STUDIES
ON
MULTI-LEADER-FOLLOWER GAMES
AND RELATED ISSUES

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by

MING HU

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Preface

In this thesis, we study two kinds of problems related to each other; the multi-leader-follower game and the equilibrium problem with equilibrium constraints, EPEC for short, respectively. In the noncooperative game theory, the multi-leader-follower game is a problem where each leader solves a Stackelberg game (single-leader-multi-follower game), which is a special kind of noncooperative Nash equilibrium problem (NEP). The Stackelberg game can be reformulated as a bilevel optimization problem called the mathematical program with equilibrium constraints (MPEC), in which the lower-level problem contains a complementarity problem (CP) or a variational inequality (VI) problem parameterized by the variable of the upper-level problem. Analogously, the multi-leader-follower game can also be reformulated as the EPEC which is to find equilibria where several players simultaneously solve their own MPECs, each of which is parameterized by decision variables of other players.

Studies on the multi-leader-follower game and EPEC have begun only recently. For example, Sherali considered a true multi-leader-follower game where each leader can exactly anticipate the aggregate followers' responses. More recently, Pang and Fukushima introduced a multi-leader-follower games which can be reformulated as the generalized Nash equilibrium problem (GNEP). Ralph and his coworkers also proposed some algorithms for the EPECs. In addition, the multi-leader-follower game and EPEC also have many applications in engineering and economics, such as the deregulated electricity markets.

The main contribution of this thesis is as follows. Firstly, a multi-leader-follower game with special structure is considered. We also study some properties of the L/F Nash equilibrium of the game. Secondly, we consider an EPEC where a parametric P-matrix linear complementarity system is contained in each player's strategy set. After reformulating the EPEC as an equivalent nonsmooth NEP, we construct a sequence of smoothed NEPs and show some convergence results. Finally, we consider a multi-leader-follower game under uncertainty. By means of the technique of robust optimization, we reformulate it as a NEP under uncertainty and show some properties of the robust L/F Nash equilibrium of the game.

Since the studies on the multl-leader-follower game and EPEC are very vigorous research fields, and there are still many problems that remain unsolved. The author hopes that this thesis will show some help for further studies on these topics and related problems.

Ming Hu
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Chapter 1

Introduction

1.1 Multi-Leader-Follower Game and Equilibrium Problem with Equilibrium Constraints

As a solid mathematical methodology to deal with many social problems, such as economics, management and political science, game theory [39, 71] studies the strategic solutions, where an individual makes a choice by taking into account the others' choices. Game theory was developed widely in 1950 as John Nash introduced the well-known concept of Nash equilibrium in non-cooperative games. The corresponding problem is particularly called the Nash game or the Nash equilibrium problem (NEP) [74, 75], which means no player can obtain any more benefit by changing his/her current strategy unilaterally (i.e., the other players have no any incentive to change their current strategies). Since then, the NEP has received more and more academic attention from a lot of researchers. It also has been playing an important role in many application areas of science, economics, signal processing and communication, engineering and so on; see [6, 72, 45, 68, 70, 88].

In the NEP, the strategy set of each player is supposed to be independent on the strategies of the other players. While in many real-world problems, the strategy set of each player may depend on the rivals' strategies, particularly in the cases that the players share some common resources or limitations, such as an electrical transmission line or a common limit on the total resources for production. We call such a game the generalized Nash equilibrium problem (GNEP) which is introduced by Debreu [21] and Arrow and Debreu [3] and can be considered an extension of the (standard) NEP. The GNEP can be employed to model more and more practical problems in electricity, gas, telecommunications, transportation and so on; see [1, 19, 44, 55, 80, 90, 93].

One of the most important GNEPs is called GNEP with shared constraints which is first introduced by Rosen [84]. It has many applications in electricity market, environmental problems, and so on [12, 92], and has been studied recently by many researchers [28, 29, 40, 53, 65, 66].

It may be necessary to mention that in the NEP (GNEP) there is no distinction among all players who choose their own strategies simultaneously by only observing but not anticipating the (re)actions of the rivals. In contrast, in the Stackelberg game [89], also called the single-leader-follower game, there is a distinctive player, called the leader, who optimizes the upper-level problem and a number of remaining players, called the followers, who optimize the lower-level problems jointly. In particular, the leader can anticipate the response of the followers, and then use this ability to select his optimal strategy. At the same time, for the given leader's strategy, all followers select their own optimal responses by completing with each other in the Nash noncooperative way.

The Stackelberg game has been studied extensively by many researchers in the wide areas of applications, such as oligopolistic market analysis [77, 87], signal processing and communication [82], optimal product design [17], and so on.

To deal with the Stackelberg game, under some suitable assumptions, we can reformulate it as a mathematical program with equilibrium constraints (MPEC) [69] by means of the first-order optimality condition of the follower's problem, where there are two types of variables called decision variable and response variable, and some or all of its constraints with respect to the lower-level variables are expressed by a variational inequality (VI) or complementarity problem (CP) parametrized by the upper-level variables. As a generalization of the MPEC, the equilibrium problem with equilibrium constraints (EPEC for short) can be formulated as a problem that consists of several MPECs, for which one seeks an equilibrium point that is achieved when all MPECs are optimized simultaneously. Several researchers have presented some practical applications of the EPEC, such as electricity markets [54, 58].

One of important applications of the EPEC is the multi-leader-follower game [22, 67, 78, 80, 86], which arises in various real-world conflict situations such as the oligopolistic competition in a deregulated electricity market. The multi-leader-follower game contains several leaders and a number of followers. Similarly to the Stackelberg game, each leader can also anticipate the response of the followers, and uses this ability to select his strategy to complete with the other leaders in the Nash noncooperative way. At the same time, for a given strategy tuple of all leaders, each follower selects his/her own optimal response by competing with the other followers in the Nash noncooperative way, too.

1.2 Overview of the Thesis

The organization of the thesis is as follows. In Chapter 2, we introduce some preliminaries. Especially, we give some notations, basic properties, and existing results that are necessary for the later discussion. In Chapter 3, we consider a class of multi-leader-follower games that satisfy some particular, but still reasonable assumptions, and show that each of these games can be formulated as the standard NEPs, and then as VIs. We establish some results on the existence and uniqueness of the equilibrium for the multi-leader-follower game. We also present some illustrative numerical examples from an electricity power market model. In Chapter 4, we particularly consider a special class of EPECs where a common parametric P-matrix linear complementarity system is contained in all players' strategy sets. After reformulating the EPEC as an equivalent nonsmooth NEP, we use a smoothing method to construct a sequence of smoothed NEPs that approximate the original problem. We consider two solution concepts, global Nash equilibrium and stationary Nash equilibrium, and establish some results about the convergence of approximate Nash equilibria. Moreover we show some illustrative numerical examples. In Chapter 5, we focus on a class of multi-leader-follower games under uncertainty with some special structure. We particularly assume that the follower's problem only contains equality constraints. By means of the robust optimization technique, we first formulate the games as robust Nash equilibrium problems, and then the generalized variational inequality (GVI) problems. We then establish some results on the existence and uniqueness of a robust leader-follower Nash equilibrium. We also apply the forward-backward splitting method to solve the GVI formulation of the problem and present some numerical examples to illustrate the behavior of robust Nash equilibria. More detailed overview of Chapters 3 – 5 will be stated in the introduction of each chapter. In Chapter 6, we summarize the thesis and mention some future issues.

Chapter 2

Preliminaries

In this chapter, we give some basic definitions and preliminary results that will be used in the thesis. We start with some notations.

2.1 Notations

We let $\Re^n$ denote the n-dimensional real Euclidean space. All vectors in $\Re^n$ are viewed as column vectors, unless otherwise stated explicitly, and $^\top$ means the transpose operation. For any vector $x \in \Re^n$, x_i denotes its i-th coordinate, also called its i-th component. The Euclidean norm of any vector $x \in \Re^n$ is denoted by

$$\|x\| := \sqrt{x^\top x} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

For two vectors $x, y \in \Re^n$, $x \perp y$ means $x^\top y = 0$. $\Re^n_+$ denotes the nonnegative orthant in $\Re^n$, that is to say,

$$\Re^n_+ := \{x \in \Re^n | x_i \geq 0, i = 1, \cdots, n\}.$$

The interior of $\Re^n_+$ is denoted by

$$\Re^n_{++} := \{x \in \Re^n | x_i > 0, i = 1, \cdots, n\}.$$

If a vector $x \in \Re^n$ consists of several subvectors, $x^1, \cdots, x^N$, it is denoted, for simplicity of notation, as $(x^1, \cdots, x^N)$ instead of $((x^1)^\top, \cdots, (x^N)^\top)^\top$. For any square matrix $A \in \Re^{n\times n}$, the notations $A \succ 0$ and $A \succeq 0$ denote the positive definiteness and positive semidefiniteness of A, respectively. We let $\mathcal{S}^n$, $\mathcal{S}^n_+$, and $\mathcal{S}^n_{++}$ denote the sets

$$\begin{aligned} \mathcal{S}^n &:= \{A \in \Re^{n\times n} | A = A^\top\}, \\ \mathcal{S}^n_+ &:= \{A \in \Re^{n\times n} | A = A^\top, A \succeq 0\}, \\ \mathcal{S}^n_{++} &:= \{A \in \Re^{n\times n} | A = A^\top, A \succ 0\}, \end{aligned}$$

respectively. For any matrix $A \in \mathcal{S}^n_+$, in the light of the ordered spectral decomposition of A, a unique matrix $A^{\frac{1}{2}} \in \mathcal{S}^n_+$ whose square is A can be defined by

$$A^{\frac{1}{2}} := U^\top (\mathrm{Diag}\sqrt{\lambda(A)})U,$$

where $U \in \Re^{n\times n}$ is some orthogonal matrix, $\lambda(A)$ means the vector consisting of n nonnegative eigenvalues (counted by multiplicity) of A, and a linear mapping $\mathrm{Diag} : \Re^n \to \mathcal{S}^n$ that maps a vector $x \in \Re^n$ to the $n \times n$ diagonal matrix with n diagonal entries $x_i, i = 1, \cdots, n$. For a nonsingular matrix $A \in \Re^{n\times n}$, we define $A^{-\top}$ by $A^{-\top} := (A^{-1})^\top = (A^\top)^{-1}$. For a differentiable real-valued function $f : \Re^n \to \Re$, its gradient $\nabla f(x)$ at $x \in \Re^n$ is the n-dimensional vector defined by

$$\nabla f(x) := \begin{pmatrix} \dfrac{\partial f(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(x)}{\partial x_n} \end{pmatrix} \in \Re^n,$$

where its i-th component $\partial f(x)/\partial x_i$ $(i = 1, \ldots, n)$ denotes the partial derivative of f at x with respect to x_i. For a real-valued function $g : \Re^n \times \Re^m \to \Re$, $g(\cdot, y) : \Re^n \to \Re$ and $g(x, \cdot) : \Re^m \to \Re$ denote the functions with variables x and y, respectively. If it is further differentiable, its partial gradients with respect to variables x and y, denoted by $\nabla_x g(x, y)$ and $\nabla_y g(x, y)$ at (x, y), can be defined in a similar way to the gradient of f. In addition, if f is twice differentiable, its Hessian matrix $\nabla^2 f(x)$ is defined by

$$\nabla^2 f(x) := \begin{pmatrix} \dfrac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix} \in \Re^{n\times n}.$$

