a simple model with two types of uncertainty, namely technological uncertainty and market uncertainty, other types of uncertainty (see [41]) and other options, such as options to abandon and expand, could be involved with practical R&D investment (see [69]). It also remains as an interesting issue for future research to incorporate incomplete information (for example, uncertainty as to rivals' behavior as investigated in [50, 65]) in the model.

Chapter 5

Real Options under Incomplete Information

5.1 Introduction

This chapter investigates real options involving incomplete information. As introduced in Section 4.1, there have a growing number of studies on strategic real options. While many studies assume complete information about the competitors, Lambrecht and Perraudin [50] consider a model involving incomplete information about the competitors' investment costs. Hsu and Lambrecht [40] introduce asymmetric and incomplete information in real options in the context of a patent race. Using the filtering theory, Bernardo and Chowdhry [3], Décamps et al. [17] and Shibata [78] have investigated models in which a firm has incomplete information about parameters of its own profit flow rather than the competitors' behavior. Furthermore, literatures [31] and [66] have examined the effect of asymmetric information between the owner and the manager in the single firm.

The effect of incomplete information is practically significant, since how accurately a firm can estimate the behaviors of rival firms has a crucial effect on whether or not its investment succeeds. The previous studies such as [50] and [40] derived the values and the optimal strategies under incomplete information simultaneously. However, in their approach, the value under incomplete information has an element of the firm's estimation and hence it may exceed the value under complete information. In order to reveal how great loss a firm may suffer due to incomplete information, we examine the value of the project from a different aspect. Actually, we regard the value derived simultaneously with the optimal stopping time under incomplete information as what the firm *believes*. We, unlike the previous studies, calculate the *real* expected payoff, which is different from the value that the firm believes. Then, we derive the loss due to incomplete information as the difference between the real expected payoff and the expected payoff in the case of complete information. This analysis is useful to unveil a risk of a firm using the real options approach under incomplete information.

This chapter examines a model with a start-up who pioneers a new market by a unique idea and technology and a large firm that will eventually take over the market from the start-up. We evaluate the start-up's loss due to incomplete information about the large firm. Then, we clarify conditions under which the start-up needs more information about the large firm. Moreover, we show that in some cases the real options strategy under incomplete information gives less expected payoff to the start-up than the zero-NPV strategy (i.e., investing when the NPV of the investment becomes positive) under the same incomplete information. Our results suggest that in some cases a firm using the real options approach to investment has a great risk of incorrect conjectures about the behaviors of its competitors. Although we consider the simple model involving two firms for the purpose of concentrating our attention on the loss due to incomplete information, the proposed method of evaluating the loss due to incomplete information could also be applied to other real options models involving several firms.

This chapter is organized as follows. After the model is introduced in Section 5.2, Section 5.3 derives the start-up's value function and optimal strategy under complete information. Section 5.4 describes our main theoretical results, which show the start-up's strategy under incomplete information, its real expected payoff, and the loss due to incomplete information. In Section 5.5, we discuss how similar results can be obtained in a general situation, although the analysis in Sections 5.3 and 5.4 limit attention to a simple situation for mathematical convenience. Section 5.6 provides several implications with numerical examples. Section 5.7 concludes the chapter.

5.2 Model

This section introduces the model treated in this chapter. We consider the start-up (leader)'s problem of determining the timing of entering the new market which may be taken over by the large firm (follower) eventually. In this problem, we will discuss how incomplete information about the large firm affects the expected payoff of the start-up. As in the previous chapter, we assume that both stochastic process and random variable in this chapter are defined on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. The model is described as follows:

Profit flows and investment costs of the two firms: The start-up can receive a profit flow

$D_s(1,0)Y(t)$ in the new market by paying an indivisible investment cost I_s, but the flow will be reduced to $D_s(1,1)Y(t)$ after the large firm's entry to the market. The sucscripts "s" mean quantities concerning the start-up. Here, $(1,0)$ (resp. $(1,1)$) denotes the situation in which only the start-up (resp. both firms) is active in the market. Quantities $I_s, D_s(1,1)$ and $D_s(1,0)$ are constants such that $I_s > 0$ and $0 \leq D_s(1,1) < D_s(1,0)$, and $Y(t)$ is the market demand satisfying the geometric Brownian motion (4.2.1) as well as the previous section.

In contrast, the large firm does not notice the existence of the potential market until the start-up's investment. The large firm can obtain a profit flow $D_l(1,1)Y(t)$ in the market by paying an indivisible investment cost I_l after the start-up's investment. The sucscripts "l" mean quantities concerning the large firm. Here, I_l and $D_l(1,1)$ are positive constants. The adapted process $Y(t)$ captures observable (and exogenous) market demand at time t, while $D_i(\cdot,\cdot)$ $(i = l, s)$ captures the endogenous change in firm cash flows resulting from the respective firms' entrance in the market.

The large firm's investment decision: The large firm does not notice the opportunity to preempt the market until the date τ_s on which the start-up invests. Then, with discount rate $r(> \mu)$, the large firm optimizes its investment time τ_l by solving the following optimal stopping problem:

$$\sup_{\substack{\tau_l \in \mathcal{T} \\ \tau_l \geq \tau_s}} E[\int_{\tau_l}^{\infty} \mathrm{e}^{-rt} D_l(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_l} I_l], \tag{5.2.1}$$

where $\mathcal{T}$ denotes the set of all $\mathcal{F}_t$ stopping times. Let us call $Q_s = D_s(1,0)/I_s$ and $Q_l = D_l(1,1)/I_l$ the efficiencies of the start-up's and the large firm's investment, respectively. The efficiencies will be influenced by the profit margin in addition to the firm's idea and technology standard. This chapter considers a situation in which the large firm sets a smaller profit margin in order to take over the market from the leader. For that reason, the large firm's efficiency is likely to be lower than that of the start-up. Let τ_l^Q denote the optimal stopping time of problem (5.2.1) with $Q_l = D_l(1,1)/I_l$ replaced by a general constant $Q(> 0)$.

The start-up's investment decision: Since the start-up does not have complete information about the efficiency of the large firm, the start-up determines its investment time τ_s assuming that the efficiency of the large firm obeys a random variable X independent of filtration $\{\mathcal{F}_t\}$. Then, the start-up believes that its expected payoff of investing at τ_s is equal to

$$E\left[\int_{\tau_s}^{\tau_l^X} \mathrm{e}^{-rt} D_s(1,0)Y(t)\mathrm{d}t + \int_{\tau_l^X}^{+\infty} \mathrm{e}^{-rt} D_s(1,1)Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s\right], \tag{5.2.2}$$

where τ_l^X represents a random variable which takes a value $\tau_l^{X(\omega)}(\omega)$ for $\omega \in \Omega$ (note that τ_l^X also depends on τ_s). The start-up finds its investment time τ_s by solving the following optimal stopping problem:

$$V(y) = \sup_{\tau_s \in \mathcal{T}} E\left[\int_{\tau_s}^{\tau_l^X} \mathrm{e}^{-rt} D_s(1,0) Y(t) \mathrm{d}t + \int_{\tau_l^X}^{+\infty} \mathrm{e}^{-rt} D_s(1,1) Y(t) \mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s\right]. \tag{5.2.3}$$

Let $V(y)$ (recall $y = Y(0)$) and τ_s^* denote the value function and the optimal stopping time in problem (5.2.3), respectively. The optimal stopping time τ_s^* is expressed in a form independent of the initial value y, as will be shown in Sections 5.3 and 5.4. Let $V(y;Q)$ and τ_s^Q be the value function and the optimal stopping time, respectively, in problem (5.2.3) with X replaced by a constant $Q(>0)$. Note that, if the start-up has the complete information on the large firm's efficiency Q_l, the start-up invests at $\tau_s^{Q_l}$ and its real expected payoff agrees with $V(y;Q_l)$.

Remark 5.2.1 For simplicity, this chapter treats the two player leader-follower game as mentioned above. Similar results can be obtained in a more practical setting that permits several followers, by assuming that the followers make joint investment. There is a possibility that the followers make joint investment even if they are non-cooperative. For details, see [42].

Literatures [22] and [42] have investigated a preemption model in which both firms attempt to become a leader assuming complete information. Unlike their model, the model studied in this chapter is a leader-follower game. Indeed, we model a situation where a small entrepreneurial firm has the advantage of pioneering a new market, while a large follower has a big power of taking over the market from the small leader. In Sections 5.3 and 5.4, we assume $0 = D_s(1,1) < D_l(1,1)$ to avoid mathematical clutter and understand the essence of the loss due to incomplete information. Since this assumption is extreme, we consider a more realistic setting, i.e., $D_s(1,1) > 0$ in Section 5.5 and explain how similar results are obtained. In the rest of the chapter, we will denote for simplicity $D_s = D_s(1,0)$ and $D_l = D_l(1,1)$ unless they cause confusion.

5.3 Case of complete information

This section derives the value function $V(y;Q)$ and the optimal stopping time τ_s^Q of the start-up who believes that the efficiency of the large firm's investment is a constant $Q(>0)$. That is, we consider problem (5.2.3) with $X \equiv Q$. As in [22] and [42], we solve the leader-follower game backwards.

First, we begin by supposing that the start-up (leader) has already invested at time τ_s, and derive the optimal stopping time τ_l^Q of the large firm (follower). That is, we consider the follower's problem. Under the assumption that the start-up has already entered the market, the large firm's problem (5.2.1) can be treated as a problem for a monopolist. Therefore, the large firm's optimal stopping time τ_l^Q in problem (5.2.1) with $Q_l = Q$ is expressed as follows:

$$\tau_l^Q = \inf\{t \geq \tau_s \mid Y(t) \geq y_M(Q)\}, \tag{5.3.1}$$

where we define

$$y_M(Q) = \frac{\beta_1(r-\mu)}{(\beta_1 - 1)Q} \qquad (Q > 0), \tag{5.3.2}$$

and β_i are β_{i0} defined by (4.2.12) and (4.2.13). In Chapters 5 and 6, we omit the subscript 0 because we no longer use β_{i1} nor β_{i2}. The function $y_M(Q)$ represents the optimal investment trigger of a monopolist with efficiency Q (see, for example, [22]).

Next, using the large firm's response τ_l^Q derived as (5.3.1), we calculate the start-up's value function $V(y; Q)$ and investment time τ_s^Q in problem (5.2.3) with $X \equiv Q$. That is, we consider the leader's problem. Before stating the proposition, we define the function

$$p(\beta_1, Q_s, Q) = \left(\frac{1}{\beta_1}\right)^{\frac{1}{\beta_1 - 1}} - \frac{Q}{Q_s} \qquad (\beta_1 > 1, Q_s > 0, Q > 0), \tag{5.3.3}$$

which values how large the start-up's efficiency Q_s is against Q. The importance of the function $p(\beta_1, Q_s, Q)$ will be mentioned after the following proposition.

Proposition 5.3.1 The start-up's value function $V(y; Q)$ and optimal stopping time τ_s^Q are given as follows. If

$$p(\beta_1, Q_s, Q) > 0, \tag{5.3.4}$$

then

$$V(y; Q) = \begin{cases} A(Q)y^{\beta_1} & (0 < y < y_M(Q_s)) \\ \dfrac{D_s y}{r-\mu} - I_s - \dfrac{D_s y_M(Q)^{-\beta_1+1} y^{\beta_1}}{r-\mu} & (y_M(Q_s) \leq y \leq y_U(Q)) \\ B(Q)y^{\beta_2} & (y > y_U(Q)), \end{cases} \tag{5.3.5}$$

and τ_s^Q is expressed as $\tau_s^Q = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(Q)]\}$ (i.e., a hitting time into the interval $[y_M(Q_s), y_U(Q)]$) regardless of the initial value $Y(0) = y$. Here, $y_M(Q_s)$ is the threshold defined by (5.3.2) with $Q = Q_s$, and $A(Q)$ is defined by

$$A(Q) = y_M(Q_s)^{-\beta_1}\left(\frac{D_s y_M(Q_s)}{r-\mu} - I_s - \frac{D_s y_M(Q)^{-\beta_1+1} y_M(Q_s)^{\beta_1}}{r-\mu}\right) \quad (Q > 0). \tag{5.3.6}$$

Moreover, for $Q > 0$ satisfying (5.3.4), $y_U(Q)$ is the threshold defined by the unique solution of the equation

$$\frac{(\beta_1 - \beta_2)Q_s y_M(Q)^{-\beta_1+1}}{r-\mu} y_U(Q)^{\beta_1} + \frac{(\beta_2 - 1)Q_s}{r-\mu} y_U(Q) - \beta_2 = 0 \quad (y_M(Q_s) < y_U(Q) < y_M(Q)), \tag{5.3.7}$$

and $B(Q)$ is defined by

$$B(Q) = y_U(Q)^{-\beta_2}\left(\frac{D_s y_U(Q)}{r-\mu} - I_s - \frac{D_s y_M(Q)^{-\beta_1+1} y_U(Q)^{\beta_1}}{r-\mu}\right). \tag{5.3.8}$$

If (5.3.4) does not hold, then $V(y;Q) = 0$ for all $y > 0$ and $\tau_s^Q = +\infty$.