For a differentiable vector-valued function $F : \Re^n \to \Re^m$, $\nabla F(x)$ denotes the transposed Jacobian matrix, that is to say,

$$\begin{aligned} \nabla F(x) &:= \Big(\nabla F_1(x), \ldots, \nabla F_m(x)\Big) \\ &= \begin{pmatrix} \dfrac{\partial F_1(x)}{\partial x_1} & \cdots & \dfrac{\partial F_m(x)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_1(x)}{\partial x_n} & \cdots & \dfrac{\partial F_m(x)}{\partial x_n} \end{pmatrix} \in \Re^{n\times m}. \end{aligned}$$

In addition, for any set X, $\mathcal{P}(X)$ denotes the set consisting of all the subsets of X. I and $\mathbf{0}$ denote the identity matrix and the zero matrix with suitable dimension, respectively. For a given set S, $\mathrm{conv}S$ denotes the convex hull of S. $\mathbb{B}(x, r)$ denotes the open Euclidean ball centered at x and with radius $r > 0$.

2.2 Background

In this section, we give some preliminary mathematical background that will be useful in the subsequent chapters. We introduce some basic concepts and properties associated with functions in Subsections 2.2.1 and 2.2.2, those associated with the VI problem and the CP in Subsection 2.2.3, and those associated with the NEP in Subsection 2.2.4. Finally, we introduce our main topic, the EPEC and the multi-leader-follower game.

2.2.1 Convexity and Monotonicity

In this section, we first focus on some properties about convexity and monotonicity of functions, which play an important role throughout the thesis, especially, in the existence and uniqueness of the (robust) L/F Nash equilibrium.

Definition 2.2.1 *For a set-valued function $\mathcal{F}:\Re^n\to\mathcal{P}(\Re^m)$, its graph is the subset of $\Re^{2n}$ given by*

$$\mathrm{gph}(\mathcal{F}) := \{(x,y)\in\Re^n\times\Re^n | x\in\Re^n, y\in\mathcal{F}(x)\}.$$

In particular, if the set-valued function $\mathcal{F}$ happens to be an extended real-valued function $f:\Re^n\to[-\infty,+\infty]$, i.e., $\mathcal{F}(x)=\{f(x)\}, \forall x\in\Re^n$, its graph can be defined by the following subset of $\Re^{n+1}$:

$$\mathrm{gph}(f) := \{(x,y)\in\Re^{n+1} | x\in\Re^n, y=f(x)\}.$$

Definition 2.2.2 *For an extended real-valued function $f:\Re^n\to[-\infty,+\infty]$, its epigraph is the subset of $\Re^{n+1}$ given by*

$$\mathrm{epi}(f) := \{(x,w) | x\in\Re^n, w\in\Re, f(x)\le w\}.$$

Definition 2.2.3 *A set X is said to be a cone if for all $x\in X$ and $\lambda>0$, we have $\lambda x\in X$.*

Definition 2.2.4 *Given a set X of $\Re^n$ and a vector x in X, a vector $y\in\Re^n$ is said to be a tangent of X at x if either $y=0$ or there exists a sequence $\{x_k\}\subseteq X$ such that $x_k\neq x$ for all k and*

$$x_k\to x, \quad \frac{x_k-x}{\|x_k-x\|}\to\frac{y}{\|y\|}.$$

The set of all tangents of X at x is called the tangent cone of X at x, and is denoted by $\mathcal{T}(x;X)$.

Definition 2.2.5 *Given a set X of $\Re^n$ and a vector $x\in X$, a vector $z\in\Re^n$ is said to be a normal of X at x if there exist sequences $\{x_k\}\subseteq X$ and $\{z_k\}$ with*

$$x_k\to x, \quad z_k\to z, \quad z_k\in\mathcal{T}(x_k;X), \quad \forall k\in\mathcal{I}.$$

The set of all normals of X at x is called the normal cone of X at x, and is denoted by $\mathcal{N}(x;X)$.

Definition 2.2.6 *A set $X \subseteq \Re^n$ is said to be convex if*

$$(1-\alpha)x + \alpha y \in X, \quad \forall x, y \in X, \forall \alpha \in [0, 1].$$

Definition 2.2.7 *Let $X \subseteq \Re^n$ be a nonempty and convex set. Then, a real-valued function $f : X \to \Re$ is said to be*

(a) *quasi-convex (on X) if its level sets*

$$L(\beta) := \{x \in \Re^n | f(x) \leq \beta\}$$

are convex for all $\beta \in \Re$.

(b) *convex (on X) if the inequality*

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$$

holds for any $x, y \in X$ and $\alpha \in [0, 1]$.

(c) *strictly convex (on X) if the inequality*

$$f((1-\alpha)x + \alpha y) < (1-\alpha)f(x) + \alpha f(y)$$

holds for any $x, y \in X$ with $x \neq y$ and $\alpha \in (0, 1)$.

(d) *strongly convex (on X) with modulus $\varepsilon > 0$ if the inequality*

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) - \frac{1}{2}\varepsilon(1-\alpha)\alpha\|x - y\|^2$$

holds for any $x, y \in X$ and $\alpha \in [0, 1]$.

It is easily seen that the following properties hold: any strongly convex function is strictly convex, and any strictly convex function is convex, but the converse is not true in general. For instance, a linear function is convex but not strictly convex, $\theta(\alpha) = e^\alpha$ is strictly convex but not strongly convex, and $\theta(\alpha) = \alpha^2$ is strongly convex. Convexity plays a crucial role in the field of optimization. For example, in the nonlinear programming problem

$$\text{minimize } f(x) \text{ subject to } g_i(x) \leq 0 \ (i = 1, \cdots, p),$$

if functions $g_1, \ldots, g_p$ and f are all convex, then any local minimum of the problem is a global minimum.

Before introducing the concept of monotonicity, we first show a property of convex functions which will be useful for our discussion later.

Proposition 2.2.1 *Let $f_i : \Re^n \to (-\infty, +\infty]$ be given functions for $i \in \mathcal{I}$, where $\mathcal{I}$ is an arbitrary index set, and consider the function $g : \Re^n \to (-\infty, +\infty]$ given by*

$$g(x) := \sup_{i \in \mathcal{I}} f_i(x).$$

If $f_i, i \in \mathcal{I}$, are convex, then g is also convex.

Now we give the definition of monotonicity for vector-valued functions from a subset of $\Re^n$ to $\Re^n$.

Definition 2.2.8 *Let $X \subseteq \Re^n$ be a nonempty convex set. Then, a vector-valued function $F : X \to \Re^n$ is said to be*

(a) *monotone (on X) if the inequality*

$$(x - y)^\top (F(x) - F(y)) \geq 0$$

holds for any $x, y \in X$.

(b) *strictly monotone (on X) if the inequality*

$$(x - y)^\top (F(x) - F(y)) > 0$$

holds for any $x, y \in X$ with $x \neq y$.

(c) *strongly monotone (on X) with modulus $\varepsilon > 0$ if the inequality*

$$(x - y)^\top (F(x) - F(y)) \geq \varepsilon \|x - y\|^2$$

holds for any $x, y \in X$.

(d) *coercive (with respect to X) if there exists a vector $x^0 \in X$ such that*

$$\lim_{x \in X, \|x\| \to \infty} \frac{F(x)^\top (x - x^0)}{\|x\|} = \infty.$$

It is easily seen that the following properties hold: any strongly monotone function is strictly monotone, and strictly monotone function is monotone, but the converse is not true in general. In particular, for the case of $n = 1$, the monotonicity in the above sense corresponds to monotonic nondecrease of real-valued functions and the strict monotonicity corresponds to monotonic increase of real-valued functions. Such monotonicity for vector-valued functions can be extended to that for set-valued functions.

Definition 2.2.9 *Let $X \subseteq \Re^n$ be a nonempty convex set. Then, a set-valued function $\mathcal{F} : X \to \mathcal{P}(X)$ is said to be*

(a) *monotone (on X) if the inequality*

$$(x-y)^\top(\xi-\eta) \geq 0$$

holds for any $x, y \in X$ and any $\xi \in \mathcal{F}(x), \eta \in \mathcal{F}(y)$.

(b) *strictly monotone (on X) if the inequality*

$$(x-y)^\top(\xi-\eta) > 0$$

holds for any $x, y \in X$ such that $x \neq y$ and any $\xi \in \mathcal{F}(x), \eta \in \mathcal{F}(y)$.

(c) *strongly monotone (on X) with modulus $\varepsilon > 0$ if the inequality*

$$(x-y)^\top(\xi-\eta) \geq \varepsilon\|x-y\|^2$$

holds for any $x, y \in X$ such that $x \neq y$ and any $\xi \in \mathcal{F}(x), \eta \in \mathcal{F}(y)$.

Moreover, $\mathcal{F}$ is said to be maximal monotone if its graph gph$(\mathcal{F})$ *is not properly contained in the graph of any other monotone functions on $\Re^n$.*

As we will see later, monotonicity plays an important role in analyzing the existence and uniqueness of a (robust) L/F Nash equilibrium for the multi-leader-follower games.

Next, we show some properties that manifest the close relations among convexity, monotonicity, and positive (semi)definiteness of matrices.

Proposition 2.2.2 *Let $X \subseteq \Re^n$ be an open convex set, and $f : X \to \Re$ be a continuously differentiable function. Then,*

(a) *f is convex if and only if ∇f is monotone.*

(b) *f is strictly convex if and only if ∇f is strictly monotone.*

(c) *f is strongly convex with modulus $\varepsilon > 0$ if and only if ∇f is strongly monotone with modulus $\varepsilon > 0$.*

Proposition 2.2.3 *Let $X \subseteq \Re^n$ be an open convex set, and $F : X \to \Re^n$ be a continuously differentiable function. Then,*

(a) *F is monotone if and only if $\nabla F(x)$ is positive semidefinite for any $x \in X$.*

(b) *F is strictly monotone if $\nabla F(x)$ is positive definite for any $x \in X$.*

(c) *F is strongly monotone if and only if $\nabla F(x)$ is uniformly positive definite for all x in X; i.e., there exists a constant $\varepsilon > 0$ such that*

$$y^\top \nabla F(x) y \geq \varepsilon\|y\|^2, \quad \forall y \in \Re^n,$$

for any $x \in X$.

Note that the converse of (b) in Proposition 2.2.3 does not hold in general. For example, though a real-valued function $F : \Re \to \Re$ defined by $F(\alpha) = \alpha^3$ is monotonically increasing on $\Re$, yet $\nabla F(\alpha) = 3\alpha^2$ is not positive when $\alpha = 0$. In the case $n = 1$, the above two propositions directly deduce the following corollary which shows a close relation between the convexity of a real-valued function and the positive (semi)definiteness of its Hessian matrix.

Corollary 2.2.1 *Let $X \subseteq \Re^n$ be an open convex set, and $f : X \to \Re$ be a twice continuously differentiable function. Then,*

(a) *f is convex if and only if $\nabla^2 f(x)$ is positive semidefinite for any $x \in X$.*

(b) *f is strictly convex if $\nabla^2 f(x)$ is positive definite for any $x \in X$.*

(c) *f is strongly convex if and only if $\nabla^2 f(x)$ is uniformly positive definite for all x in X; i.e., there exists a constant $\varepsilon > 0$ such that*

$$y^\top \nabla^2 f(x) y \geq \varepsilon \|y\|^2, \quad \forall y \in \Re^n$$

for any $x \in X$.

2.2.2 Subdifferential and Epiconvergence

In this subsection, we focus on some properties of subdifferential and epiconvergence for real-valued functions. We will also introduce some concepts and results on set-valued functions which will be necessary in the discussion later.

Definition 2.2.10 *Let $f : \Re^n \to \Re$ be a convex function. A vector $d \in \Re^n$ is a subgradient of f at a point $x \in \Re^n$ if*

$$f(z) \geq f(x) + (z - x)^\top d, \quad \forall z \in \Re^n.$$

The set of all subgradients of a convex function f at $x \in \Re^n$ is called the subdifferential of f at x, and is denoted by $\partial f(x)$.

By the concept of subdifferential of convex function f, we can define the corresponding subdifferential function $\partial f : \Re^n \to \mathcal{P}(\Re^n)$, which plays an crucial role in optimization. In particular, there is a close relation between monotonicity of subdifferential mappings and convexity of functions.

Proposition 2.2.4 *Let $f : \Re^n \to \Re$ be a convex function. Then, its corresponding subdifferential mapping $\partial f : \Re^n \to \mathcal{P}(\Re^n)$ is monotone. If f is strictly (strongly) convex, then ∂f is strictly (strongly) monotone.*

Next, we give the definition of directional differentiability for real-valued functions.

Definition 2.2.11 *A function $f : \Re^n \to \Re$ is said to be directionally differentiable at x in the direction d, if the limit*

$$f'(x; d) := \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

exists, and we call the limit $f'(x; d)$ the directional derivative of f at x in the direction d. We say that f is directionally differentiable at x if it is directionally differentiable at x in all directions.

In particular, if f is differentiable, $f'(x; d)$ is equal to $\nabla f(x)^\top d$, where $\nabla f(x)$ is the gradient of f at x. As the similar way, the directional derivative of $f : \Re^{n_{\rm I}} \times \Re^{n_{\rm II}} \to \Re$ at $x^{\rm I} \in \Re^{n_{\rm I}}$ in the direction $d^{\rm I} \in \Re^{n_{\rm I}}$, denoted by $f'_{x^{\rm I}}(x^{\rm I}, x^{\rm II}; d^{\rm I}) \in \Re$, can be also defined as follows:

$$f'_{x^{\rm I}}(x^{\rm I}, x^{\rm II}; d^{\rm I}) := \lim_{\alpha \downarrow 0} \frac{f(x^{\rm I} + \alpha d^{\rm I}, x^{\rm II}) - f(x^{\rm I}, x^{\rm II})}{\alpha}.$$

Definition 2.2.12 [18] *For each $\nu = 1, \cdots, N$, function $f : \Re^{n_{\rm I}+n_{\rm II}} \to \Re$ is said to be regular if, for all $x^{\rm I} \in \Re^{n_{\rm I}}, x^{\rm II} \in \Re^{n_{\rm II}}$ and $d^{\rm I} \in \Re^{n_{\rm I}}$, the directional derivative $f'_{x^{\rm I}}(x^{\rm I}, x^{\rm II}; d^{\rm I})$ exists and satisfies*

$$f'_{x^{\rm I}}(x^{\rm I}, x^{\rm II}; d^{\rm I}) = \limsup_{\substack{z^{\rm I} \to x^{\rm I} \\ \alpha \downarrow 0}} \frac{f(z^{\rm I} + \alpha d^{\rm I}, x^{\rm II}) - f(z^{\rm I}, x^{\rm II})}{\alpha}.$$

Definition 2.2.13 *Let $f : \Re^n \to \Re$ be a given real-valued function and x a given point in $\Re^n$. The function f is said to be locally Lipschitz at $\bar{x}$ if there exists a constant κ and a positive scalar ε such that the following inequality holds:*

$$|f(x) - f(y)| \le \kappa \|x - y\|, \quad \forall x, y \in \bar{x} + \mathbb{B}(0, \varepsilon).$$

A vector-valued function $F : \Re^n \to \Re^m$ is called locally Lipschitz at $\bar{x}$, if its each component function $F_i : \Re^n \to \Re, i = 1, \cdots, m$ is locally Lipschitz at $\bar{x}$.

Definition 2.2.14 *Let $F : \Omega \to \Re^m$ be a locally Lipschitz function at $x \in \Omega$, where Ω is a open set in $\Re^n$. The Clarke generalized Jocobian of F is defined as:*

$$\partial F(x) := {\rm conv}\{G \in \Re^{n \times m} | G = \lim_{k \to \infty} \nabla F(x^k), \textit{ for some sequence } \{x^k\} \to x, x^k \in \mathcal{D}_F\},$$

where $\mathcal{D}_F$ denotes the set consisting of points at which F is differentiable. In the case $m = 1$, that is G is a real-valued function $g : \Re^n \to \Re$, $\partial g(x)$ is called the generalized gradient of g at x.

In the second part, we introduce some concepts and properties associated with epiconvergence, which play an important role in the subject of approximation in optimization.

Definition 2.2.15 *A sequence of sets* $\{C_k\}_{k\in\mathcal{N}}$, *where* $C_k \subseteq \Re^n$ *for all* $k \in \mathcal{I}$, *where* $\mathcal{I}$ *is an arbitrary index set, is said to converge, in the Painlevé-Kuratowski sense* [83], *to a set* $C \subseteq \Re^n$, *denoted by* $C_k \to C$, *if*

(a) *any cluster point of a sequence* $\{x^k\}_{k\in\mathcal{I}}$, *where* $x^k \in C_k$ *for all* $k \in \mathcal{I}$, *belongs to* C;

(b) *for each* $x \in C$, *one can find a sequence* $\{x^k\}_{k\in\mathcal{I}}$ *such that* $x^k \in C_k$ *for every* $k \in \mathcal{I}$ *and* $x^k \to x$.

The theory of set convergence provides a convenient tool to study the approximation issues in optimization. In particular, we can define the concept of epiconvergence through the convergence of epigraphs.

Definition 2.2.16 *A sequence of functions* $\{f_k\}_{k\in\mathcal{I}}$ *on* X, *where* $f_k : \Re^n \to \Re$ *for each* k *and* X *is a set in* $\Re^n$, *is said to epiconverge to a function* $f : \Re^n \to \Re$ *on* X, *when* $\mathrm{epi} f_k \to \mathrm{epi} f$ *as* $k \to \infty$.

For checking the epiconvergence, the following proposition is useful.

Proposition 2.2.5 *Let* $\{f_k\}_{k\in\mathcal{I}}$ *be any sequence of functions on* $\Re^n$. *Then,* f_k *epiconverges to* f *on* $X \subseteq \Re^n$ *if and only if at each point* $x \in X$ *one has*

$$\begin{cases} \liminf\limits_{k\to\infty} f_k(x^k) \geq f(x) & \text{for every sequence } \{x^k\} \text{ such that } x^k \to x, \\ \limsup\limits_{k\to\infty} f_k(x^k) \leq f(x) & \text{for some sequence } \{x^k\} \text{ such that } x^k \to x. \end{cases} \tag{2.2.1}$$

It is easy to see from (2.2.1) that we have actually $f_k(x^k) \to f(x)$ for at least one sequence $\{x^k\}_{k\in\mathcal{I}}$ such that $x^k \to x$. If this property holds for any such sequence $\{x^k\}$, we say the sequence of functions $\{f_k\}_{k\in\mathcal{I}}$ converges continuously to function f.