Proof Taking account of (5.3.1), we can compute (5.2.2) as follows:

$$\begin{aligned}
&E\left[\int_{\tau_s}^{\tau_l^Q} \mathrm{e}^{-rt} D_s Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s\right]\\
&= E\left[\mathrm{e}^{-r\tau_s}\left(D_s E^{Y(\tau_s)}\left[\int_0^{\tau_l^Q} \mathrm{e}^{-rt} Y(t)\mathrm{d}t\right] - I_s\right)\right] && (5.3.9)\\
&= E\left[\mathrm{e}^{-r\tau_s}\left(D_s E^{Y(\tau_s)}\left[\int_0^{+\infty} \mathrm{e}^{-rt} Y(t)\mathrm{d}t - \int_{\tau_l^Q}^{+\infty} \mathrm{e}^{-rt} Y(t)\mathrm{d}t\right] - I_s\right)\right]\\
&= E\left[\mathrm{e}^{-r\tau_s}\left(D_s E^{Y(\tau_s)}\left[\int_0^{+\infty} \mathrm{e}^{-rt} Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_l^Q} E^{Y(\tau_l^Q)}\left[\int_0^{+\infty} \mathrm{e}^{-rt} Y(t)\mathrm{d}t\right]\right] - I_s\right)\right] && (5.3.10)\\
&= E\left[\mathrm{e}^{-r\tau_s}\left(D_s E^{Y(\tau_s)}\left[\int_0^{+\infty} \mathrm{e}^{-rt} Y(t)\mathrm{d}t - \frac{\mathrm{e}^{-r\tau_l^Q}\max(Y(0), y_M(Q))}{r-\mu}\right] - I_s\right)\right]\\
&= E\left[\mathrm{e}^{-r\tau_s}\left(\frac{D_s Y(\tau_s)}{r-\mu} - \frac{D_s \max(Y(\tau_s), y_M(Q))^{-\beta_1+1} Y(\tau_s)^{\beta_1}}{r-\mu} - I_s\right)\right], && (5.3.11)
\end{aligned}$$

where we use the strong Markov property (e.g. see [68]) of the geometric Brownian motion $Y(t)$ to deduce (5.3.9) and (5.3.10), and use the formula of the expectation involving a hitting time (e.g. see [22]) to deduce (5.3.11). Here, for a random variable Z, $E^{Y(\tau_i)}[Z]$ denotes a random variable $G(Y(\tau_i))$, where for $y' > 0$, $G(y')$ is defined as an expectation $E[Z]$ in the case where $Y(t)$ starts at $Y(0) = y'$. Thus, problem (5.2.3) with $X \equiv Q$ is equivalent to $\sup_{\tau_s} E\left[\mathrm{e}^{-r\tau_s} f(Y(\tau_s);Q)\right]$, where

$$f(y;Q) = \frac{D_s y}{r-\mu} - \frac{D_s \max(y, y_M(Q))^{-\beta_1+1} y^{\beta_1}}{r-\mu} - I_s. \tag{5.3.12}$$

Consider the case where $f(y;Q) \le 0$ for all $y > 0$. In this case, the value function and the optimal stopping time are trivially given by $V(y;Q) = 0$ and $\tau_s^Q = +\infty$, respectively, for all $y > 0$. Now, let us derive a necessary and sufficient condition for $f(y;Q) \le 0$ to hold for all $y > 0$. Since $f(y;Q)$ is concave for $y \in [0, y_M(Q)]$ by $\beta_1 > 1$ and $f(y;Q) = -I_s$ holds for $y = 0$ and $y \ge y_M(Q)$, $f(y;Q)$ $(y > 0)$ takes the maximum value at $y = \beta_1^{-1/(\beta_1-1)} y_M(Q)$, which is the unique solution of $\partial f(y;Q)/\partial y = 0$ for $y \in [0, y_M(Q)]$. Since we have $f(\beta_1^{-1/(\beta_1-1)} y_M(Q);Q) = D_s p(\beta_1, Q_s, Q)/Q$ by (5.3.2), (5.3.3) and (5.3.12), we can deduce that $p(\beta_1, Q_s, Q) \le 0$ is a necessary and sufficient condition for $f(y;Q) \le 0$ to hold for all $y > 0$. Thus, if $p(\beta_1, Q_s, Q) \le 0$, we have $V(y;Q) = 0$ and $\tau_s^Q = +\infty$.

Next, we consider the case where $p(\beta_1, Q_s, Q) > 0$. In this case, if we can check that the right-hand side of (5.3.5), denoted $\phi(y)$, is a continuously differentiable function satisfying the following conditions:

$$
\begin{aligned}
&\frac{\sigma^2 y^2}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}(y) + \mu y \frac{\mathrm{d}\phi}{\mathrm{d}y}(y) - r\phi(y) \begin{cases} \leq 0 \text{ for all } y > 0, \\ = 0 \text{ for all } y \notin [y_M(Q_s), y_U(Q)], \end{cases} \\
&\phi(y) - f(y) \begin{cases} \geq 0 \text{ for all } y > 0, \\ = 0 \text{ for all } y \in [y_M(Q_s), y_U(Q)], \end{cases} \\
&\lim_{y \downarrow 0} \phi(y) = \lim_{y \to +\infty} \phi(y) = 0, \\
&\phi(y) : \text{twice continuously differentiable at any } y \neq y_M(Q_s), y_U(Q),
\end{aligned}
\tag{5.3.13}
$$

then we obtain the value function $V(y;Q) = \phi(y)$ and the optimal stopping time $\tau_s^Q = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(Q)]\}$ via the relation between optimal stopping and variational inequalities (for details see [68]). Note that the thresholds $y_M(Q)$ and $y_U(Q)$ are defined so that $\phi(y)$ is continuously differentiable at the thresholds (i.e, value matching and smooth pasting, see also [22]). Since we can check all the conditions for $\phi(y)$ by direct calculation, we obtain the proposition. □

Remark 5.3.1 Until the large firm's efficiency Q exceeds the solution of $p(\beta_1, Q_s, Q) = 0$, inequality (5.3.4) holds, and $y_U(Q)$ and $V(y;Q)$ monotonically decrease with Q. On the contrary, we have $y_U(Q) \to +\infty$ and $\tau_s^Q \to \inf\{t \geq 0 \mid Y(t) \geq y_M(Q_s)\}$ as $Q \downarrow 0$; this means that the stopping time τ_s^Q tends to the optimal stopping time of a monopolist.

Remark 5.3.2 By taking $Q = Q_l$ in Proposition 5.3.1, we obtain the real expected payoff $V(y;Q_l)$ of the start-up who has complete information about the efficiency of the large firm.

We explain equation (5.3.5) in Proposition 5.3.1. The interval $[y_M(Q_s), y_U(Q)]$ represents the stopping (investment) region, where the start-up immediately invests. The value function $V(y;Q)$ in this region consists of two components, namely the monopoly profit $D_s y/(r-\mu) - I_s$ and the subtracter $D_s y_M(Q)^{-\beta_1+1} y^{\beta_1}/(r-\mu)$ which represents the effect of takeover by the follower. The remaining parts $(0, y_M(Q_s))$ and $(y_U(Q), \infty)$ represent the continuation region, where the start-up delays its investment until the market demand $Y(t)$ reaches one of the thresholds. However, the reasons why the start-up defers its investment in the two regions are completely different. When $Y(t)$ lies in the region $(0, y_M(Q_s))$, the start-up waits until the market demand level $y_M(Q_s)$ is achieved so as to obtain a good profit from the investment. On the other hand, when $Y(t)$ is in the region $(y_U(Q), \infty)$, the start-up waits for the market demand $Y(t)$ to *fall down* to $y_U(Q)$ so as to prevent the follower from investing too early. Note that the start-up's investment trigger $y_M(Q_s)$ remains unchanged from that of the monopolist regardless of the large firm's efficiency Q, as far as inequality (5.3.4) is satisfied. The value functions $A(Q)y^{\beta_1}$ and $B(Q)y^{\beta_1}$ in the continuation regions mean the values of the options to invest at the triggers $y_M(Q_s)$ and $y_U(Q)$, respectively.

As the presence of the upper trigger $y_U(Q)$ is a distinctive feature of our leader-follower model, we will explain it in more detail. Proposition 5.3.1 suggests that if the initial value $Y(0) = y$ is larger than $y_U(Q)$, the start-up should delay its investment until the market demand $Y(t)$ drops to the threshold $y_U(Q)$. Note that $Y(t)$ could decrease from the initial value y to the threshold $y_U(Q)$ even with a positive drift μ in (4.2.1) because of the positive volatility σ in (4.2.1). Even if the start-up makes its investment in the case of $Y(t) > y_U(Q)$, the large firm is quite likely to enter the market before the start-up gains a sufficient cash flow. Thus, the start-up defers its investment when the market demand is great.

Inequality (5.3.4) can be interpreted as a prerequisite condition for the start-up's investment. In fact, the start-up's expected payoff never becomes positive for any time t and any value of $Y(t)$, unless (5.3.4) holds. Now we examine how the prerequisite condition (5.3.4) is changed by the values of parameters μ, r and σ. We can see from (4.2.12) that $\partial\beta_1/\partial\sigma < 0, \lim_{\sigma\to+\infty}\beta_1 = 1, \lim_{\sigma\downarrow 0}\beta_1 = r/\mu > 1, \partial\beta_1/\partial\mu < 0$ and $\partial\beta_1/\partial r > 0$ (see [22]). Since $p(\beta_1, Q_s, Q)$ is monotonically increasing for $\beta_1 > 1$ by (5.3.3), the prerequisite condition (5.3.4) becomes more restrictive, i.e., the opportunity for the start-up to invest is more likely to be lost, as the drift μ and the volatility σ (resp. the discount rate r) in the market increase (resp. decrease). This is because the start-up's investment opportunity is greatly affected by the large firm. That is, increases in the drift and volatility raise the probability of the large firm's entry. Consequently, it is harder for the start-up to find the opportunity to obtain enough profits before the takeover by the big follower. Moreover, we have $p(\beta_1, Q_s, Q) \downarrow 1/\mathrm{e} - Q/Q_s$ as $\beta_1 \downarrow 1$, and $p(\beta_1, Q_s, Q) \uparrow 1 - Q/Q_s$ as $\beta_1 \to +\infty$ by (5.3.3). Hence, if the start-up's efficiency Q_s is e times larger than that of the large firm, the prerequisite condition (5.3.4) always holds and the start-up's entrepreneurial activity is absolutely valuable for any values of parameters μ, r and σ. If the start-up's efficiency Q_s is less than that of the large firm, on the other hand, then the prerequisite condition never holds, which means that the start-up has no opportunity to make the entrepreneurial activity.

Finally, it should be noted that the complete information version has an element of incomplete information, because the large firm does not learn about the investment opportunity until the start-up makes its investment. The start-up knows this and uses its informational advantage in determining its optimal investment. The next section describes our main results, which evaluate the start-up's loss due to incomplete information about the efficiency of the large firm's investment

5.4 Loss due to incomplete information

This section evaluates the start-up's loss due to incomplete information about the efficiency of the large firm's investment by the following procedure:

Step 1: Derive the value function $V(y)$ and the optimal stopping time τ_s^* in problem (5.2.3) which the start-up believes.

Step 2: Calculate the real expected payoff $\tilde{V}(y)$ of the start-up who invests at time τ_s^* calculated in Step 1.

Step 3: Derive $W(y) = V(y; Q_l) - \tilde{V}(y)$, which is the difference between the expected payoff of the start-up who invests at time $\tau_s^{Q_l}$ under complete information and that of the start-up who invests at (wrong) time τ_s^* due to incomplete information.

The quantity $W(y)$ calculated in Step 3 is regarded as the loss due to incomplete information. Most of the existing works concerning real options under incomplete information consider only Step 1, namely the optimal strategy and the value that the firm *believes* under incomplete information. We however consider the real payoff in Step 2 and then compare the real payoff (*which is different from the value in Step 1*) and the value under complete information in Step 3. In the above procedure, we examine the loss which the firm suffers due to incomplete information. The proposed method may also be applied to other real options models involving incomplete information. The loss due to incomplete information is identified as the value of information about the rival firm, and hence it tells us whether the firm should conduct a further survey on the rival firm or not. Sections 5.4.1, 5.4.2, and 5.4.3 describe Steps 1, 2, and 3, respectively.

5.4.1 The start-up's strategy under incomplete information

The start-up determines its investment time, believing that the large firm's efficiency obeys a random variable X independent of the filtration $\{\mathcal{F}_t\}$. We call the random variable X the start-up's estimation of the large firm's efficiency. Here we assume that $X > 0$ and $E[X^{\beta_1 - 1}] < +\infty$. We define the function $g(y)$ by

$$g(y) = \frac{D_s y}{r - \mu} - I_s - \frac{D_s E\left[\max(y_M(X), y)^{-\beta_1 + 1}\right] y^{\beta_1}}{r - \mu} \qquad (y > 0). \tag{5.4.1}$$

It can be seen that $g(y)$ is equal to the expectation (5.2.2) with $\tau_s = 0$, namely the payoff (that the start-up believes) by the immediate investment. Recall that y is the initial market demand $y = Y(0)$. Generally, it is hard to derive an explicit form of the value function and the optimal stopping time in problem (5.2.3). However, we can show that problem (5.2.3) is reducible to the problem with $Q = \tilde{Q}_l$ in Section 5.3, where we define $\tilde{Q}_l = E[X^{\beta_1-1}]^{1/(\beta_1-1)}$, provided that the following condition holds:

Condition (a): The inequality $g(y) \leq V(y; \tilde{Q}_l)$ holds for all $y > 0$.

The quantity $\tilde{Q}_l$ features the start-up's strategy under incomplete information as will be shown in the following proposition. In relation to (5.4.1), we define

$$\tilde{g}(y) = \frac{D_s y}{r-\mu} - I_s - \frac{D_s E\left[y_M(X)^{-\beta_1+1}\right] y^{\beta_1}}{r-\mu} \qquad (y > 0). \tag{5.4.2}$$

From the definitions of $g(y), \tilde{g}(y), \tilde{Q}_l$ and Proposition 5.3.1, it immediately follows that

$$\tilde{g}(y) \leq g(y) \qquad (y > 0), \tag{5.4.3}$$

$$\tilde{g}(y) = V(y; \tilde{Q}_l) \qquad (y_M(Q_s) < y < y_U(\tilde{Q}_l),\ p(\beta_1, Q_s, \tilde{Q}_l) > 0). \tag{5.4.4}$$

Using this property, we can show the following proposition, which is the key to evaluating the loss due to incomplete information.