Definition 2.2.17 *A sequence of functions* $\{f_k\}_{k\in\mathcal{I}}$ *is said to converge continuously to a function* $f : \Re^n \to \Re$ *at a point* $x \in X \subseteq \Re^n$ *if* $f_k(x^k) \to f(x)$ *for any sequence* $\{x^k\}_{k\in\mathcal{I}} \subseteq X$ *converging to* x, *and to converge continuously on* $X \subseteq \Re^n$ *if this is true at every* $x \in X$.

The next proposition gives a sufficient condition for a sequence of functions to epiconverge.

Proposition 2.2.6 *If a sequence of functions* $\{f_k\}_{k\in\mathcal{I}}$ *converges continuously to* $f : \Re^n \to \Re$ *on* $X \subseteq \Re^n$, *then it epiconverges to* f *on* X.

The concept of epiconvergence plays an essential role in studying approximation issues for optimization problems. To study approximation issues for the NEPs, which contain multiple objective functions associated with all players' optimization problems, Gürkan and Pang [46] introduced the concept of multi-epiconvergence for a sequence of families of functions, as a generalization of epiconvergence.

Definition 2.2.18 *A sequence of families of functions* $\{\{f_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$*, where* $f_{\nu,k} : \Re^n \to \Re$ *for each* ν *and* k*, is said to multi-epiconverge to the family of functions* $\{f_\nu\}_{\nu=1}^N$*, where* $f_\nu : \Re^n \to \Re$ *for each* ν*, on the set* $X = \prod_{\nu=1}^N X^\nu \subseteq \Re^n$*, if the following two conditions hold for every* $\nu = 1, \cdots, N$ *and every* $x = (x^1, \cdots, x^N) \in X$*:*

(a) *For every sequence* $\{x^{-\nu,k}\}_{k\in\mathcal{I}} \subseteq X^{-\nu} := \prod_{\nu'=1,\nu'\neq\nu}^N X^{\nu'}$ *converging to* $x^{-\nu}$*, there exists a sequence* $\{x^{\nu,k}\}_{k\in\mathcal{I}} \subseteq X^\nu$ *converging to* x^ν *such that*

$$\limsup_{k\to\infty} f_{\nu,k}(x^{\nu,k}, x^{-\nu,k}) \leq f_\nu(x^\nu, x^{-\nu}).$$

(b) *For every sequence* $\{x^k\}_{k\in\mathcal{I}} \subseteq X$ *converging to* x*,*

$$\liminf_{k\to\infty} f_{\nu,k}(x^k) \geq f_\nu(x).$$

The following proposition establishes the strong relation between multi-epiconvergence and epiconvergence.

Proposition 2.2.7 [46] *A sequence of families of functions* $\{\{f_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$ *multi-epiconverges to the family of functions* $\{f_\nu\}_{\nu=1}^N$ *on the set* $X = \prod_{\nu=1}^N X^\nu$ *if and only if for every* $\nu = 1, \cdots, N$ *and every sequence* $\{x^{-\nu,k}\}_{k\in\mathcal{I}} \subseteq X^{-\nu}$ *converging to some* $x^{-\nu,\infty} \in X^{-\nu}$*, the sequence of functions* $\{\psi_{\nu,k}\}_{k\in\mathcal{I}}$*, where each* $\psi_{\nu,k} : \Re^{n_\nu} \to \Re$ *is defined by*

$$\psi_{\nu,k}(x^\nu) := f_{\nu,k}(x^\nu, x^{-\nu,k}),\ x^\nu \in X^\nu,$$

epiconverges to the function $\psi_{\nu,\infty} : \Re^{n_\nu} \to \Re$ *defined by*

$$\psi_{\nu,\infty}(x^\nu) := f_\nu(x^\nu, x^{-\nu,\infty}),\ x^\nu \in X^\nu$$

on the set X^ν*.*

2.2.3 Variational Inequality and Complementarity Problems

In this subsection, we focus on some basic concepts and properties of the VI, its special case called the CP, and one of its generalizations called the GVI.

Definition 2.2.19 *The variational inequality (VI) denoted by VI*(K, F) *is a problem of finding a vector* $x \in K$ *such that*

$$F(x)^\top (y - x) \geq 0, \quad \forall y \in K, \tag{2.2.2}$$

where K *is a nonempty closed convex subset of* $\Re^n$ *and* $F : \Re^n \to \Re^n$ *is a continuous mapping. The set of solutions to this problem is denoted SOL*(K, F)*.*

As we can see from the definition of the VI, a vector $x \in K$ solves the VI(K, F)(2.2.2) if and only if $-F(x)$ is a normal vector to K at x, or equivalently,

$$0 \in F(x) + \mathcal{N}(x; K),$$

In the VI(K, F) (2.2.2), suppose that the feasible set K can be represented as

$$K := \{x \in \Re^n | h(x) = 0, g(x) \leq 0\},$$

where $h : \Re^n \to \Re^p$ is an affine function and $g : \Re^n \to \Re^m$ is a continuously differentiable convex function. Then under a suitable constraint qualification, a solution x of the VI(K, F) (2.2.2), along with some Lagrange multipliers $\mu \in \Re^p$ and $\delta \in \Re^m$, satisfies the following KKT system:

$$\begin{aligned} L(x, \mu, \lambda) &= 0, \\ h(x) &= 0, \\ g(x) \leq 0, \lambda \geq 0, \lambda^\top g(x) &= 0. \end{aligned} \tag{2.2.3}$$

where

$$L(x, \mu, \lambda) := F(x) + \nabla h(x)\mu + \nabla g(x)\lambda,$$

is the VI Lagrangian function of the VI(K, F) (2.2.2).

Conversely, if a triple $(x, \mu, \lambda) \in \Re^n \times \Re^p \times \Re^m$ satisfies the KKT system (2.2.3), then x solves the VI(K, F) (2.2.2). We call the triple (x, μ, λ) a KKT triple of the VI(K, F) (2.2.2) if and only if it satisfies the above KKT conditions. Moreover, the corresponding x-vector is called a KKT point. For simplicity, when there is no possibility of confusion, we shall often refer to a KKT triple simply as a KKT point.

Applications of the VI can be found in various areas, such as transportation systems, mechanics, and economics; see [36, 73].

As to the existence and uniqueness of a solution in the VI, a number of results are known. One of the most fundamental results relies on the compactness of set K. Other existence results can be obtained by imposing another condition, such as coerciveness of F, instead of the compactness of K. On the other hand, under some monotonicity assumptions on F, we have the following two results on the uniqueness of a solution:

Proposition 2.2.8 *If F is strictly monotone on K, and the VI(K, F) has at least one solution, then the solution is unique.*

Proposition 2.2.9 *If F is strongly monotone on K, then there exists a unique solution to the VI(K, F).*

As a very large class of problems, the VI contains many problems as its special cases, such as the system of equations, the convex programming problem, and the CP. For example, when $K = \Re^n$, the VI(K, F) (2.2.2) is equivalent to the system of equations:

$$F(x) = 0.$$

When $K = \Re^n_+$, the VI(K, F) (2.2.2) is equivalent to the complementarity problem (CP) denoted by CP(F), which is to find a vector $x \in \Re^n$ such that

$$F(x) \geq 0, \quad x \geq 0, \quad F(x)^\top x = 0. \tag{2.2.4}$$

When F is an affine function given by $F(x) = Mx + q$ with a square matrix $M \in \Re^{n\times n}$ and a vector $q \in \Re^n$, the CP(F) (2.2.4) becomes the linear complementarity problem (LCP) denoted by LCP(M, q), which is to find a vector $x \in \Re^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad (Mx + q)^\top x = 0. \tag{2.2.5}$$

One of important special cases of the LCP(M, q) (2.2.5) is that M is a P-matrix. A matrix $M \in \Re^{n\times n}$ is said to be a P-matrix, if its all principal minors are positive, or equivalently,

$$\max_{1\leq i\leq n} x_i(Mx)_i > 0 \quad \text{for all } x(\neq 0) \in \Re^n.$$

The LCP with a P-matrix has many good properties.

Proposition 2.2.10 *A matrix $M \in \Re^{n\times n}$ is a P-matrix if and only if the LCP(M, q) has a unique solution for any vector $q \in \Re^n$.*

As a generalization of P-matrix, a matrix $M \in \Re^{n\times n}$ is said to be a P_0-matrix, if its all principal minors are nonnegative, or equivalently,

$$\max_{1\leq i\leq n} x_i(Mx)_i \geq 0.$$

It is well-known that a matrix $M \in \Re n \times n$ is P_0-matrix if and only if for each $\varepsilon > 0$, $M + \varepsilon I$ is a P-matrix.

For the VI, there exists several important generalizations, one of which is called the generalized variational inequality defined as follows.

Definition 2.2.20 *The generalized variational inequality (GVI), denoted by GVI$(K, \mathcal{F})$, is a problem of finding a vector $x \in K$ such that*

$$\exists\, \xi \in \mathcal{F}(x), \quad \xi^\top(y - x) \geq 0, \quad \forall\, x \in K, \tag{2.2.6}$$

where $K \subseteq \Re^n$ is a nonempty closed convex set and $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is a set-valued mapping.

It is easy to see that if the set-valued function $\mathcal{F}$ happens to be a vector-valued function $F : \Re^n \to \Re^n$, i.e., $\mathcal{F}(x) = \{F(x)\}$, then GVI (2.2.6) reduces to the VI(K, F) (2.2.2).

As a generalization of the VI, the GVI also has wide applications in various areas, such as transportation systems, mechanics, and economics [33, 35]. The GVI also shares some similar properties with the VI. For example, a vector $x \in K$ solves the GVI$(K, \mathcal{F})$(2.2.6) if and only if

$$0 \in \mathcal{F}(x) + \mathcal{N}(x; K).$$

We also have the following proposition:

Proposition 2.2.11 *Suppose that the set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is strictly monotone on S. Then the GVI* (2.2.6) *has at most one solution.*

The following definition of copositivity is also useful in this thesis.

Definition 2.2.21 *Let C be a cone in $\Re^n$. A metrix $M \in \Re^{n\times n}$ is said to be*

(a) *copositive on C if*

$$x^\top Mx \geq 0, \quad \forall x \in C;$$

(b) *strictly copositive on C if*

$$x^\top Mx \geq 0, \quad \forall x \in C \setminus \{0\}.$$

2.2.4 Nash Equilibrium Problems

In this subsection, we describe some concepts and properties about the NEP which has close relations with our main problems, and its generalization, the GNEP. First, we introduce the N-person non-cooperative NEP and its solution concepts called the global (pure strategy) Nash equilibrium and stationary Nash equilibrium.

The well-known NEP is to seek the so-called Nash equilibrium, which is one of the most popular solution concept in an N-person non-cooperative game and defined as follows: Suppose that each player has chosen a (pure) strategy. If there is no player can reduce any more payment by changing his/her current strategy unilaterally (i.e., the other players have no any incentive to change their current strategies).

More precisely, in the NEP denoted by NEP$(\theta_\nu, X^\nu)_{\nu=1}^N$, there are N players labelled by integers $\nu = 1, \cdots, N$. Player ν's strategy is denoted by vector $x^\nu \in \Re^{n_\nu}$ and his/her payoff function $\theta_\nu(x)$ depends on all players' strategies, which are collectively denoted by the vector $x \in \Re^n$ consisting of subvectors $x^\nu \in \Re^{n_\nu}$, $\nu = 1, \cdots, N$, and $n = n_1 + \cdots + n_N$. Player ν's strategy set $X^\nu \subseteq \Re^{n_\nu}$ is independent of the other players' strategies which are denoted collectively as $x^{-\nu} = (x^1, \cdots, x^{\nu-1}, x^{\nu+1}, \cdots, x^N) \in \Re^{n_{-\nu}}$, where $n_{-\nu} := n - n_\nu$. For every fixed but arbitrary vector $x^{-\nu} \in X^{-\nu}$, which consists of all the other players' strategies, player ν solves the following optimization problem for his own variable x^ν:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & x^\nu \in X^\nu, \end{array} \tag{2.2.7}$$

where we denote the player ν's payoff function $\theta_\nu(x) = \theta_\nu(x^\nu, x^{-\nu})$ to emphasize the particular role of x^ν in this problem. The global (pure strategy) Nash equilibrium can be formally defined as follows.

Definition 2.2.22 *A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in X := \prod_{\nu=1}^N X^\nu$ is called a global (pure strategy) Nash equilibrium if, for each $\nu = 1, \cdots, N$,*

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}) \quad \forall x^\nu \in X^\nu.$$

That is to say, for each $\nu = 1, \cdots, N$, $x^{*,\nu}$ is an optimal solution of the following optimization problem in the variable x^ν with $x^{-\nu}$ fixed at $x^{*,\nu}$:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{*,-\nu}) \\ \text{subject to} & x^\nu \in X^\nu. \end{array}$$

On the other hand, a tuple of strategies x^* is called a stationary (pure strategy) Nash Equilibrium if for all $\nu = 1, \cdots, N$, $x^{*,\nu}$ is a stationary point for the optimization problem (2.2.7) with $x^{-\nu} = x^{*,-\nu}$, where a stationary point means that it satisfies a first-order optimality condition for the problem. Assuming the differentiability of the payoff functions θ_ν and the convexity of the strategy sets X^ν, a stationary Nash equilibrium is characterized as a tuple $x^* = (x^{*,\nu})_{\nu=1}^N \in X$ that satisfies the following conditions for all $\nu = 1, \cdots, N$:

$$\nabla_{x^\nu}\theta_\nu(x^{*,\nu}, x^{*,-\nu})^\top (x^\nu - x^{*,\nu}) \geq 0 \quad \forall x^\nu \in X^\nu.$$

If, in addition, θ_ν are convex with respect to x^ν for all ν, then a stationary Nash equilibrium reduces to a global Nash equilibrium.

For the NEP, Nash [74] showed the following well-known result about the existence of a global Nash equilibrium through the Kakutani's fixed point theorem [60].

Proposition 2.2.12 *Suppose that for each player ν,*

(a) *the strategy set X^ν is nonempty, convex and compact;*

(b) *the objective function $\theta_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \to \Re$ is continuous;*

(c) *the objective function θ_ν is convex with respect to x^ν.*

Then, the NEP$(\theta_\nu, X^\nu)_{\nu=1}^N$ comprised of the players' problems (2.2.7) *has at least one global Nash equilibrium.*

There are several different ways to compute a global Nash equilibrium.

(a) Under a suitable constraint qualification, for each $\nu = 1, \cdots, N$, if $X^\nu := \{x^\nu \in \Re^\nu | h^\nu(x^\nu) = 0, g^\nu(x^\nu) \leq 0\}$, where the continuously differentiable functions $g^\nu : \Re^{n_\nu} \to \Re^{s_\nu}$ and θ_ν are convex with respect to x^ν, and $h^\nu : \Re^{n_\nu} \to \Re^{t_\nu}$ is affine, then computing a global Nash equilibrium of NEP$(\theta_\nu, X^\nu)_{\nu=1}^N$ comprised of the players' problems (2.2.7) is equivalent to solving a mixed complementarity problem which is obtained by concatenating the following KKT system of each player:

$$\begin{aligned} \nabla\theta_{x^\nu}(x^\nu, x^{-\nu}) + \nabla h^\nu(x^\nu)\mu^\nu + \nabla g^\nu(x^\nu)\lambda^\nu = 0, \\ h^\nu(x^\nu) = 0, \\ g^\nu(x^\nu) \leq 0, \lambda^\nu \geq 0, (\lambda^\nu)^\top g^\nu(x^\nu) = 0, \end{aligned}$$

with the Lagrangian multipliers $\mu^\nu \in \Re^{t_\nu}$ and $\lambda^\nu \in \Re^{s_\nu}$. The mixed complementarity problem can be solved by some methods, such as the PATH solver [11, 23, 37, 81].

(b) Under some suitable assumptions, a NEP can be equivalently reformulated as a VI by combining the first order optimality conditions arising from each player's optimization problem (2.2.7) [32, 47, 48]. The solutions of a NEP and a VI also have a close relation stated as follows:

Proposition 2.2.13 [32, Proposition 1.4.2] *Consider the NEP$(\theta_\nu, X^\nu)_{\nu=1}^N$ comprised of the players' problems* (2.2.7). *If each strategy set X^ν is a nonempty, closed and convex subset of $\Re^{n_\nu}$ and, for each fixed $x^{-\nu}$, the objective function $\theta_\nu(x^\nu, x^{-\nu})$ is convex and continuously differentiable with respect to x^ν, then a strategy tuple $x = (x^1, \cdots, x^N)$ is a global Nash equilibrium if and only if x solves the VI(X, F), where*

$$X \equiv \prod_{\nu=1}^{N} X^\nu \quad \textit{and} \quad F(x) := (\nabla_{x^\nu}\theta_\nu(x))_{\nu=1}^N.$$

(c) In addition, in the light of technique of approximation, we may compute a global Nash equilibrium by the convergence of a sequence of approximate global Nash equilibria. For example, under the assumption of multi-epiconvergence, Gürkan and Pang [46] showed a sufficient condition for the convergence of approximate global Nash equilibria as follows:

Proposition 2.2.14 [46, Theorem 1] *Suppose that the sequence of families of functions $\{\{\theta_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$ multi-epiconverges to the family of functions $\{\theta_\nu\}_{\nu=1}^N$ on the set $X = \prod_{\nu=1}^N X^\nu$. If the sequence $\{x^k\}_{k\in\mathcal{I}}$, where each $x^k = (x^{\nu,k})_{\nu=1}^N$ is a global Nash equilibrium of the NEP$(\theta_{\nu,k}, X^\nu)_{\nu=1}^N$, converges to $x^\infty = (x^{\nu,\infty})_{\nu=1}^N$, then x^∞ is a global Nash equilibrium of the NEP$(\theta_\nu, X^\nu)_{\nu=1}^N$ comprised of the players' problems* (2.2.7).

The GNEP is a generalization of the standard NEP, in which each player's (pure) strategy set may depend on the strategies of the other players. For a N-person non-cooperative GNEP, where player ν solves the following optimization problem:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} & x^\nu \in S^\nu(x^{-\nu}), \end{array}$$

where $S^\nu(x^{-\nu}) \subseteq \Re^{n_\nu}$ is the player ν's strategy set depending on the other players' strategies $x^{-\nu}$, a generalized global Nash equilibrium can be formally defined as follows.

Definition 2.2.23 *A tuple of strategies $x^* := (x^{*,\nu})_{\nu=1}^N \in S(x) := \prod_{\nu=1}^N S^\nu(x^{-\nu})$ is called a global generalized Nash equilibrium if, for each $\nu = 1, \cdots, N$,*

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^\nu, x^{*,-\nu}) \quad \forall x^\nu \in S^\nu(x^{-\nu}).$$

That is to say, for each $\nu = 1, \cdots, N$, $x^{*,\nu}$ is an optimal solution of the following optimization problem in the variable x^ν with $x^{-\nu}$ fixed at $x^{*,\nu}$:

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{*,-\nu}) \\ \text{subject to} & x^\nu \in S^\nu(x^{*,-\nu}). \end{array}$$

Under some suitable conditions, the existence of a generalized global Nash equilibrium can be established.

Proposition 2.2.15 [3, Lemma] *Let a GNEP be given and suppose that*

(a) *There exist N nonempty, convex, and compact sets $K^\nu \subseteq \Re^{n_\nu}$ such that for each $x \in \Re$ with $x^\nu \in K^\nu$ for each ν, $S^\nu(x^{-\nu})$ is nonempty, closed and convex, $S^\nu(x^{-\nu}) \subseteq K^\nu$, and S^ν, as a set-valued mapping, is continuous.* [1]

(b) *For each player ν, the payoff function $\theta_\nu(\cdot, x^{-\nu})$ is quasi-convex on $S^\nu(x^{-\nu})$.*

Then a generalized global Nash equilibrium exists.