Proposition 5.4.1 Assume that Condition (a) holds. The value function $V(y)$ and the optimal stopping time τ_s^* in problem (5.2.3) which the start-up believes are given as $V(y) = V(y; \tilde{Q}_l)$ and $\tau_s^* = \tau_s^{\tilde{Q}_l}$ for all $y > 0$, respectively, where $V(y; \tilde{Q}_l)$ and $\tau_s^{\tilde{Q}_l}$ are given in Proposition 5.3.1.

Proof Note that

$$\begin{aligned}
& E\left[\int_{\tau_s}^{\tau_l^X} \mathrm{e}^{-rt} D_s Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s\right] \\
&= \int_0^{+\infty} E\left[\int_{\tau_s}^{\tau_l^X} \mathrm{e}^{-rt} D_s Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s \mid X = Q\right] \mathrm{d}\Psi_X(Q) \\
&= \int_0^{+\infty} E\left[\int_{\tau_s}^{\tau_l^Q} \mathrm{e}^{-rt} D_s Y(t)\mathrm{d}t - \mathrm{e}^{-r\tau_s} I_s\right] \mathrm{d}\Psi_X(Q) && (5.4.5) \\
&= \int_0^{+\infty} E\left[\mathrm{e}^{-r\tau_s} f(Y(\tau_s); Q)\right] \mathrm{d}\Psi_X(Q) && (5.4.6) \\
&= E\left[\mathrm{e}^{-r\tau_s} g(Y(\tau_s))\right], && (5.4.7)
\end{aligned}$$

where $\Psi_X(Q)$ denotes the distribution of X, and f and g are defined by (5.3.12) and (5.4.1), respectively. Here, (5.4.5) and (5.4.7) follow from the independence between X and $Y(t)$, and (5.4.6) follows from the strong Markov property as in Proof of Proposition 5.3.1.

First, we consider the case where $g(y) \le 0$ for all $y > 0$. In this case, apparently, the value function and the optimal stopping time are given by $V(y) = 0$ and $\tau_s^* = +\infty$, respectively, for all $y > 0$. Since $\tilde{g}(y) \le 0$ holds for all $y > 0$ by (5.4.3), $V(y; \tilde{Q}_l) = 0$ and $\tau_s^{\tilde{Q}_l} = +\infty$ hold for all $y > 0$. This implies $V(y) = V(y; \tilde{Q}_l)$ and $\tau_s^* = \tau_s^{\tilde{Q}_l}$ for all $y > 0$.

Next, let us assume that there exists some $\hat{y} > 0$ such that $g(\hat{y}) > 0$. We have $V(\hat{y}; \tilde{Q}_l) > 0$ by Condition (a) (i.e., $g(y) \le V(y; \tilde{Q}_l)$ for all $y > 0$). Then, we can deduce that $p(y, Q_s, \tilde{Q}_l) > 0$, taking into consideration that $V(y; \tilde{Q}_l) = 0$ holds for all $y > 0$ whenever $p(y, Q_s, \tilde{Q}_l) \le 0$ by Proposition 5.3.1. We have only to check the conditions (5.3.13) with Q and f replaced by $\tilde{Q}_l$ and g for $\phi(y) = V(y; \tilde{Q}_l)$ (i.e., the right-hand side of (5.3.5) with Q replaced by $\tilde{Q}_l$). The conditions (5.3.13) except for the second can be checked directly. Condition (a) ensures $\phi(y) - g(y) \ge 0$ for all $y > 0$. By (5.4.3) and (5.4.4), for all $y \in [y_M(Q_s), y_U(\tilde{Q}_l)]$, we have $\phi(y) - g(y) = \tilde{g}(y) - g(y) \le 0$, where $\tilde{g}(y)$ is defined by (5.4.2). These imply the second condition. Therefore, we obtain $V(y) = \phi(y)$ and $\tau_s^* = \inf\{t \ge 0 \mid Y(t) \in [y_M(Q_s), y_U(\tilde{Q}_l)]\}$ via the relation between optimal stopping and variational inequalities (e.g., see [68]). □

Remark 5.4.1 Condition (a) is likely to hold when the support of X is not very wide. In particular, we can easily show that Condition (a) always holds whenever X is a constant.

Remark 5.4.2 Figure 5.1 illustrates the function $V(y) = V(y; \tilde{Q}_l)$ together with the functions $g(y)$ and $\tilde{g}(y)$ under Condition (a). In particular, we observe that $V(y) = V(y; \tilde{Q}_l) = g(y) = \tilde{g}(y)$ holds for $y \in [y_M(Q_s), y_U(\tilde{Q}_l)]$.

Proposition 5.4.1 shows the value function and the optimal stopping time of the start-up with estimation X. It should be noted that the value $V(y)$ is just the one *believed* by the start-up and is different from the real expected payoff of the investment, $\tilde{V}(y)$, which will be calculated in the next subsection. By Proposition 5.4.1, the start-up with the estimation X takes the same strategy as that of the start-up with the constant estimation $\tilde{Q}_l$ under Condition (a). That is, the start-up's estimation of the large firm's efficiency is completely characterized by the single quantity $\tilde{Q}_l = E[X^{\beta_1 - 1}]^{1/(\beta_1 - 1)}$ independently of its distribution. However, this is not always true in a general case without Condition (a). In the rest of the chapter, we will restrict our attention to the case where Condition (a) is satisfied.

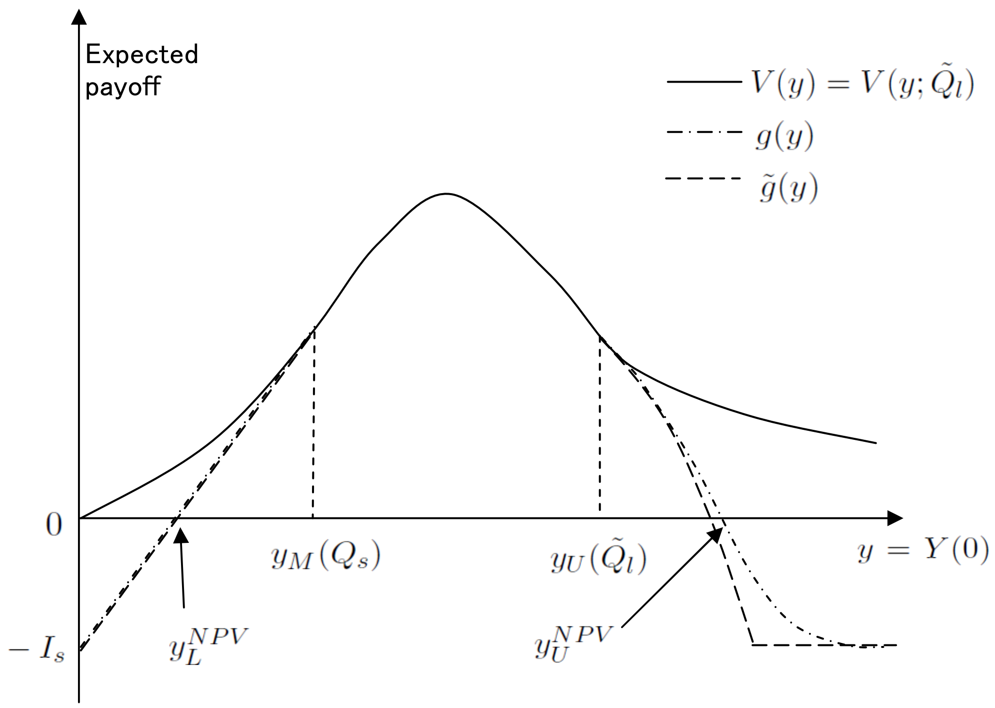

Figure 5.1: $g(y), \tilde{g}(y)$ and $V(y) = V(y; \tilde{Q}_l)$.

5.4.2 The real expected payoff of the start-up

This subsection derives the real expected payoff $\tilde{V}(y)$ of the start-up who invests at time τ_s^* calculated in Proposition 5.4.1. Since the large firm's real efficiency is Q_l, its real investment time is equal to (5.3.1) with $Q = Q_l$, i.e.,

$$\tau_l^{Q_l} = \inf\{t \geq \tau_s^* \mid Y(t) \geq y_M(Q_l)\}. \tag{5.4.8}$$

Then, the start-up's real expected payoff $\tilde{V}(y)$ becomes

$$\tilde{V}(y) = E\left[\int_{\tau_s^*}^{\tau_l^{Q_l}} \mathrm{e}^{-rt} D_s Y(t) \mathrm{d}t - \mathrm{e}^{-r\tau_s^*} I_s\right]. \tag{5.4.9}$$

We can show the following proposition by calculating the expectation (5.4.9).

Proposition 5.4.2 Assume that Condition (a) holds. The real expected payoff $\tilde{V}(y)$ of the start-up who invests at τ_s^* is given as follows. If $p(\beta_1, Q_s, \tilde{Q}_l) > 0$, then

$$\tilde{V}(y) = \begin{cases} \tilde{A}(Q_l) y^{\beta_1} & (0 < y < y_M(Q_s)) \\ \dfrac{D_s y}{r-\mu} - I_s - \dfrac{D_s \max(y, y_M(Q_l))^{-\beta_1+1} y^{\beta_1}}{r-\mu} & (y_M(Q_s) \leq y \leq y_U(\tilde{Q}_l)) \\ \tilde{B}(\tilde{Q}_l) y^{\beta_2} & (y > y_U(\tilde{Q}_l)), \end{cases} \tag{5.4.10}$$

where $y_M(\cdot)$ is defined by (5.3.2), $y_U(\tilde{Q}_l)$ is the unique solution of equation (5.3.7) with $Q = \tilde{Q}_l$, and $\tilde{A}(Q_l)$ and $\tilde{B}(\tilde{Q}_l)$ are defined by

$$\tilde{A}(Q_l) = y_M(Q_s)^{-\beta_1} \left(\frac{D_s y_M(Q_s)}{r-\mu} - I_s - \frac{D_s \max(y_M(Q_s), y_M(Q_l))^{-\beta_1+1} y_M(Q_s)^{\beta_1}}{r-\mu}\right), \tag{5.4.11}$$

$$\tilde{B}(\tilde{Q}_l) = y_U(\tilde{Q}_l)^{-\beta_2} \left(\frac{D_s y_U(\tilde{Q}_l)}{r-\mu} - I_s - \frac{D_s \max(y_U(\tilde{Q}_l), y_M(Q_l))^{-\beta_1+1} y_U(\tilde{Q}_l)^{\beta_1}}{r-\mu}\right). \tag{5.4.12}$$

If $p(\beta_1, Q_s, \tilde{Q}_l) \leq 0$, then $\tilde{V}(y) = 0$ for all $y > 0$.

Proof We have only to compute the expectation (5.4.9). First, we assume $p(\beta_1, Q_s, \tilde{Q}_l) \leq 0$. In this case, we have $\tau_s^* = \tau_s^{\tilde{Q}_l} = +\infty$ by Propositions 5.3.1 and 5.4.1, and hence we have also $\tau_l^{Q_l} = +\infty$ by (5.4.8). Thus, $\tilde{V}(y) = 0$ holds for all $y > 0$. Next, let us assume $p(\beta_1, Q_s, \tilde{Q}_l) > 0$. In this case, we have

$$\tau_s^* = \tau_s^{\tilde{Q}_l} = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(\tilde{Q}_l)]\} \tag{5.4.13}$$

by Propositions 5.3.1 and 5.4.1. By the strong Markov property, (5.4.9) is equal to (5.3.11) with τ_s and Q replaced by τ_s^* and Q_l, respectively. That is, we have $\tilde{V}(y) = E\left[\mathrm{e}^{-r\tau_s^*} f(Y(\tau_s^*); Q_l)\right]$, where

f is defined by (5.3.12). Since $Y(\tau_s^*)$ is a constant such that

$$Y(\tau_s^*) = \begin{cases} y_M(Q_s) & (0 < y < y_M(Q_s)) \\ y & (y_M(Q_s) \leq y \leq y_U(\tilde{Q}_l)) \\ y_U(\tilde{Q}_l) & (y > y_U(\tilde{Q}_l)) \end{cases} \tag{5.4.14}$$

by (5.4.13), we have

$$\tilde{V}(y) = f(Y(\tau_s^*); Q_l) E\left[\mathrm{e}^{-r\tau_s^*}\right]. \tag{5.4.15}$$

Thus, by applying the formula of the expectation involving a hitting time to (5.4.15), we obtain the formula of $\tilde{V}(y)$ given in the proposition. □

Remark 5.4.3 Propositions 5.3.1, 5.4.1 and 5.4.2 ensure that $\tilde{V}(y) = V(y; Q_l) = V(y; \tilde{Q}_l) = V(y)$ under Condition (a), whenever $\tilde{Q}_l = Q_l$.

We make a brief explanation about Proposition 5.4.2. If $p(\beta_1, Q_s, \tilde{Q}_l) > 0$, then the start-up invests as soon as the market demand $Y(t)$ reaches the investment region $[y_M(Q_s), y_U(\tilde{Q}_l)]$. Then it obtains the expected cash flow (5.4.10), but (5.4.10) may be negative if the start-up's estimation of the large firm's efficiency is far from correct. Otherwise, the start-up makes a decision of never investing because it considers no value of the project due to the presence of the big follower.

5.4.3 The start-up's loss due to incomplete information

We evaluate the start-up's loss $W(y) = V(y; Q_l) - \tilde{V}(y)$ due to incomplete information about the large firm's efficiency. The loss $W(y)$ varies according to the relation between $\tilde{Q}_l$ and Q_l. Note that $y_M(\cdot)$ is monotonically decreasing by definition (5.3.2).

Case (U): $\tilde{Q}_l < Q_l$ The start-up underestimates the large firm's efficiency, and $y_M(\tilde{Q}_l) > y_M(Q_l)$ holds with respect to the large firm's entry trigger.

Case (C): $\tilde{Q}_l = Q_l$ The start-up correctly estimates the large firm's efficiency, and $y_M(\tilde{Q}_l) = y_M(Q_l)$ holds with respect to the large firm's entry trigger.

Case (O): $\tilde{Q}_l > Q_l$ The start-up overestimates the large firm's efficiency, and $y_M(\tilde{Q}_l) < y_M(Q_l)$ holds with respect to the large firm's entry trigger.

Proposition 5.4.3 Assume that Condition (a) holds. The start-up's loss $W(y)$ due to incomplete information is given as follows.

Case (U): $\tilde{Q}_l < Q_l$

Case (U.1): $p(\beta_1, Q_s, \tilde{Q}_l) \leq 0$ $\quad W(y) = 0$ for all $y > 0$.

Case (U.2): $p(\beta_1, Q_s, \tilde{Q}_l) > 0$ and $p(\beta_1, Q_s, Q_l) \leq 0$ $\quad W(y) = -\tilde{V}(y)$ for all $y > 0$.

Case (U.3): $p(\beta_1, Q_s, Q_l) > 0$

$$W(y) = \begin{cases} 0 & (0 < y < y_U(Q_l)) \\ B(Q_l)y^{\beta_2} - \dfrac{D_s y}{r-\mu} + I_s + \dfrac{D_s \max(y, y_M(Q_l))^{-\beta_1+1} y^{\beta_1}}{r-\mu} & (y_U(Q_l) \leq y \leq y_U(\tilde{Q}_l)) \\ \left(B(Q_l) - \tilde{B}(\tilde{Q}_l)\right) y^{\beta_2} & (y > y_U(\tilde{Q}_l)). \end{cases}$$

Case (C): $\tilde{Q}_l = Q_l$ $\quad W(y) = 0$ for all $y > 0$.

Case (O): $\tilde{Q}_l > Q_l$

Case (O.1): $p(\beta_1, Q_s, Q_l) \leq 0$ $\quad W(y) = 0$ for all $y > 0$.

Case (O.2): $p(\beta_1, Q_s, Q_l) > 0$ and $p(\beta_1, Q_s, \tilde{Q}_l) \leq 0$ $\quad W(y) = V(y; Q_l)$ for all $y > 0$.

Case (O.3): $p(\beta_1, Q_s, \tilde{Q}_l) > 0$

$$W(y) = \begin{cases} 0 & (0 < y < y_U(\tilde{Q}_l)) \\ \dfrac{D_s y}{r-\mu} - I_s - \dfrac{D_s y_M(Q_l)^{-\beta_1+1} y^{\beta_1}}{r-\mu} - \tilde{B}(\tilde{Q}_l) & (y_U(\tilde{Q}_l) \leq y \leq y_U(Q_l)) \\ \left(B(Q_l) - \tilde{B}(\tilde{Q}_l)\right) y^{\beta_2} & (y > y_U(Q_l)). \end{cases}$$

Here, $y_U(\cdot)$ is the unique solution of equation (5.3.7), and $B(Q_l)$ and $\tilde{B}(\tilde{Q}_l)$ are defined by (5.3.8) with $Q = Q_l$ and (5.4.12), respectively.

Proof In Case (C) (i.e., $\tilde{Q}_l = Q_l$), we have $\tilde{V}(y) = V(y; Q_l)$ and hence $W(y) = V(y; Q_l) - \tilde{V}(y) = 0$. By (5.3.3), in Case (U) (i.e., $\tilde{Q}_l < Q_l$) we have $p(\beta_1, Q_s, Q_l) < p(\beta_1, Q_s, \tilde{Q}_l)$, while in Case (O) (i.e., $\tilde{Q}_l > Q_l$) we have $p(\beta_1, Q_s, \tilde{Q}_l) < p(\beta_1, Q_s, Q_l)$. Therefore, we can further classify Cases (O) and (U) into six regions. Then, we can easily calculate $W(y) = V(y; Q_l) - \tilde{V}(y)$ from Propositions 5.3.1 and 5.4.2 in each case. □

Let us mention how the start-up suffers the loss due to incomplete information in each case of the above proposition. Needless to say, in Case (C) the start-up's strategy becomes optimal as $\tau_s^* = \tau_s^{\tilde{Q}_l} = \tau_s^{Q_l}$, and hence the start-up suffers no loss for any initial value. In Cases (U.1) and (O.1), the prerequisite condition for the start-up's investment does not actually hold (i.e., $p(\beta_1, Q_s, Q_l) \leq 0$), and the start-up never attempts to invest. As a result, in these cases the start-up's never investment strategy is optimal, and the loss is always zero.

Cases (U.2) and (O.2) correspond, respectively, to the case where the start-up attempts to invest although the prerequisite condition does not actually hold and the case where the start-up never attempts to invest although the prerequisite condition actually holds. Due to this misjudgment of the opportunity to invest, the start-up suffers the loss for any initial value $y > 0$. Note that $W(y)$ in Case (U.2) is positive by $\tilde{V}(y) < 0$ for all $y > 0$.

In Cases (U.3) and (O.3), the prerequisite condition actually holds, and also the start-up attempts to invest. Thus, in these cases, unlike in Cases (U.2) and (O.2), the start-up's judgement of the investment opportunity is correct. The start-up makes its investment at $\tau_s^{\tilde{Q}_l} = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(\tilde{Q}_l)]\}$, though the optimal investment timing $\tau_s^{Q_l}$ is given as $\inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(Q_l)]\}$. In Case (U.3), since $y_U(\tilde{Q}_l) > y_U(Q_l)$, the start-up makes the investment earlier than $\tau_s^{Q_l}$ and suffers the loss $W(y)$ when $y > y_U(Q_l)$; contrarily, in Case (O.3), since $y_U(\tilde{Q}_l) < y_U(Q_l)$, the start-up makes the investment later than $\tau_s^{Q_l}$ and suffers the loss $W(y)$ when $y > y_U(\tilde{Q}_l)$. The loss in the second region in Case (U.3) can be interpreted as the value of the option to defer the investment minus the value of the immediate investment. On the other hand, the loss in the second region in Case (O.3) represents the value of the immediate investment minus the value of the option to defer the investment

Corollary 5.4.1 Suppose that Condition (a) holds. Also assume that the random variable X has a support $(0, Q_U]$ for some constant Q_U, and that the large firm's real efficiency Q_l satisfies $Q_l \in (0, Q_U]$. If conditions $p(\beta_1, Q_s, Q_U) > 0$ and $y \leq y_U(Q_U)$ are satisfied, then the start-up suffers no loss due to incomplete information. Here, $y_U(Q_U)$ is defined as the unique solution of equation (5.3.7) with $Q = Q_U$.

The first condition means that it is certain that the efficiency of the start-up's investment, Q_s, is sufficiently greater than that of the large firm, Q_l. The second condition means that the initial market demand $Y(0) = y$ cannot generate great profit immediately. Thus, Proposition 5.4.3 suggests that more detailed information about the large firm is of little value when the start-up's efficiency is much better than that of the large firm in the new market that is small for the present.

The expected payoff $\tilde{V}(y)$ obtained by the real options strategy τ_s^* may generate less profit than the expected payoff $\tilde{V}_{NPV}(y)$ obtained by the zero-NPV strategy (which means to invest when the NPV of the investment becomes positive) under the same estimation X. To see this, consider the function $g(y)$ defined by (5.4.1) and assume that the equation $g(y) = 0$ $(y > 0)$ has exactly two solutions denoted $0 < y_L^{NPV} < y_U^{NPV}$ as shown in Figure 5.1. This assumption holds in most cases. Then, the start-up that employs the zero-NPV strategy invests at $\tau_s^{NPV} = \inf\{t \geq 0 \mid Y(t) \in [y_L^{NPV}, y_U^{NPV}]\}$, although the start-up that takes the real options strategy invests at $\tau_s^* = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_U(\tilde{Q}_l)]\}$. Since $y_L^{NPV} < y_M(Q_s) < y_U(\tilde{Q}_l) < y_U^{NPV}$ as observed in Figure 5.1, the zero-NPV timing $\tau_s^{NPV}$is not later than the real options timing τ_s^*. We define Q_{NPV}

as the unique solution of $y_U(Q) = y_U^{NPV}$. Taking into consideration that the zero-NPV timing is expressed as $\inf\{t \geq 0 \mid Y(t) \in [y_L^{NPV}, y_U(Q_{NPV})]\}$, we can show the following corollary.

Corollary 5.4.2 Suppose that Condition (a) holds. Also assume that the equation $g(y) = 0$ ($y > 0$) has exactly two solutions. Then, $\tilde{V}_{NPV}(y) > \tilde{V}(y)$ holds if one of the following three conditions is satisfied in Case (O.3) (i.e., $\tilde{Q}_l > Q_l$ and $p(\beta_1, Q_s, \tilde{Q}_l) > 0$):

- $Q_{NPV} < Q_l$ and $y > y_U(\tilde{Q}_l)$
- $Q_l \leq Q_{NPV}, \tilde{B}(\tilde{Q}_l) < \tilde{B}(Q_{NPV})$ and $y > y_U(\tilde{Q}_l)$
- $Q_l \leq Q_{NPV}, \tilde{B}(Q_{NPV}) \leq \tilde{B}(\tilde{Q}_l)$ and $y_U(\tilde{Q}_l) < y < y_C$

Here, $y_U(\tilde{Q}_l)$ is the unique solution of equation (5.3.7) with $Q = \tilde{Q}_l$ and $\tilde{B}(\cdot)$ is defined by (5.4.12). Moreover, y_C is the unique solution of the equation

$$\frac{D_s}{r-\mu}\max(y_C, y_M(Q_l))^{-\beta_1+1} y_C^{\beta_1} + \tilde{B}(\tilde{Q}_l) y_C^{\beta_2} - \frac{D_s}{r-\mu} y_C + I_s = 0 \quad (y_U(Q_l) < y_C \leq y_U(Q_{NPV})),$$

which is obtained as the intersection of the graphs of two functions $\tilde{V}_{NPV}(y) = D_s y/(r-\mu) - I_s - D_s \max(y, y_M(Q_l))^{-\beta_1+1} y^{\beta_1}/(r-\mu)$ and $\tilde{V}(y) = \tilde{B}(\tilde{Q}_l) y^{\beta_2}$.

5.5 General setting

This section makes a brief explanation about results in the general situation where the large follower does not completely annihilate the small firm, i.e., we assume that $0 < D_s(1,1) < D_s(1,0)$. In practice, a small entrepreneurial firm (or its unique technology) may tend to be bought-out by a large follower. The analysis in this section could be useful in such a situation by reinterpreting the expected profit $E[\int_{\tau_l}^{+\infty} \mathrm{e}^{-rt} D_s(1,1) Y(t) \mathrm{d}t]$ after the large firm's entry time τ_l as the reward which the start-up gains by the buy-out.

The difference from the results of the previous sections only consists in the fact that the start-up's investment policy involves one more investment trigger. First, let us consider the case of complete information. The start-up's investment strategy can be written as $\tau_s^Q = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_s), y_{U_1}(Q)] \cup [y_{U_2}(Q), +\infty)\}$ under a similar condition (but much more complicated since it involves $D_s(1,1)$ in addition to β_1, Q_s, and Q) to the prerequisite condition in Proposition 5.3.1. The additional stopping (investment) region $[y_{U_2}(Q), +\infty)$ represents the start-up's investment allowing the large firm's immediate follow. Indeed, for any sufficiently large market demand $Y(t)$, the start-up obtains the positive profit $D(1,1)Y(t)/(r-\mu) - I_s$ (note $D_s(1,1) > 0$) in spite of the large firm's immediate follow. Note that this region is not important in terms of clarifying the feature of the small firm's strategy as the leader. The start-up does not have to wait forever for large $Y(t)$, though for halfway $Y(t) \in (y_{U_1}(Q), y_{U_2}(Q))$, it delays the investment until $Y(t)$ either falls to $y_{U_1}(Q)$ or rises to $y_{U_2}(Q)$. The condition (called, hereafter, the preemptive condition), which is obtained by modifying the prerequisite condition, determines whether the start-up takes a preemptive action. Recall that the prerequisite condition determines whether the start-up completely gives up. If the preemptive condition is not satisfied, the start-up's strategy can be expressed as $\tau_s^Q = \inf\{t \geq 0 \mid Y(t) \geq y_M(D_s(1,1)/I_s))\}$. In this case, the start-up gives up any entrepreneurial action, instead of completely giving up the investment.

Next, we consider the case of incomplete information. Similar results to those in Sections 5.4.1 and 5.4.2 are obtained by modifying the definition of $g(y)$. With the definition of $\tilde{Q}_l$ unchanged, we can still classify the start-up's strategy into Cases (U), (C), and (O), according to the relation between Q_l and $\tilde{Q}_l$. Therefore, the essence of the results about the start-up's loss due to incomplete information is preserved. Indeed, the results differ from those in Section 5.4.3 only in that the loss $W(y)$ always becomes zero for a sufficiently large initial value y because the start-up's optimal investment strategy allows the large firm's immediate follow. There is little difference between the results in the previous sections and those of the general situation when the initial value y is small, and therefore the same statement as in Corollary 5.4.1 holds.

5.6 Numerical examples

This section shows economic implications, using some numerical examples, of the theoretical results given in Sections 5.3 and 5.4. Unless otherwise noted, in what follows we set the start-up's parameters as $D_s = 0.04, I_s = 1, \mu = 0, \sigma = 0.2$, and $r = 0.04$ as in the standard parameter values in [22]. Then, we have $Q_s = 0.04, \beta_1 = 2$ and $\beta_2 = -1$. Moreover, the investment trigger for the monopolist is calculated as $y_M(Q_s = 0.04) = 2$, which is twice as big as the Marshallian trigger (i.e., the point where the NPV is zero).