[1] A set-valued mapping $\Phi : \Re^n \to \mathcal{P}(\Re^m)$ is continuous at a point $x \in \Re^n$ if Φ is both upper and lower semicontinuous at x. Φ is upper semicontinuous at x if for every $\varepsilon > 0$ there exists an open neighborhood $\mathcal{V}$ of x such that

$$\bigcup_{y \in \mathcal{V}} \Phi(y) \subseteq \Phi(x) + \mathbb{B}(0, \varepsilon).$$

Φ is lower semicontinuous at x if for every open set $\mathcal{U}$ such that $\Phi(x) \cap \mathcal{U} \neq \emptyset$, there exists an open neighborhood $\mathcal{V}$ of x such that for each $y \in \mathcal{V}$, $\Phi(x) \cap \mathcal{U} \neq \emptyset$. Φ is continuous in a domain if it is continuous at every point in the domain.

2.2.5 Equilibrium Problems with Equilibrium Constraints and Multi-Leader-Follower Games

In this subsection, we introduce our main topics, the EPEC and the multi-leader-follower game. Before this, we first introduce the MPEC and the Stackelberg game, since they carry over to our main problems.

The MPEC is generally an optimization problem with two types of variables called decision variable, $x \in \Re^n$, and response variable, $y \in \Re^m$, in which some or all of its constraints with respect to the lower-level variables are expressed by a VI or a CP parameterized by the upper-level variables. More precisely, this problem is defined as follows.

$$\begin{array}{ll} \text{minimize} & f(x,y) \\ \text{subject to} & (x,y) \in Z, \\ & y \text{ solves } \mathrm{VI}(C(x), F(x,\cdot)), \end{array}$$

where $f : \Re^{n+m} \to \Re$ and $F : \Re^{n+m} \to \Re^m$ are given functions, Z is a closed subset of $\Re^{n+m}$, and $C : \Re^n \to \mathcal{P}(\Re^m)$ is a set-valued mapping from $\Re^n$ to a non-empty closed convex set of $\Re^m$. For each feasible point $x \in \Re^n$, y solves $\mathrm{VI}(C(x), F(x,\cdot))$ if and only if $y \in C(x)$ and the following inequalities

$$(v-y)^\top F(x,y) \geq 0, \quad \forall v \in C(x)$$

holds.

The MPEC can be regarded as a generalization of the bilevel optimization problem in which the lower-level problem is represented as the first-order optimality condition of the lower-level problem [32]. More precisely, if for each fixed feasible point x, $F(x,y) = \nabla_y \theta(x,y)$, i.e., $F(x,\cdot)$ is the partial gradient mapping with respect to the second argument of a real-valued continuously differentiable function $\theta : \Re^{n+m} \to \Re$, θ is convex with respect to x, and $C(x) \subseteq \Re^m$ is a convex set, then the $\mathrm{VI}(C(x), F(x,\cdot))$ is equivalent to the following constrained optimization problem:

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \theta_\nu(x,y) \\ \text{subject to} & y \in C(x). \end{array}$$

As a special case of the bilevel optimization problem, the Stackelberg game has a close relation with the MPEC. In the Stackelberg game, also called the single-leader-multi-follower game, there is a distinctive player, called the leader, who optimizes the upper-level problem and a number of remaining players, called the followers, who optimize the lower-level problems jointly. In particular, the leader can anticipate the response of the followers, and then uses this ability to select his optimal strategy. At the same time, for the given leader's strategy, all followers select their own optimal responses by competing with each other in the Nash noncooperative way. More precisely, a general Stackelberg game with one leader and M followers can be represented as follows:

Leader's Problem.

$$\begin{aligned} &\underset{x}{\text{minimize}} && \theta(x,y) \\ &\text{subject to} && x \in X. \end{aligned} \tag{2.2.8}$$

Follower ω's Problem ($\omega = 1, \cdots, M$).

$$\begin{aligned} &\underset{y^\omega}{\text{minimize}} && \gamma_\omega(x, y^\omega, y^{-\omega}) \\ &\text{subject to} && y^\omega \in K^\omega(x). \end{aligned} \tag{2.2.9}$$

Here, we denote $\gamma_\omega(x,y) = \gamma_\omega(x, y^\omega, y^{-\omega})$ to emphasize the particular role of y^ω in the problem of follower ω. We may also define an equilibrium point for the Stackelberg game as follows: Suppose that all players (the leader and the followers) have chosen their own strategy. Then, there is no player who can reduce any more payment by changing his/her current strategy unilaterally (i.e., the other players have no any incentive to change their current strategies).

In the Stackelberg game, the leader chooses his/her strategy from the strategy set $X \subseteq \Re^n$. For each of the leader's strategy $x \in X$, the followers compete in the Nash noncooperative way, in which we assume that each follower's strategy set $K^\omega(x) \subseteq \Re^{m_\omega}$ is closed and convex, and for any given and arbitrary x^* and $y^{*,-\omega}$, the function $\gamma_\omega(x^*, \cdot, y^{*,-\omega})$ is convex and continuously differentiable with respect to $y^\omega \in K(x^*)$. Let $m = \sum_{\omega=1}^{M} m_\omega$. By Proposition (2.2.13), the above Stackelberg game comprised of problems (2.2.8) and (2.2.9) can be equivalently reformulated as the following MPEC:

$$\begin{aligned} &\text{minimize} && \theta(x,y) \\ &\text{subject to} && x \in X, \\ & && y \text{ solves } \mathrm{VI}(C(x), F(x,\cdot)), \end{aligned}$$

where for $y \in \Re^m$ and $x \in X$,

$$F(x,y) := (\nabla_{y^\omega} \gamma_\omega(x,y))_{\omega=1}^{M},$$

and

$$C(x) := \prod_{\omega=1}^{M} K^\omega(x).$$

The EPEC can be looked on as a generalization of the NEP or GNEP where each player solves his/her own MPEC simultaneously, and the equilibrium constraints consisting of a VI or CP parameterized by decision variable x may be different from those of the other players. Here, we consider an EPEC with shared identical equilibrium constraints. More precisely, such EPEC can be represented as follows:

Player ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{array}{ll} \text{minimize} & f_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} & (x^\nu, y) \in Z^\nu, \\ & y \text{ solves VI}(C(x), F(x, \cdot)), \end{array} \tag{2.2.10}$$

where $y \in \Re^m$ is the shared response variable. For each player $\nu = 1, \cdots, N$, $f_\nu : \Re^{n+m} \to \Re$ and $F : \Re^{n+m} \to \Re^m$ are given functions, Z^ν is a closed subset of $\Re^{n_\nu+m}$, and $C : \Re^n \to \mathcal{P}(\Re^m)$ is a set-valued mapping from $\Re^n$ to a non-empty closed convex set of $\Re^m$.

As a generalization of the Stackelberg game, in the multi-leader-follower game, there are several leaders in the upper-level problem. Similarly to the Stackelberg game, each leader can also anticipate the response of the followers, and uses this ability to select his strategy to compete with the other leaders in the Nash noncooperative way. At the same time, for a given strategy tuple of all leaders, each follower selects his/her own optimal response by competing with the other followers in the Nash noncooperative way, too. More precisely, a general multi-leader-follower game with N leaders and M followers can be represented as follows:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} & x^\nu \in X^\nu. \end{array} \tag{2.2.11}$$

Follower ω's Problem ($\omega = 1, \cdots, M$).

$$\begin{array}{ll} \underset{y^\omega}{\text{minimize}} & \gamma_\omega(x, y^\omega, y^{-\omega}) \\ \text{subject to} & y^\omega \in K^\omega(x). \end{array} \tag{2.2.12}$$

Here, we denote $\theta_\nu(x, y) = \theta_\nu(x^\nu, x^{-\nu}, y)$ and $\gamma_\omega(x, y) = \gamma_\omega(x, y^\omega, y^{-\omega})$ to emphasize the particular role of x^ν and y^ω in the problems of leader ν and follower ω, respectively.

Similarly to the analysis on the relation between the MPEC and the Stackelberg game, by Proposition (2.2.13) the multi-leader-follower game comprised of problems (2.2.11) and (2.2.12) also can be equivalently reformulated as an EPEC with shared identical equilibrium constraints under some suitable assumptions.

In this thesis, we consider one of most interesting and important multi-leader-follower games where all leaders share an identical follower, which can be represented as follows:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned}$$

Follower's Problem.

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \gamma(x, y) \\ \text{subject to} \quad & y \in K(x). \end{aligned}$$

2.3 Nonsmooth Equation Reformulation and Smoothing Method

In this section, we introduce the concept and some basic properties of an important complementarity function called the Fischer-Burmeister (FB) function [38], which can be used to reformulate a VI or a CP as an equivalent system of equations. We also introduce a smoothing function for the FB function.

Definition 2.3.1 *A function $\psi : \Re^2 \to \Re$ is called a complementarity function, if for any pair $(a, b) \in \Re^2$,*

$$\psi(a, b) = 0 \iff a \geq 0,\, b \geq 0,\, ab = 0;$$

equivalently, ψ is a complementarity function if the union of the set of its zeros is the two nonnegative semiaxes.

One of most important complementarity functions is called Fischer-Burmeister (FB) function, denoted by ψ_{FB}, which plays a central role in the computation of solutions of the CP and is defined as follows:

$$\psi_{\text{FB}} := \sqrt{a^2 + b^2} - (a + b), \quad \forall (a, b) \in \Re^2.$$

The FB function has many good properties. For example, it is convex, differentiable everywhere except the origin. Moreover, ψ_{FB}^2 is continuously differentiable on $\Re^2$.