To begin with, we consider the start-up's investment strategy with complete information for a range of the large firm's efficiency $Q_l = Q$. We observe from Figure 5.2 that, if the large firm's efficiency $Q_l = Q$ is less than half of the start-up's efficiency $Q_s = 0.04$, the prerequisite condition (5.3.4) for the start-up's investment is satisfied. The vertical dotted line at $Q = 0.02$ in Figure 5.2 divides the whole region into two subregions that correspond to the case where the prerequisite condition (5.3.4) holds and the case of never investing. The former subregion is further divided into three regions as shown in Figure 5.2. We can observe that the start-up's investment region becomes larger as the large firm's efficiency $Q_l = Q$ decreases.

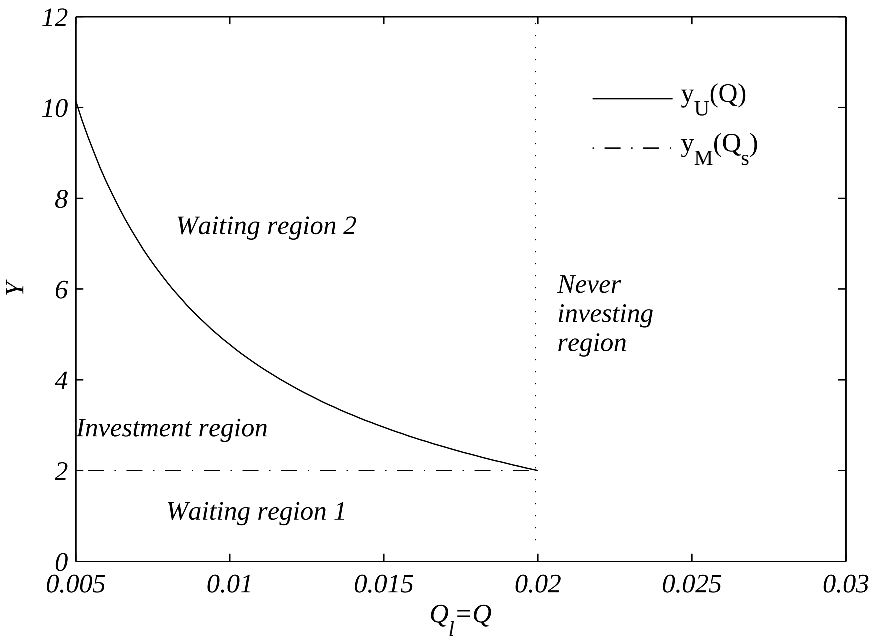

Figure 5.2: Investment triggers for various $Q_l = Q$.

Let us turn our attention to the strategy and the loss $W(y)$ of the start-up with estimations X of the large firm's efficiency. We examine two different cases: (1) the prerequisite condition (5.3.4) is really satisfied, in which case we set the large firm's real efficiency $Q_l = 0.01$, and (2) the prerequisite condition (5.3.4) does not hold, in which case we set $Q_l = 0.03$. Tables 5.1 and 5.2 show the resulting values in cases (1) and (2), respectively. The second column in the tables represents the resulting cases defined in Proposition 5.4.3. The notation $\emptyset$ in the third column means that the start-up never invests regardless of the value of the market demand $Y(t)$. Note that $\tilde{Q}_l = E[X]$ because of $\beta_1 = 2$. In addition, Condition (a) given in Section 5.4.1 is always satisfied in the examples. The tables show the losses $W(y)$ for different initial values $y = 1, 4$ and 10. Note that, for $Q_l = 0.01$, $y = 1, 4$ and 10 lie in the waiting region 1, the investment region, and the waiting region 2 in Figure 5.2, respectively.

In Table 5.1, the losses $W(1)$ and $W(4)$ are zero for several distributions of X in spite of the misestimation. This corresponds to the fact that for a small initial market demand no loss occurs, which is shown in Proposition 5.4.3 and Corollary 5.4.1. For a large initial market demand, comparing $W(10) = 1.55, 0.41$ and $0.23, 0.097$ in Table 5.1 suggests that the loss in the overestimation case (O.3) is smaller than that in the underestimation case (U.3). Taking into account that the value in the case of complete information is $V(10, 0.01) = 0.55$, even a small estimation error (in particular, in the underestimation case) causes a serious problem to the start-up. Indeed, the losses $W(10) = 1.55$ and 0.41 correspond to 280 and 75 percents of $V(10, 0.01) = 0.55$, respectively. In the case where the prerequisite condition does not hold, on the other hand, a small estimation error causes no problem to the start-up. This can be seen from $W(y) = 0$ in many rows above and below $X = Q_l = 0.03$ in Table 5.2. In fact, the start-up can make a correct judgement of never investing even if it has a small estimation error.

From the above observation, we obtain the following implications about the start-up's investment policy under incomplete information. The start-up needs more accurate estimation of the large firm's efficiency in the case where it tries to invest than in the case of never investing. In addition, the start-up that attempts to invest should make a modest estimate because the loss in the underestimation case is likely to be much larger than that in the overestimation case. The start-up's strategy in the underestimation case may cause a loss larger than the project value in the case of complete information, and therefore its confident investment policy is excessively risky.

Finally, we show interesting numerical comparative static results with respect to the volatility σ in the underlying market demand $Y(t)$. We set the large firm's efficiency $Q_l = 0.01$. Figure 5.3 illustrates the start-up's value function $V(y; Q_l = 0.01)$ in the case of complete information for $\sigma = 0, 0.1, 0.2$ and 0.3. For an initial value around the investment region (approximately $[1.5, 4.5]$ in Figure 5.3) a lower volatility generates a higher value of the start-up's investment, while for an initial value in the waiting regions (especially, the waiting region 2 such as $y \approx 10$) a higher volatility is beneficial to the start-up.

This result is intuitive. Note that the possibilities of both firm's entries in the waiting regions become smaller as the volatility in the market becomes lower. In other words, the value of the option to delay the investment is monotonic with respect to the volatility in the market. Then, a lower volatility has both positive and negative effects on the start-up. The positive one is that a lower volatility leads the large firm to delay its entry. The negative one is that a lower volatility decreases the start-up's option value of waiting. Since around the investment region the start-up can invest soon even if the volatility in the market is small, the negative effect in the start-up's waiting region is not important. As a result, around the investment region, a lower volatility increases the start-up's value by the positive effect. Far from the investment region, on the other hand, the negative effect is dominant, because the start-up tends to wait for a long time until its investment. In consequence, far from the investment region, a lower volatility decreases the start-up's value by the negative effect.

Figure 5.4 illustrates the relative loss $W(y)/V(y;Q_l)$ of the start-up with incomplete information $X = [0.005, 0.01]$ and $[0.01, 0.015]$ for various values of σ. As shown in Proposition 5.4.3 and Corollary 5.4.1, the relative loss is zero for a small initial value y in Figure 5.4. Observe that, for a large initial value y, the relative loss is constant with respect to y. This is because $W(y)/V(y;Q_l)$ equals $(B(Q_l) - \tilde{B}(\tilde{Q}_l))y^{\beta_2}/B(Q_l)y^{\beta_2} = (B(Q_l) - \tilde{B}(\tilde{Q}_l))/B(Q_l)$ by Proposition 5.4.3. In Figure 5.4, a lower volatility increases both the relative loss $W(y)/V(y;Q_l)$ and the absolute loss $W(y)$. We observed that the same property holds for most other parameter values than the presented example.

We can interpret this property as follows. The start-up's investment decision involves two different types of uncertainty; namely the market volatility and the estimation of the large firm's efficiency. Intuitively, in the market with high volatility, a small estimation error does not make a big difference in the loss. However, if the uncertainty in the market demand is less, the start-up's payoff is more decisively determined by its investment policy. Naturally, the start-up also needs to take a more accurate investment policy.

5.7 Conclusion

This chapter has investigated the effect of incomplete information in the model in which a start-up with a unique idea and technology pioneers a new market that will be taken over by a large firm eventually. The main contribution of this chapter is to evaluate the start-up's loss due to incomplete information about the large firm. The proposed method could be applied in other real options models involving several firms. The results obtained in this chapter can be summarized as follows.

If the start-up's efficiency is much better than that of the large firm and the current market demand cannot generate great profit immediately, then the start-up requires no further survey on the large firm's efficiency. On the other hand, information about the large firm's efficiency is valuable in the market that can readily generate great profit, even if the start-up's efficiency is much better than that of the large firm. In this case, it is quite likely that the start-up's immediate investment does not produce much income for the start-up before the large firm's entry.

Table 5.1: The loss for uniform distributions X and $Q_l = 0.01$.

X	Case	Investment region	$W(1)$	$W(4)$	$W(10)$
$[0.0025, 0.0075]$	(U.3)	$[2, 10.14]$	0	0	1.55
$[0.005, 0.01]$	(U.3)	$[2, 6.57]$	0	0	0.41
$Q_l = 0.01$	(C)	$[2, 4.77]$	0	0	0
$[0.01, 0.015]$	(O.3)	$[2, 3.69]$	0	0.089	0.097
$[0.0125, 0.0175]$	(O.3)	$[2, 2.95]$	0	0.36	0.23
$[0.015, 0.02]$	(O.3)	$[2, 2.42]$	0	0.58	0.34
$[0.0175, 0.0225]$	(O.2)	$\emptyset$	0.13	1	0.55
$[0.02, 0.025]$	(O.2)	$\emptyset$	0.13	1	0.55

Table 5.2: The loss for uniform distributions X and $Q_l = 0.03$.

X	Case	Investment region	$W(1)$	$W(4)$	$W(10)$
$[0.0125, 0.0175]$	(U.2)	$[2, 2.95]$	0.13	0.74	0.37
$[0.015, 0.02]$	(U.2)	$[2, 2.42]$	0.13	0.47	0.23
$[0.0175, 0.0225]$	(U.1)	$\emptyset$	0	0	0
$[0.02, 0.025]$	(U.1)	$\emptyset$	0	0	0
$[0.0225, 0.0275]$	(U.1)	$\emptyset$	0	0	0
$[0.025, 0.03]$	(U.1)	$\emptyset$	0	0	0
$Q_l = 0.03$	(C)	$\emptyset$	0	0	0
$[0.03, 0.035]$	(O.1)	$\emptyset$	0	0	0
$[0.0325, 0.0375]$	(O.1)	$\emptyset$	0	0	0

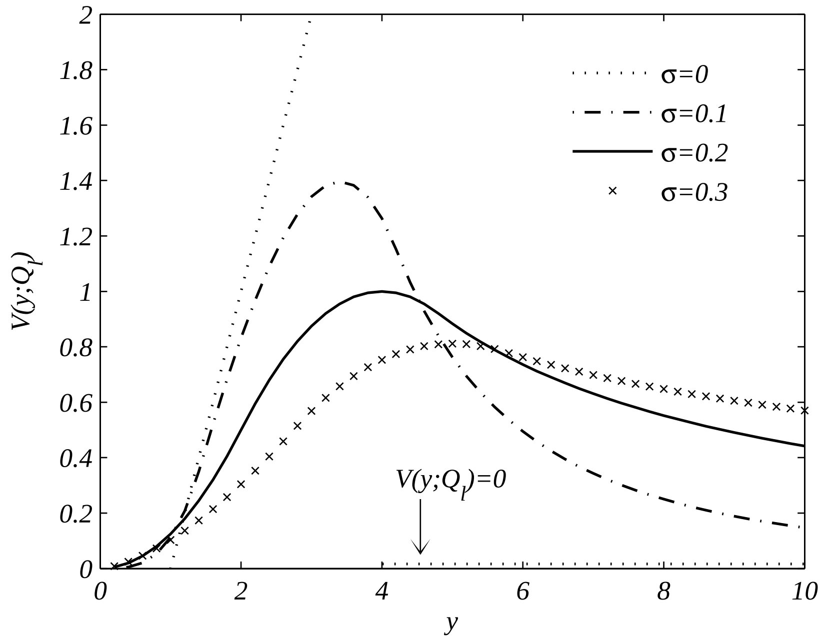

Figure 5.3: $V(y;Q_l = 0.01)$ for $\sigma = 0, 0.1, 0.2$ and 0.3.

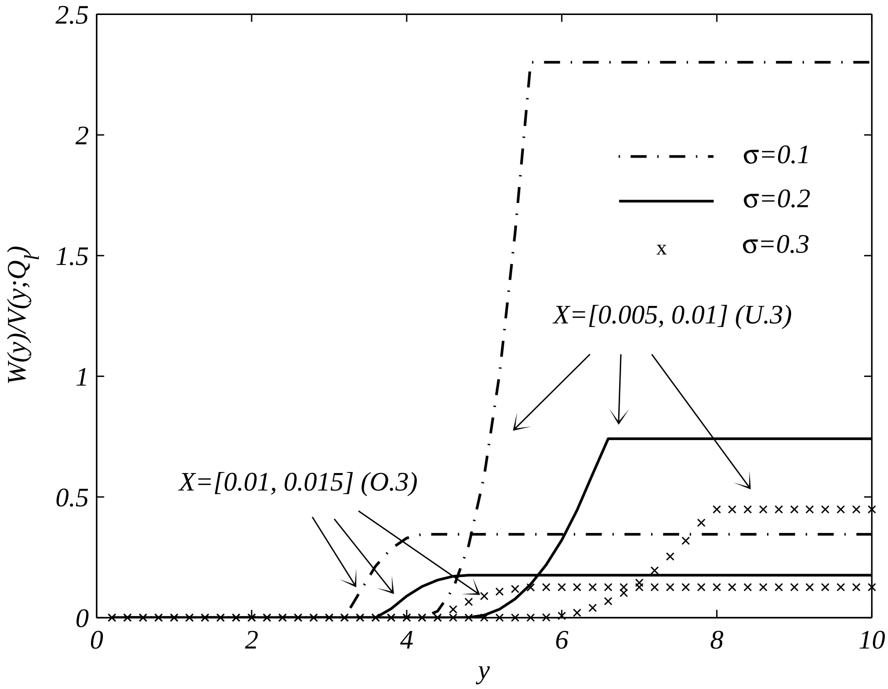

Figure 5.4: $W(y)/V(y;Q_l = 0.01)$ for $\sigma = 0.1, 0.2$ and 0.3.

When it is doubtful that the start-up's efficiency overwhelms that of the large firm, information about the large firm's efficiency is always valuable regardless of the current market demand. The reason for this is that there is a possibility that the investment opportunity for the start-up does not exist in the market, in addition to the same risk as in the previous case, that is, the possibility that the start-up obtains little profit before the large firm's entry.

Furthermore, under incomplete information, the expected payoff of the start-up investing at the zero-NPV trigger could become greater than that of the start-up following the real options approach. In numerical examples, we have also observed some interesting features of the loss due to incomplete information such as the property that the loss in the overestimation case tends to be smaller than that in the underestimation case.

In the real world, a small entrepreneurial firm that has a unique idea and technology but is not competitive in the market may want to sell its idea and technology to a large firm, rather than pioneering the market by itself. Then, the value function which the start-up believes can be interpreted as a reward which the start-up demands for its idea and technology. As revealed in this chapter, the value of the investment which the start-up believes under incomplete information is generally different from the real value of the investment. Because of this gap, negotiations between the start-up and the large firm may not go smoothly. It remains as an interesting issue of future research to reveal the effect of incomplete information in such a negotiation problem of a firm having an option to sell its idea and technology to the rival firm.

Chapter 6

Real Options under Asymmetric Information

6.1 Introduction

This chapter focuses on agency conflicts between the owner and the manager in a decentralized firm. While most literatures (cf., Sections 4.1 and 5.1) have focused on the strategic interaction with rival firms, Grenadier and Wang [31] investigated investment timing in a decentralized firm where the owner (principal) delegates the investment decision to the manager (agent) who holds private information by combining the real options approach and contract theory (for contract theory, see the standard textbook [11]). In their model, asymmetric information changes the investment behavior of the firm from the first-best no-agency case because the owner designs the contract to provide bonus-incentive for the manager to truthfully reveal private information. Similar real options models with agency conflicts have been also studied in [8, 56].

Although these models consider only the *carrot* (i.e., giving bonus-incentive to the manager) as a measure to deal with agency conflicts, the owner can usually use not only the *carrot*, but also the *stick* (i.e., auditing and fining the manager). Naturally, the impact of auditing has been clarified in other contexts (e.g., [1, 85]).

In this chapter, we incorporate an auditing mechanism into a model following [31]. As far as the purpose of the chapter is concerned, we limit their original setup involving both hidden information and action to only the case of hidden information. We assume that the owner can utilize an auditing system that fines the manager when a false report is detected, where the higher the cost the owner pays in auditing, the greater becomes the probability of detection. We show that the optimal contract is determined among three feasible types of contracts: the bonus-incentive only mechanism, the joint bonus-incentive and auditing mechanism, and the auditing only mechanism. This is according to the relation between the auditing cost and the amount of the penalty. Although a similar auditing technology is introduced in [79], the findings in this chapter are more comprehensive and help connect previous results in this area. Indeed, our solution includes the results of [31] in the bonus-incentive only region, the results of [79] in the joint bonus-incentive and auditing region, and the first-best no-agency solution as the limit of the auditing only region.

Furthermore, our results also give good account of the real life relationship between audit and bonus-incentive as follows. In the case where the manager may commit seriously outright frauds such as embezzlement, the owner tries to prevent the manager's offense by using only audit, and stern punishment by the law improves the social welfare. On the other hand, in the case where the manager's private benefit is not necessarily illegal, the owner is likely to prefer a bonus system such as stock options, and an owner's too great demand may decrease the social benefit.

The chapter is organized as follows. Section 6.2 provides a brief review of [31]. Section 6.3 incorporates the auditing technology into the model and derives the owner's optimal contract after allowing for audit. The final section discusses economic implications of our results involving both audit and bonus-incentive. Section 6.4 concludes this chapter.

6.2 Preliminaries

This section provides a brief review of the results in [31]. First, let us explain the setup. We consider a decentralized firm that faces the investment timing decision of a single project. We assume that the owner (principal) delegates the decision to the manager (agent) and that both the owner and the manager are risk neutral. Then, in this chapter, the discout factor $r(>\mu)$ is equal to the risk-free rate. While both the owner and the manager know the investment cost I, the value of the project consists of two components, namely the value $Y(t)$ that is observable to both the owner and the manager, and the value θ that is privately observed only by the manager. Thus, the total value of the project is $Y(t)+\theta$. For simplicity, we assume that the observable value $Y(t)$ at time t obeys the geometric Brownian motion (4.2.1) defined in Section 4.2.

The private component θ potentially takes on two possible values, θ_1 or θ_2, with $0 \le \theta_2 < \theta_1 < I$. We denote $\Delta\theta = \theta_1 - \theta_2$. Before contracting, both the owner and the manager know that the probability of drawing a higher quality project θ_1 equals κ. Immediately after making a contract with the owner at time 0, the manager privately observes whether the project is of a higher quality

θ_1 or a lower quality θ_2. Although the manager's one-time effort, which cannot be observed by the owner, changes the likelihood κ in [31], we exclude the effect of the hidden action from the original setup.

As a benchmark, we examine the case where there is no delegation of the exercise decision and the owner observes the true value of θ. In this case, the owner's problem with given $\theta = \theta_i$ $(i = 1, 2)$ becomes the following optimal stopping problem:

$$U(y;\theta_i) = \sup_{\tau_i \in \mathcal{T}} E[\mathrm{e}^{-r\tau_i}(Y(\tau_i) + \theta_i - I)]. \tag{6.2.1}$$

Recall that y is the initial value $Y(0)$ and $\mathcal{T}$ is a set of all $\mathcal{F}_t$ stopping times (see Section 4.2). In this chapter, it is always assumed that the initial value y is sufficiently low so that the firm has to wait for its exercise condition to be met. Using the standard method (see [22]), we obtain the value function $U(y;\theta_i)$ and the optimal stopping time τ_i^* of problem (6.2.1) for $\theta = \theta_i$ as follows:

$$\begin{aligned} U(y;\theta_i) &= \left(\frac{y}{y_i^*}\right)^{\beta_1} (y_i^* + \theta_i - I) \\ \tau_i^* &= \inf\{t \geq 0 \mid Y(t) \geq y_i^*\} \\ y_i^* &= \frac{\beta_1}{\beta_1 - 1}(I - \theta_i). \end{aligned} \tag{6.2.2}$$

Recall that $\beta_1 = \beta_{10}$ is the characteristic root defined by (4.2.12).

The threshold y_i^* is the optimal investment trigger for the owner who observes the value θ_i at time 0. Thus, the ex ante value of the owner's option in the first-best no-agency setting (denoted $\pi_{\mathrm{o}}^*(y)$) becomes:

$$\begin{aligned} \pi_{\mathrm{o}}^*(y) &= \kappa U(y;\theta_1) + (1-\kappa) U(y;\theta_2) \\ &= \kappa \left(\frac{y}{y_1^*}\right)^{\beta_1} (y_1^* + \theta_1 - I) + (1-\kappa) \left(\frac{y}{y_2^*}\right)^{\beta_1} (y_2^* + \theta_2 - I). \end{aligned} \tag{6.2.3}$$

Now, let us turn to the principal-agent setting without auditing. In this case, the owner has the option to invest, but delegates the exercise decision to the manager. At time 0, the owner offers the manager a contract that commits the owner to pay the manager at the time of exercise. We assume no opportunity for renegotiation exists. Although the commitment may lead to ex post inefficiency in investment timing, it increases the ex ante value of the project. In fact, if the owner makes no contract with the manager, the owner's ex ante option value becomes (6.2.3) with $q = 0$. This is because the manager hands the owner θ_2 and makes $\theta_1 - \theta_2$ his/her own when the true value is θ_1. As discussed in [31], the optimal contract is included in a mechanism $\mathcal{M}^{\mathrm{GW}} = \{(y_i, w_i) \mid i = 1, 2\}$ in which the owner pays the manager the bonus w_i at the investment time when the manager exercises the investment at time $\tau_i = \inf\{t \geq 0 \mid Y(t) \geq y_i\}$. Here the superscript "GW" refers to the solution of [31]. Since the revelation principle (see [11]) ensures that the manager who observes θ_i faithfully invests at the trigger y_i, the optimal contract is the solution of the problem of maximizing the owner's ex ante option value of the investment:

$$\begin{aligned}
&\text{maximize}_{y_i,w_i} && \kappa\left(\frac{y}{y_1}\right)^{\beta_1}(y_1+\theta_1-I-w_1)+(1-\kappa)\left(\frac{y}{y_2}\right)^{\beta_1}(y_2+\theta_2-I-w_2)\\
&\text{subject to} && \kappa\left(\frac{y}{y_1}\right)^{\beta_1}w_1+(1-\kappa)\left(\frac{y}{y_2}\right)^{\beta_1}w_2\geq 0\\
& && w_i\geq 0\quad (i=1,2)\\
& && \left(\frac{y}{y_1}\right)^{\beta_1}w_1-\left(\frac{y}{y_2}\right)^{\beta_1}(w_2+\Delta\theta)\geq 0\\
& && \left(\frac{y}{y_2}\right)^{\beta_1}w_2-\left(\frac{y}{y_1}\right)^{\beta_1}(w_1-\Delta\theta)\geq 0,
\end{aligned} \tag{6.2.4}$$

where $y_i > y$. In the constraints of problem (6.2.4), the first and second inequalities correspond to the ex ante participation constraint and the ex post limited-liability constraints, respectively, while the last two inequalities are the ex post incentive-compatibility constraints. The incentive-compatibility constraint means that with a truthful report, the manager who observes $\theta = \theta_1$ (resp. $\theta = \theta_2$) obtains the expected payoff $(y/y_1)^{\beta_1}w_1$ (resp. $(y/y_2)^{\beta_1}w_2$), which is larger than the expected payoff for a false report, $(y/y_2)^{\beta_1}(w_2+\Delta\theta)$ (resp. $(y/y_1)^{\beta_1}(w_1-\Delta\theta)$).

In problem (6.2.4), it can be shown that the bonus $w_2 = 0$ and only the third inequality (i.e., the incentive-compatibility condition for the manager who observes the better project value θ_1) binds. Then, the optimal solution $\{(y_i^{\mathrm{GW}}, w_i^{\mathrm{GW}}) \mid i = 1,2\}$ becomes:

$$(y_1^{\mathrm{GW}}, w_1^{\mathrm{GW}}) = \left(y_1^*, \left(\frac{y_1^*}{y_2^{\mathrm{GW}}}\right)^{\beta_1}\Delta\theta\right) \tag{6.2.5}$$

$$(y_2^{\mathrm{GW}}, w_2^{\mathrm{GW}}) = \left(\frac{\beta_1}{\beta_1-1}\left(I-\theta_2+\frac{\kappa\Delta\theta}{1-\kappa}\right), 0\right). \tag{6.2.6}$$

For further details, see the solution for the hidden information only region in [31]. It is worth noting that the trigger for the higher quality project, y_1^{GW}, remains unchanged from the first-best trigger y_1^* defined by (6.2.2), while the trigger for the lower quality project, y_2^{GW}, is larger than the first-best trigger y_2^* defined by (6.2.2). This is because the owner attempts to decrease the information rent to the manager who observes the higher quality project by deferring investment timing for the lower quality project. The owner's and manager's ex ante option values, $\pi_{\mathrm{o}}^{\mathrm{GW}}$ and $\pi_{\mathrm{m}}^{\mathrm{GW}}$, respectively, are obtained by substituting the optimal solution $\{(y_i^{\mathrm{GW}}, w_i^{\mathrm{GW}}) \mid i = 1,2\}$ into the objective function and the right-hand side of the first inequality of the constraints of problem (6.2.4).

This is the result in the hidden information case of [31]. In the next section, we extend their analysis to a case allowing the owner to audit the manager at a cost.

6.3 Theoretical results

This section derives the optimal contract involving bonus-incentive and audit. The owner detects the real value of θ at probability d_i by paying the auditing cost $c(d_i)$ for the manager's report $\theta = \theta_i$ when the manager executes the project. We assume that the manager is fined the penalty $\Gamma(> 0)$ for cheating when a false report is detected. In general, as discussed in [2], the society could suffer different damages, according to the types of the punishment. However, we do not have to care whether the owner can receive the total amount of the fine, Γ from the manager, or whether a part of Γ represents the manager's disutility from other punishment such as dismissal from the managerial post. This is because the manager does not in any case cheat the owner in the optimal contract in our setting.

Here, the cost function $c(d_i)$ and the penalty Γ are given exogenously. We assume that the cost function $c(d_i)$ satisfies conditions:

$$c(0) = 0, \tag{6.3.1}$$

$$\lim_{d_i \uparrow 1} c(d_i) = +\infty, \tag{6.3.2}$$

$$c'(d_i) > 0 \ (d_i \in [0, 1)), \tag{6.3.3}$$

$$c''(d_i) > 0 \ (d_i \in [0, 1)). \tag{6.3.4}$$

The conditions (6.3.1) and (6.3.3) are explicit from the property of auditing. The assumption (6.3.2) is realistically intuitive because no auditing can always detect the manager's false report. The condition (6.3.4), which is a little bit technical, ensures the convexity of the cost function.