By the FB function, we can reformulate the KKT system (2.2.3) as the following system of nonlinear equations:

$$\Phi_{\text{FB}}(x, \mu, \lambda) = 0,$$

where $\Phi_{\text{FB}} : \Re^{n+p+m} \to \Re^{n+p+m}$ is defined by

$$\Phi_{\text{FB}}(x, \mu, \lambda) := \begin{pmatrix} L(x, \mu, \lambda) \\ -h(x) \\ \psi_{\text{FB}}(-g_1(x), \lambda_1) \\ \vdots \\ \psi_{\text{FB}}(-g_m(x), \lambda_m) \end{pmatrix}.$$

The system of nonlinear equations is associated with the problem of minimizing its natural merit function, i.e.,

$$\text{minimize} \quad \theta_{\text{FB}}(x, \mu, \lambda), \tag{2.3.1}$$

where

$$\theta_{\text{FB}}(x, \mu, \lambda) := \frac{1}{2}\Phi_{\text{FB}}(x, \mu, \lambda)^\top \Phi_{\text{FB}}(x, \mu, \lambda).$$

Although the function ψ_{FB} is not differentiable at the origin, the function θ_{FB} is continuously differentiable on $\Re^{n+p+m}$ when F is a continuously differentiable function, and g and h are both twice continuously differentiable functions [26].

Proposition 2.3.1 *If F is a continuously differentiable function and g and h are both twice continuously differentiable functions, then function θ_{FB} is continuously differentiable.*

Facchinei, Fischer and Kanzow [25] showed conditions under which a stationary point of θ_{FB} is a KKT triple by the proposition below.

Proposition 2.3.2 *Let $(x^*, \mu^*, \lambda^*) \in \Re^{n+p+m}$ be a stationary point of* (2.3.1). *It holds that $\theta_{\text{FB}}(x^*, \mu^*, \delta^*) = 0$; i.e.,$(x^*, \mu^*, \lambda^*)$ is a KKT triple of the VI(K, F)* (2.2.2) *under conditions* (a), (b), *and either one of* (c) *and* (d) *stated below:*

(a) *$\nabla_x L(x^*, \mu^*, \lambda^*)$ is positive semidefinite on $\Re^n$;*

(b) *$\nabla_x L(x^*, \mu^*, \lambda^*)$ is positive definite on the null space of the gradients*

$$\{\nabla h_j(x^*) : j = 1, ..., p\} \cup \{\nabla g_i(x^*) : i \in \mathcal{I}(x^*)\},$$

where $\mathcal{I}(x^*) := \{i | g_i(x^*) = 0\}$;

(c) *$\nabla h(x^*)$ has full row rank;*

(d) *h is an affine function and there exists x satisfying $h(x) = 0$.*

By the above Proposition 2.3.1, we can ensure $\nabla_x L(x, \mu, \lambda)$ is positive semidefinite under some special conditions. But in some general cases, for example, when function F is monotone, h is affine and g is convex and nonlinear, we also cannot ensure the partial Jacobian is positive semidefinite since multipliers λ_i may be negative and large enough. Therefore, it seems to be reasonable to consider the following constrained problem:

$$\begin{aligned} &\text{minimize} \quad \theta_{\text{FB}}(x, \mu, \lambda) \\ &\text{subject to} \quad \lambda \geq 0. \end{aligned} \tag{2.3.2}$$

Based on this constrained problem, Facchinei, Fischer, Kanzow, and Peng [27] showed the following result.

Proposition 2.3.3 *Let* $(x^*, \mu^*, \lambda^*) \in \Re^{n+p+m}$ *be a stationary point of* (2.3.2)*. It holds that* (x^*, μ^*, λ^*) *is a solution of KKT system* (2.2.3) *under conditions* (a), (b), *and either one of* (c) *and* (d) *stated below:*

(a) $\nabla_x L(x^*, \mu^*, \lambda^*)$ *is positive semidefinite on* $\Re^n$*;*

(b) $\nabla_x L(x^*, \mu^*, \lambda^*)$ *is strictly copositive on the cone*

$$\begin{aligned} C(x^*, \lambda^*) := \{\nu \in \Re^n : \nabla h(x^*)\nu = 0,\\ \nabla g_i(x^*)^\top \nu \geq 0, \quad \forall i \in \mathcal{I}_{00} \cup \mathcal{I}_<,\\ \nabla g_i(x^*)^\top \nu \geq 0, \quad \forall i \in \mathcal{I}_+ \cup \mathcal{I}_R\}; \end{aligned}$$

(c) $\nabla h(x^*)$ *has full row rank;*

(d) h *is an affine function,*

where

$$\begin{aligned} \mathcal{I} &:= \{1, ..., m\},\\ \mathcal{I}_0 &:= \{i \in \mathcal{I} : g_i(x^*) = 0 \leq \lambda_i^*\},\\ \mathcal{I}_< &:= \{i \in \mathcal{I} : g_i(x^*) < 0 = \lambda_i^*\},\\ \mathcal{I}_R &:= \mathcal{I} \setminus (\mathcal{I}_0 \cup \mathcal{I}_<),\\ \mathcal{I}_{00} &:= \{i \in \mathcal{I}_0 : \lambda_i^* = 0\},\\ \mathcal{I}_+ &:= \{i \in \mathcal{I}_0 : \lambda_i^* > 0\}. \end{aligned}$$

Since the FB function is not differentiable at origin as mentioned earlier, it is hard to deal with the nonsmooth equation $\psi_{FB}(a,b) = 0$ directly. Smoothing is an effective technique to overcome such difficulties [30, 42, 79]. Generally, for a nondifferentiable function $\psi : \Re^n \to \Re^m$, we call a function $\psi_\mu : \Re^n \to \Re^m$ with a parameter $\mu > 0$ the smoothing function, if it satisfies the following properties:

(a) ψ_μ is differentiable for any $\mu > 0$;

(b) $\lim_{\mu \downarrow 0} \psi_\mu(x) = \psi(x)$, for each $x \in \Re^n$.

Particularly, for the FB function, Fischer [38] introduced the following smoothing Fischer-Burmeister (SFB) function:

$$\psi_\mu(a,b) := a + b - \sqrt{a^2 + b^2 + 2\mu^2},$$

with some parameter $\mu \geq 0$.

Similar to the FB function, the SFB function has the following property:

$$\psi_\mu(a,b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = \mu^2.$$

More interestingly, for any fixed $\mu > 0$, the SFB function ψ_μ is continuously differentiable on the whole space $\Re^2$. Furthermore, the SFB function ψ_μ coincides with the FB function ψ when $\mu = 0$.

In view of the above good properties, the SFB function has many applications, such as [14, 16, 41, 57].

Chapter 3

Variational Inequality Formulation for a Class of Multi-Leader-Follower Games

3.1 Introduction

In this chapter, we consider a simplified multi-leader-follower game with one follower, which still has wide applications; see [54, 58, 80].

The multi-leader-follower game has recently been studied by some researchers and used to model several problems in applications. Pang and Fukushima [80] introduced a class of remedial models for the multi-leader-follower game, that can be formulated as a GNEP with convexified strategy sets; they also proposed some oligopolistic competition models in electricity power markets, that led to multi-leader-follower games. Based on the strong s-tationarity conditions of each leader in a multi-leader-follower game, Leyffer and Munson [67] derived a family of nonlinear complementarity problem (NCPs), nonlinear programming problem, and MPEC formulations of the multi-leader-follower game. They also reformulated the game as a square nonlinear complementarity problem by imposing an additional restriction. Outrata [78] converted a kind of multi-leader-follower games to the EPECs and presented some KKT type of necessary conditions for equilibria. Sherali [86] considered a special multi-leader-follower game where each leader anticipates the response explicitly by the aggregate follower reaction curve. He also showed the existence and uniqueness of the equilibrium of the game called generalized-Stackelberg-Nash Cournot equilibrium, and then proposed a numerical approach to find a generalized-Stackelberg-Nash Cournot equilibrium. Unlike the above deterministic multi-leader-follower games, DeMiguel and Xu [22] considered a stochastic multi-leader-follower game. They introduced a new concept called stochastic multiple-leader Stackelberg-Nash-Cournot equilibrium and showed the existence and uniqueness results under some assumptions. They proposed also a numerical approach to seek the Stackelberg-Nash-Cournot equilibrium with a sample average approximation method. Even with these efforts, we still have to face a lot of problems when we deal with the multi-leader-follower game, because of the inherent difficulties such as the lack of convexity and the failure of constraint qualifications.

Under some particular assumptions on the cost functions of both leaders and follower, as well as the constraints in the follower's problem, we show that the game can be reduced to a NEP, which may further be reformulated as a VI. Moreover, under suitable assumptions, we establish the existence and uniqueness of a L/F Nash equilibrium. We also consider an optimization reformulation of the VI and give some conditions that guarantee a stationarity point of the optimization problem to be a L/F Nash equilibrium. Finally, we present illustrative numerical examples of an electricity power market that consists of two or three firms as the leaders and the independent system operator (ISO) as the follower.

The organization of the chapter is as follows. In Section 3.2, we introduce the particular multi-leader-follower game considered in the chapter, and formulate it as a NEP. In Section 3.3, we reformulate it as a VI and show some conditions that ensure the existence and uniqueness of a L/F Nash equilibrium. In Sections 3.4, we give an application in an electricity power market, and show some illustrative numerical examples from this model in Section 3.5.

3.2 Multi-Leader-Follower Games with Special Structure

In this section, we concentrate on a multi-leader-follower game with the following special structure:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y) := \omega_\nu(x^\nu, x^{-\nu}) + \varphi_\nu(x^\nu, y) \\ \text{subject to} \quad & g^\nu(x^\nu) \le 0, \ h^\nu(x^\nu) = 0. \end{aligned}$$

Follower's Problem.