In this setting, the contract is designed as a mechanism $\mathcal{M}^{\mathrm{A}} = \{(y_i, w_i, d_i) \mid i = 1, 2\}$, where the auditing level d_i for the manager's report $\theta = \theta_i$ is added to the mechanism $\mathcal{M}^{\mathrm{GW}}$. Here the superscript "A" refers to the solution of the setting allowing audit. The optimal contract to the owner becomes the solution of the following problem:

$$\begin{aligned}
\text{maximize}_{y_i,w_i,d_i} \quad & \kappa\left(\frac{y}{y_1}\right)^{\beta_1}(y_1+\theta_1-I-w_1-c(d_1))+(1-\kappa)\left(\frac{y}{y_2}\right)^{\beta_1}(y_2+\theta_2-I-w_2-c(d_2))\\
\text{subject to} \quad & \kappa\left(\frac{y}{y_1}\right)^{\beta_1}w_1+(1-\kappa)\left(\frac{y}{y_2}\right)^{\beta_1}w_2 \geq 0\\
& w_i \geq 0 \quad (i=1,2)\\
& d_i \geq 0 \quad (i=1,2)\\
& \left(\frac{y}{y_1}\right)^{\beta_1}w_1-\left(\frac{y}{y_2}\right)^{\beta_1}(w_2+\Delta\theta-d_2\Gamma) \geq 0\\
& \left(\frac{y}{y_2}\right)^{\beta_1}w_2-\left(\frac{y}{y_1}\right)^{\beta_1}(w_1-\Delta\theta-d_1\Gamma) \geq 0,
\end{aligned} \tag{6.3.5}$$

where $y_i > y$. The first two constraints of problem (6.3.5) are the same as those of problem (6.2.4), while the incentive-compatibility constraints of problem (6.3.5) include an additional term, the expected penalty $d_i\Gamma$. It can be easily checked that the revelation principle holds in this case, as in the previous setting, even if the penalty Γ is counted in the owner's profit. Therefore, the owner does not make any contract to allow the manager to report untruthfully. Let us check this revelation principle. Assume that the owner makes the contract $\{(\tilde{y}_i, \tilde{w}_i, \tilde{d}_i) \mid (i=1,2)\}$ that leads the manager who observes the higher quality θ_1 to falsely report θ_2, and the manager who observes the lower quality θ_2 to truthfully report θ_2. Then, the owner obtains the expected payoff $(y/\tilde{y}_2)^{\beta_1}(\tilde{y}_2+\theta_2-I-\tilde{w}_2+\kappa\tilde{d}_2\Gamma-c(\tilde{d}_2))$, but the same expected payoff can be realized by a feasible solution $(y_1, w_1, d_1) = (\tilde{y}_2, \tilde{w}_2+\Delta\theta-\tilde{d}_2\Gamma, \tilde{d}_2), (y_2, w_2, d_2) = (\tilde{y}_2, \tilde{w}_2, \tilde{d}_2)$ of problem (6.3.5). Similarly, we can show that the owner's expected payoffs in other types of contracts allowing the manager's dishonest behavior are dominated by the maximum value of problem (6.3.5). Thus, the revelation principle always holds, and we only have to derive the optimal solution of problem (6.3.5).

Proposition 6.3.1 The optimal contract $\{(y_i^{\mathrm{A}}, w_i^{\mathrm{A}}, d_i^{\mathrm{A}}) \mid i=1,2\}$ in the setting with auditing is given as follows:

Case (I): $0 < \Gamma \leq (1-\kappa)c'(0)/\kappa$ (*bonus-incentive only region*)

$$\begin{aligned}
(y_1^{\mathrm{A}}, w_1^{\mathrm{A}}, d_1^{\mathrm{A}}) &= (y_1^*, w_1^{\mathrm{GW}}, 0)\\
(y_2^{\mathrm{A}}, w_2^{\mathrm{A}}, d_2^{\mathrm{A}}) &= (y_2^{\mathrm{GW}}, 0, 0),
\end{aligned}$$

Case (II): $(1-\kappa)c'(0)/\kappa < \Gamma \leq \max\{\Delta\theta, (1-\kappa)c'(\Delta\theta/\Gamma)/\kappa\}$ (*joint bonus-incentive and auditing* *region*)

$$(y_1^{\mathrm{A}}, w_1^{\mathrm{A}}, d_1^{\mathrm{A}}) = \left(y_1^*, \left(\frac{y_1^*}{y_2^{\mathrm{A}}}\right)^{\beta_1} (\Delta\theta - d_2^{\mathrm{A}}\Gamma), 0\right)$$
$$(y_2^{\mathrm{A}}, w_2^{\mathrm{A}}, d_2^{\mathrm{A}}) = \left(\frac{\beta_1}{\beta_1 - 1}\left(I - \theta_2 + \frac{\kappa}{1-\kappa}(\Delta\theta - d_2^{\mathrm{A}}\Gamma) + c(d_2^{\mathrm{A}})\right), 0, (c')^{-1}\left(\frac{\kappa\Gamma}{1-\kappa}\right)\right),$$

Case (III): $\Gamma > \max\{\Delta\theta, (1-\kappa)c'(\Delta\theta/\Gamma)/\kappa\}$ (*auditing only region*)

$$(y_1^{\mathrm{A}}, w_1^{\mathrm{A}}, d_1^{\mathrm{A}}) = (y_1^*, 0, 0)$$
$$(y_2^{\mathrm{A}}, w_2^{\mathrm{A}}, d_2^{\mathrm{A}}) = \left(\frac{\beta_1}{\beta_1 - 1}(I - \theta_2 + c(d_2^{\mathrm{A}})), 0, \Delta\theta/\Gamma\right).$$

Proof Note that in problem (6.3.5), the first constraint is induced by the second constraints $w_i \geq 0$ $(i = 1, 2)$, and y^{β_1} can be ignored. We solve the problem (6.3.5) without the final constraint (the incentive-compatibility constraint for the manager who observes the bad value $\theta = \theta_2$), and then also check that the obtained solution satisfies the removed constraint. Let $\{(y_i^{\mathrm{A}}, w_i^{\mathrm{A}}, d_i^{\mathrm{A}}) \mid i = 1, 2\}$ be the optimal solution of problem (6.3.5) without the final constraint. It immediately follows that $w_2^{\mathrm{A}} = 0, d_1^{\mathrm{A}} = 0$, and

$$\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1} w_1^{\mathrm{A}} - \left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1} (\Delta\theta - d_2^{\mathrm{A}}\Gamma) = 0. \tag{6.3.6}$$

Let λ_i $(i = 1, 2, 3)$ denote the Lagrangian multipliers associated with the remaining constraints $w_1 \geq 0, d_2 \geq 0$, and (6.3.6), respectively. That is, we form the Lagrangian:

$$\mathcal{L}(y_1, y_2, w_1, d_2) = \kappa \left(\frac{1}{y_1}\right)^{\beta_1} (y_1 + \theta_1 - I - w_1) + (1-\kappa)\left(\frac{1}{y_2}\right)^{\beta_1} (y_2 + \theta_2 - I - c(d_2))$$
$$+\lambda_1 w_1 + \lambda_2 d_2 + \lambda_3 \left(\left(\frac{1}{y_1}\right)^{\beta_1} w_1 - \left(\frac{1}{y_2}\right)^{\beta_1} (\Delta\theta - d_2\Gamma)\right).$$

The Karush-Kuhn-Tucker conditions are (6.3.6),

$$\frac{\partial\mathcal{L}}{\partial y_1} = \kappa\left((-\beta_1 + 1)\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1} - \beta_1(\theta_1 - I - w_1^{\mathrm{A}})\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1+1}\right) - \lambda_3\beta_1 w_1^{\mathrm{A}}\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1+1} = 0, \tag{6.3.7}$$

$$\frac{\partial\mathcal{L}}{\partial y_2} = (1-\kappa)\left((-\beta_1 + 1)\left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1} - \beta_1(\theta_2 - I - c(d_2^{\mathrm{A}}))\left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1+1}\right) + \lambda_3\beta_1(\Delta\theta - d_2^{\mathrm{A}}\Gamma)\left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1+1}$$
$$= 0, \tag{6.3.8}$$

$$\frac{\partial\mathcal{L}}{\partial w_1} = -\kappa\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1} + \lambda_1 + \lambda_3\left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1} = 0, \tag{6.3.9}$$

$$\frac{\partial\mathcal{L}}{\partial d_2} = -(1-\kappa)c'(d_2^{\mathrm{A}})\left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1} + \lambda_2 + \lambda_3\Gamma\left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1} = 0, \tag{6.3.10}$$

and

$$\lambda_1 w_1^{\mathrm{A}} = \lambda_2 d_2^{\mathrm{A}} = 0,\ \lambda_i \geq 0\ (i = 1, 2, 3). \tag{6.3.11}$$

Let us now derive the solution of (6.3.6)–(6.3.11), depending on whether λ_i equals zero. If $\lambda_1 = \lambda_2 = 0$, we have $\lambda_3 = \kappa$ and the solution in Case (II). If $\lambda_1 > 0$ and $\lambda_2 = 0$, from (6.3.6)–(6.3.10) we have the solution in Case (III) with

$$\lambda_1 = \left(\frac{1}{y_1^{\mathrm{A}}}\right)^{\beta_1} \left(\kappa - \frac{(1-\kappa)c'(d_2^{\mathrm{A}})}{\Gamma}\right)$$

and

$$\lambda_3 = \frac{(1-\kappa)c'(d_2^{\mathrm{A}})}{\Gamma}.$$

If $\lambda_1 = 0$ and $\lambda_2 > 0$, we obtain the solution in Case (I) with

$$\lambda_2 = \left(\frac{1}{y_2^{\mathrm{A}}}\right)^{\beta_1} ((1-\kappa)c'(0) - \kappa\Gamma)$$

and

$$\lambda_3 = \kappa.$$

If $\lambda_i > 0$ $(i = 1, 2)$, from (6.3.6) we have $\Delta\theta = 0$, which contradicts $\Delta\theta > 0$. Taking into account that the conditions $\lambda_i \geq 0$ $(i = 1, 2, 3), d_2^{\mathrm{A}} < 1$ that results from $\lim_{d_2 \uparrow 1} c(d_2) = +\infty$, and the condition under which the solution of $c'(d_2^{\mathrm{A}}) = \kappa\Gamma/(1-\kappa)$ exists, we can show that for a given Γ, the solution satisfying the Karush-Kuhn-Tucker conditions (6.3.6)–(6.3.11) is uniquely determined as the statement of Proposition 6.3.1. Furthermore, the solution explicitly satisfies the final constraint in problem (6.3.5). □

In Proposition 6.3.1, and as intuitively expected, the owner neither gives any bonus for the manager's bad report θ_2 nor audits the manager's good report θ_1. The investment trigger of the high quality project does not change from that of the no-agency setting as in the result by [31]. However, the other components of the contract and the owner's strategy changes, depending on the auditing cost $c(d_i)$ and the amount of the penalty, Γ. Indeed, the contract is classified into three regions. The solution changes from Case (I) to Case (III) via Case (II) as the penalty Γ becomes larger, as observed in Figures 6.1 and 6.2. In the numerical example, we set the parameter values $\alpha = 0, r = 0.04, \sigma = 0.2$, and $I = 1$ as in [22], and set $y = 0.5, \theta_1 = 0.5, \theta_2 = 0, \kappa = 0.5$, and $c(d_i) = 0.5d_i/(1-d_i)$. Figure 6.1 depicts the expected discounted cost of the bonus and auditing, $(y/y_1^{\mathrm{A}})_1^{\beta} w_1^{\mathrm{A}}, (y/y_2^{\mathrm{A}})_1^{\beta} c(d_2^{\mathrm{A}})$, together with the investment trigger y_2^{A}. We do not illustrate quantities $y_1^{\mathrm{A}}, w_2^{\mathrm{A}}$, and d_1^{A} because of $y_1^{\mathrm{A}} = y_1^*, w_2^{\mathrm{A}} = d_1^{\mathrm{A}} = 0$.

Case (I) is likely to hold if the marginal cost of auditing is high relative to the penalty Γ, or if the probability of drawing the better project θ_1 is low. In this case, the owner pays the whole information rent to the manager without auditing, since the auditing technology does not work at all. Conversely, in Case (III), where the penalty is severe and the auditing cost is not so high, the owner uses only the auditing system without giving any bonus to the manager. Case (II) is the intermediate case. In this case, both bonus-incentive and audit are effective by the trade-off between the auditing cost and the amount of the penalty.

Let us explain the relation between Proposition 6.3.1 and the results from previous studies. The solution in Case (I) coincides with that of [31], i.e., (6.2.5) and (6.2.6). Then, it is readily seen that $y_2^* < y_2^{\mathrm{A}} \leq y_2^{\mathrm{GW}}$, $\pi_{\mathrm{o}}^{\mathrm{GW}}(y) \leq \pi_{\mathrm{o}}^{\mathrm{A}}(y) < \pi_{\mathrm{o}}^*(y)$, and $\pi_{\mathrm{m}}^{\mathrm{A}}(y) \leq \pi_{\mathrm{m}}^{\mathrm{GW}}(y)$, where $\pi_{\mathrm{o}}^{\mathrm{A}}$ and $\pi_{\mathrm{m}}^{\mathrm{A}}$ denote the owner's and manager's ex ante option values defined by

$$\begin{aligned}\pi_{\mathrm{o}}^{\mathrm{A}}(y) &= \kappa \left(\frac{y}{y_1^*}\right)^{\beta_1} (y_1^* + \theta_1 - I - w_1^{\mathrm{A}}) + (1-\kappa)\left(\frac{y}{y_2^{\mathrm{A}}}\right)^{\beta_1} (y_2^{\mathrm{A}} + \theta_2 - I - c(d_2^{\mathrm{A}}))\\ \pi_{\mathrm{m}}^{\mathrm{A}}(y) &= \kappa \left(\frac{y}{y_1^*}\right)^{\beta_1} w_1^{\mathrm{A}}.\end{aligned}$$

Note that $\pi_{\mathrm{o}}^{\mathrm{A}}$ and $\pi_{\mathrm{m}}^{\mathrm{A}}$ are exactly the same as $\pi_{\mathrm{o}}^{\mathrm{GW}}$ and $\pi_{\mathrm{m}}^{\mathrm{GW}}$, respectively, in Case (I). If we assume that $\Gamma = \Delta\theta$ and $c'(0) = 0$, the solution is classified into Case (II) and agrees with that of [79] where the owner can control the penalty Γ to a limit $\Delta\theta$. Moreover, the solution in Case (III) converges to that of the first-best no-agency case by letting $\Gamma \to +\infty$. This appears to correspond to Proposition 4 in [1].

We can see from Figure 6.1 that y_2^{A} monotonically decreases to $y_2^* = 2$ for penalties Γ. Figure 6.2 indicates that the owner's (resp. manager's) ex ante option value $\pi_{\mathrm{o}}^{\mathrm{A}}$ (resp. $\pi_{\mathrm{m}}^{\mathrm{A}}$) monotonically increases (resp. decreases) for Γ. These results can also be easily proved, and therefore the proofs are omitted. In particular, the monotone increase in the owner's option value with respect to the amount of the penalty is consistent with the maximal punishment principle (see [1, 79]). That is, the owner imposes the maximum penalty if he/she can determine the penalty within some limit.

Following [31], we define the social loss π_{loss}^{A} by

$$\pi_{\mathrm{loss}}^{A} = \pi_{\mathrm{o}}^* - (\pi_{\mathrm{o}}^{\mathrm{A}} + \pi_{\mathrm{m}}^{\mathrm{A}}).$$

It should be noted that the social loss $\pi_{\mathrm{loss}}^{\mathrm{A}}$ does not need to involve the social cost of the punishment, which varies between different kinds of punishment as mentioned in [2], since the optimal contract precludes cheating. The social loss $\pi_{\mathrm{loss}}^{\mathrm{A}}$, unlike the owner's option value, is not necessarily monotonic for Γ as observed in Figure 6.2, although for sufficiently large Γ, it monotonically decreases to zero (cf. Proposition 4 in [1]). This means that a halfway penalty could only be of benefit to the owner and brings with it inefficiency in terms of social surplus, while a severe penalty improves social efficiency.

6.4 Economic insights

This section gives the motivation of considering both audit and bonus-incentive in the contract described in the previous section as well as the economic insights extracted from the results we obtained in the previous section. To do so, we shall place our discussion in a real-life context.

In practice, it is common that firms employ independent auditing systems in addition to their internal ones. There are three types of independent audits, that is, financial statement audit, operational audit, and compliance audit. We shall now relate our results to the cases of financial statement audit and operational audit.

By the means of financial statement audit, the owner can ensure the accuracy of the firm's financial statements in its business report. In other words, the owner attempts to prevent the manager from making false statements regarding the firm's financial status and figures. When the manager's outright frauds such as window-dressing settlement and embezzlement are detected by the audit, the manager is severely punished by the relevant law (e.g., The Commercial Code, The Corporation Law, The Securities and Exchange Law, The Penal Code, etc. in the case of Japan). A well-known example is the WorldCom scandal in which its managers were punished for its window-dressing settlement which led to higher reported profits than the actual profits. Another example is the Aramark case in which its managers were charged with the embezzlement involving underreporting of the company's vending machine revenues.

For this kind of audit, the penalty Γ is expected to be much larger than $\Delta\theta$, since Γ includes not only the compensation but also further punishment by the law. Hence, we believe that Case (III) in Proposition 3.1 is appropriate for this situation, even if the auditing cost c is somewhat high. Here, the owner simply tries to set the auditing level high enough to prevent the manager from making false financial statements. Another logical finding we obtained under Case (III) is that severe penalty Γ brings small social loss $\pi^{\mathrm{A}}_{\mathrm{loss}}$ (see Figure 6.2).

On the other hand, Cases (I) and (II) are more likely to apply to operational audit than financial statement audit. The purpose of operational audit is usually to assess the manager's efficiency, instead of detecting explicitly illegal frauds of the manager. When the manager's inefficient waste is pointed out by the audit, the owner can force the manager to make improvements by transferring

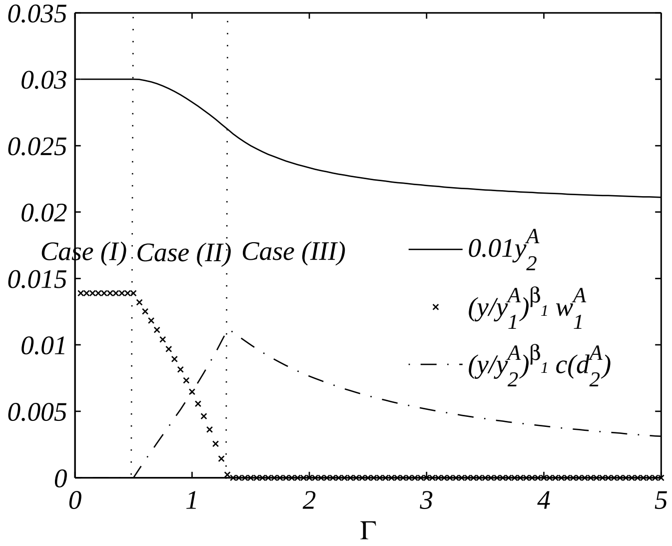

Figure 6.1: $y_2^{\mathrm{A}}, w_1^{\mathrm{A}}$ and $c(d_2^{\mathrm{A}})$ for penalties Γ.

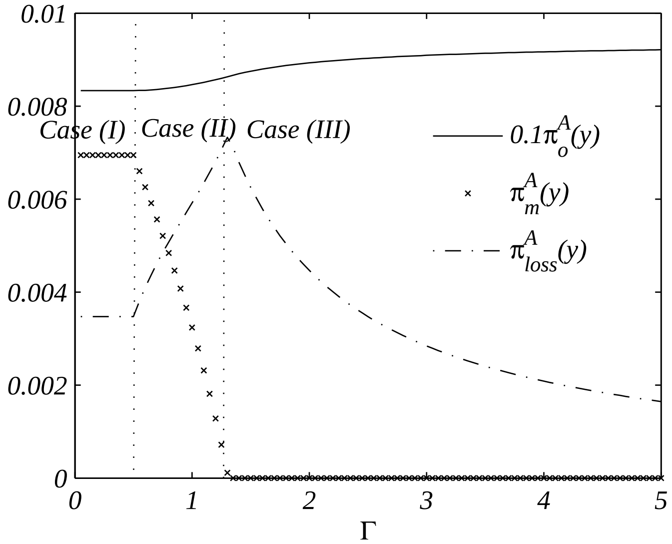

Figure 6.2: $\pi_{\mathrm{o}}^{\mathrm{A}}(y), \pi_{\mathrm{m}}^{\mathrm{A}}(y)$ and π_{loss}^{A} for penalties Γ.

a portion of the manager's profit to the owner. That is, in this case, Γ can be interpreted as the owner's punishment imposed on the manager rather than the punishment by the law. The owner cannot claim Γ larger than $\Delta\theta$, because the manager's act is not necessarily illegal in this case. Accordingly, it is quite likely that the owner prefers the bonus system to operational audit depending on the auditing cost. The bonus system often takes the form of stock options in real life. Note that in this situation the social welfare may be lost if the owner demands too much profit from the manager who does not execute any illegal fraud. This intuition is captured by our results in Case (II) where the social loss $\pi^{\mathrm{A}}_{\mathrm{loss}}$ does not necessarily show a monotone decrease with respect to Γ (see Figure 6.2).

6.5 Conclusion

In this chapter, we investigated the effects of both audit and bonus-incentive as a measure to deal with agency conflicts between the owner and the manager with private information. We showed that the owner's optimal contract is determined among three feasible types of contracts, according to the relation between the auditing cost and the amount of the penalty. This solution not only helps unify several previous results in this area, but also give good account of the real life relationship between audit and bonus-incentive.

Chapter 7

Conclusion

In this thesis, we have shown new results about the duality and bounds on risk-neutral probabilities in option pricing based on prices of other derivatives. Furthermore, in the context of strategic real options, we have analyzed (i) the option to choose both the type and the timing of the projects, (ii) the loss due to incomplete information about the competitor, and (iii) the bonus and audit system to deal with agency conflicts between the owner and the manager. The results obtained in this thesis are summarized as follows:

(Chapter 2) We clarified financial meanings of duality in the problem of finding the derivative price range from the observed prices of other derivatives in terms of the buy-and-hold hedging.

(Chapter 3) We derived analytical bounds on risk-neutral cumulative distribution functions of the underlying asset price from the observed prices of call and put options. Moreover, we computed the bounds from the Nikkei 225 option data in Japan so that we could capture the property of the bounds.

(Chapter 4) We investigated the simultaneous effects of the competition on the investment timing and the choice of the project type in the context of R&D investment. In particular, we showed that a lower-standard technology is likely to appear for the reason that the preemption game due to the competition hastens the investment timing.

(Chapter 5) We evaluated the start-up's loss due to incomplete information about the efficiency of the large firm that subsequently takes over the market from the start-up. We elucidated in which cases and how greatly the firm suffers the loss due to incomplete information.

(Chapter 6) We studied the problem of the owner who designs the contract consisting of bonus and audit so that the manager with private information makes investment at efficient timing.

The optimal contract is determined among three feasible types of contracts, depending on whether the audit is effective or not, and whether the manager's private benefit is legal or illegal. The result connects previous works in this area because the result for each type of contracts includes a previous result.

As we summarized above, we have made several contributions to the study on financial and real options. However, there are some problems that remain unsolved. In the following, we give some future issues related to each section of the thesis.

(Chapters 2 and 3) Sections 2 and 3 treated only European type derivatives. As of now, similar results have yet to be obtained for American type derivatives. As shown in Section 2, the problem of finding the derivative price range from the observed prices of other derivatives is equivalent to the problem of hedging the derivative with a buy-and-hold portfolio. In most cases, we cannot practice complete dynamic hedging for complicated derivatives such as American type derivatives. Thus, providing a simple hedging strategy for American type derivatives, which is easy to execute, will be very useful in practice.

(Chapter 4) The R&D investment often takes a form of multi-step investment, though Section 4 did not consider such a case. This is because the multi-step investment reduces the risk in R&D projects which is much higher than that in other projects. However, a large-scale investment in a lump may be favored by a firm with the fear of preemption by the competitor. So far, no studies have clarified the effects of the competition on a multi-step R&D investment.

(Chapter 5) Section 5 did not consider the situation where the large firm buys out the small entrepreneur. It is an interesting issue to investigate how incomplete information influences a firm's M&A strategy, and in which case and how greatly the firm suffers the loss due to incomplete information. Recently, M&A has frequently caused a social problem, and therefore theoretical analysis about M&A needs to be conducted more and more.

(Chapter 6) Section 6 investigated agency conflicts between two players, namely the owner and the manager, in a single firm. Another important player in corporate finance is the creditor. By structuring a real options model that captures the relationship of three players, i.e., the shareholder (owner), the manager, and the creditor, we will be able to understand the interactions among financing, the capital structure, and the investment decision.

Appendix A

Appendix of Chapter 4

A.1 The stopping time game and its equilibrium

In this paper, we adopt the concept of the stopping time game introduced in [24] because of its intuitive simplicity. We make a brief explanation of the concept by [24] below. See [24] for further details.

The stopping time game proceeds as follows. In the absence of an action by either player, the game environment evolves according to the stochastic process (4.2.1). If a firm has not invested until time t, its action set is $A_t = \{0, 1, 2\}$, where 0 stands for *no entry* and $1, 2$ for *invest* in technology $1, 2$ respectively. If a firm has already invested before time t, then the action set A_t is the null. Investment by one of the firms (called leader) terminates the game and determines the (expected) payoff of both firms because the other (called follower) necessarily takes the optimal response. We assume that the simultaneous action yields the expected payoff $(L(Y) + F(Y))/2$ to both firms (we take $\alpha = 1/2$ in p. 746 in [24]). That is, one of the firms prove to invest infinitesimally earlier than the other even if both attempt to invest at the same timing. The remaining one must take the optimal response as a follower. The probability that a firm is chosen as a leader is fair, i.e., $1/2$.

A strategy for a firm is generally defined as a mapping from the history of the game H_t to the action set A_t. Here, at time t the history H_t has two components: the sample path of the stochastic process (4.2.1) and the actions of two players up to time t. Since the stochastic process (4.2.1) is Markovian, we restrict attention to Markovian strategies and a Markovian perfect equilibrium. Then, in equilibrium, at time $\inf\{t \geq 0 \mid Y(t) \geq y_P\}$ both firms attempt to invest in technology i satisfying $L_i(y_P) \geq L_j(y_P)$ for $j \neq i$. Under the assumption only one of the firms is actually allowed to invest at time $\inf\{t \geq 0 \mid Y(t) \geq y_P\}$ and the other invests at time $\tau^*_{F_i}$ as a follower.

Several studies such as [29, 87] use the above concept by [24], but there is another stream [42, 82, 83] that has tried to elucidate the possibility of mistaken simultaneous investment. They introduces a more complex strategy space of the firms, instead of the technical assumption that the simultaneous action yields the payoff $(L(Y) + F(Y))/2$. Even in their approach, the outcome still holds true under Assumption A due to rent equalization. They suggest that, without Assumption A both firms may make simultaneous investment mistakenly, and of course the analysis without Assumption A is an interesting issue in future research direction.

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