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \gamma(x, y) := \psi(y) - \sum_{\nu=1}^{N} \varphi_\nu(x^\nu, y) \\ \text{subject to} \quad & y \in \mathcal{Y}. \end{aligned}$$

Here, all functions are assumed to be twice continuously differentiable functions. In particular, each function $\omega_\nu : \Re^n \to \Re$ is convex with respect to x^ν. Functions $g^\nu : \Re^{n_\nu} \to \Re^{s_\nu}$ are all convex and $h^\nu : \Re^{n_\nu} \to \Re^{t_\nu}$ are all affine. Functions ψ, φ_ν and set $\mathcal{Y}$ are assumed to have the following explicit representations:

$$\begin{aligned} \varphi_\nu(x^\nu, y) &\equiv (x^\nu)^\top D_\nu y, \ \nu = 1, \cdots, N, \\ \psi(y) &\equiv \frac{1}{2} y^\top B y + c^\top y, \\ \mathcal{Y} &\equiv \{y \in R^m | Ay + a = 0\}. \end{aligned}$$

Here, $D_\nu \in \Re^{n_\nu \times m}, \nu = 1, \cdots, N, c \in \Re^m$, $a \in \Re^p$, and $B \in \mathcal{S}^m_{++}$. Moreover, $A \in \Re^{p\times m}$ has full row rank. Note that the strategy set of the follower's problem $\mathcal{Y}$ is an affine subset of $\Re^m$.

Then the above multi-leader-follower game can be written as follows:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} \quad \omega_\nu(x^\nu, x^{-\nu}) + (x^\nu)^T D_\nu y \\ &\text{subject to} \quad g^\nu(x^\nu) \le 0, \ h^\nu(x^\nu) = 0. \end{aligned} \tag{3.2.1}$$

Follower's Problem.

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \frac{1}{2} y^\top B y + c^\top y - \sum_{\nu=1}^{N} (x^\nu)^\top D_\nu y \\ &\text{subject to} \quad Ay + a = 0. \end{aligned} \tag{3.2.2}$$

In this game, the objective functions of N leaders and the follower contain some related terms. Specifically, the second term of the objective function appears in the follower's objective function in the negated form. Therefore, the game partly contains a kind of zero-sum structure between each leader and the follower. An application of such special multi-leader-follower games will be presented with some illustrative numerical examples later.

In the remainder of the paper, for simplicity, we will mainly consider the following game with two leaders, labelled I and II. The results presented below can be extended to the case of more than two leaders in a straightforward manner.

Leader I's Problem.

$$\begin{aligned} &\underset{x^{\mathrm{I}}}{\text{minimize}} \quad \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^\top D_{\mathrm{I}} y \\ &\text{subject to} \quad g^{\mathrm{I}}(x^{\mathrm{I}}) \le 0, \ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0. \end{aligned}$$

Leader II's Problem.

$$\begin{aligned} &\underset{x^{\mathrm{II}}}{\text{minimize}} \quad \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ &\text{subject to} \quad g^{\mathrm{II}}(x^{\mathrm{II}}) \le 0, \ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0. \end{aligned}$$

Follower's Problem.

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \frac{1}{2} y^\top B y + c^\top y - (x^{\mathrm{I}})^\top D_{\mathrm{I}} y - (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ \text{subject to} \quad & Ay + a = 0. \end{aligned}$$

Since the follower's problem is a strictly convex quadratic programming problem with equality constraints, it is equivalent to finding a pair $(y, \lambda) \in \Re^{m \times p}$ satisfying the following KKT system of linear equations:

$$\begin{aligned} By + c - (D_{\mathrm{I}})^\top x^{\mathrm{I}} - (D_{\mathrm{II}})^\top x^{\mathrm{II}} + A^\top \lambda = 0, \\ Ay + a = 0. \end{aligned} \tag{3.2.3}$$

Note that, under the given assumptions, a KKT pair (y, λ) exists uniquely for each $(x^{\mathrm{I}}, x^{\mathrm{II}})$ and is denoted by $(y(x^{\mathrm{I}}, x^{\mathrm{II}}), \lambda(x^{\mathrm{I}}, x^{\mathrm{II}}))$. By direct calculations, we have

$$\begin{aligned} y(x^{\mathrm{I}}, x^{\mathrm{II}}) =& - B^{-1}c - B^{-1}A^\top (AB^{-1}A^\top)^{-1}(a - AB^{-1}c) \\ &+ [B^{-1}(D_{\mathrm{I}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{I}})^\top] x^{\mathrm{I}} \\ &+ [B^{-1}(D_{\mathrm{II}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{II}})^\top] x^{\mathrm{II}}, \\ \lambda(x^{\mathrm{I}}, x^{\mathrm{II}}) =& (AB^{-1}A^\top)^{-1}(a - AB^{-1}c) + (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{I}})^\top x^{\mathrm{I}} \\ &+ (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{II}})^\top x^{\mathrm{II}}. \end{aligned}$$

Substituting $y(x^{\mathrm{I}}, x^{\mathrm{II}})$ for y in the leaders' problems, the leaders' objective functions can be rewritten as

$$\begin{aligned} \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^\top D_{\mathrm{I}} y = \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{I}})^\top D_{\mathrm{I}} r \\ + (x^{\mathrm{I}})^\top D_{\mathrm{I}} G x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} H x^{\mathrm{II}}, \end{aligned} \tag{3.2.4}$$

$$\begin{aligned} \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^\top D_{\mathrm{II}} y = \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + (x^{\mathrm{II}})^\top D_{\mathrm{II}} r \\ + (x^{\mathrm{II}})^\top D_{\mathrm{II}} G x^{\mathrm{I}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} H x^{\mathrm{II}}, \end{aligned} \tag{3.2.5}$$

where $G \in \Re^{m \times n_{\mathrm{I}}}$, $H \in \Re^{m \times n_{\mathrm{II}}}$, and $r \in \Re^m$ are given by

$$\begin{aligned} G &= B^{-1}(D_{\mathrm{I}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{I}})^\top, \\ H &= B^{-1}(D_{\mathrm{II}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1} AB^{-1}(D_{\mathrm{II}})^\top, \\ r &= -B^{-1}c - B^{-1}A^\top (AB^{-1}A^\top)^{-1}(a - AB^{-1}c). \end{aligned}$$

Let the functions defined by (3.2.4) and (3.2.5) be denoted as $\Theta_{\mathrm{I}} : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \Re$ and $\Theta_{\mathrm{II}} : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \Re$, respectively. Then we can formulate the above multi-leader-follower game as the following NEP, which we call the $\mathrm{NEP}(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$, where $X^\nu = \{x^\nu : g^\nu(x^\nu) \leq 0, h^\nu(x^\nu) = 0\}, \nu = \mathrm{I}, \mathrm{II}$:

Leader I's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{I}}}{\text{minimize}} & \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} & g^{\mathrm{I}}(x^{\mathrm{I}}) \le 0,\ h^{\mathrm{I}}(x^{\mathrm{I}}) = 0. \end{array}$$

Leader II's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{II}}}{\text{minimize}} & \Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} & g^{\mathrm{II}}(x^{\mathrm{II}}) \le 0,\ h^{\mathrm{II}}(x^{\mathrm{II}}) = 0. \end{array}$$

Remark 3.2.1 Instead of solving the KKT system (3.2.3) explicitly, we may leave it as additional constraints in each leader's problem. This results in a GNEP, which has attracted much attention recently [31, 80].

3.3 Existence and Uniqueness of L/F Nash Equilibrium

In this section, we further reformulate the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ derived in the previous section as a variational inequality, and discuss the existence and uniqueness of a L/F Nash equilibrium. Throughout this section, we assume the sets X^{I} and X^{II} are nonempty.

In the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$, the Hessian matrices of the leaders' objective functions are calculated as

$$\begin{aligned} \nabla^2_{x^{\mathrm{I}}} \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= \nabla^2_{x^{\mathrm{I}}} \omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + 2(D_{\mathrm{I}} B^{-\frac{1}{2}}) P (D_{\mathrm{I}} B^{-\frac{1}{2}})^\top, \\ \nabla^2_{x^{\mathrm{II}}} \Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) &= \nabla^2_{x^{\mathrm{II}}} \omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + 2(D_{\mathrm{II}} B^{-\frac{1}{2}}) P (D_{\mathrm{II}} B^{-\frac{1}{2}})^\top, \end{aligned}$$

where matrix $P \in \Re^{m \times m}$ is given by

$$P = I - B^{-\frac{1}{2}} A^\top (A B^{-1} A^\top)^{-1} A B^{-\frac{1}{2}}.$$

Since P is a projection matrix, it is symmetric, idempotent, and positive semidefinite. By the given assumption, functions $\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ and $\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})$ are both convex with respect to x^{I} and x^{II}, respectively. Therefore the leaders' objective functions in the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ are both convex with respect to x^{I} and x^{II}, respectively.

By Proposition 2.4, we can reformulate the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ as the following VI denoted by the VI$(X, \hat{F})$:

Find a vector $x^* \in X := X^{\mathrm{I}} \times X^{\mathrm{II}}$ such that

$$\hat{F}(x^*)^\top (x - x^*) \ge 0 \quad \text{for all } x \in X, \tag{3.3.1}$$

where $x = (x^{\mathrm{I}}, x^{\mathrm{II}}) \in \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}}$, $X^\nu = \{x^\nu : g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0\}, \nu = \mathrm{I}, \mathrm{II}$ and the function $\hat{F} : \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}} \to \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}}$ is defined by

$$\hat{F}(x) := \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla_{x^{\mathrm{II}}}\Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} = \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + D_{\mathrm{I}}r + 2D_{\mathrm{I}}Gx^{\mathrm{I}} + D_{\mathrm{I}}Hx^{\mathrm{II}} \\ \nabla_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) + D_{\mathrm{II}}r + D_{\mathrm{II}}Gx^{\mathrm{I}} + 2D_{\mathrm{II}}Hx^{\mathrm{II}} \end{pmatrix}. \quad (3.3.2)$$

Here we notice that matrices $D_{\mathrm{I}}G$ and $D_{\mathrm{II}}H$ are symmetric from the definitions of G and H, respectively.

The VI Lagrangian function is written as

$$\hat{L}(x, \mu, \lambda) = \hat{F}(x) + \begin{pmatrix} \sum_{i=1}^{s_{\mathrm{I}}} \lambda_i^{\mathrm{I}} \nabla g_i^{\mathrm{I}}(x^{\mathrm{I}}) \\ \sum_{i=1}^{s_{\mathrm{II}}} \lambda_i^{\mathrm{II}} \nabla g_i^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{t_{\mathrm{I}}} \mu_j^{\mathrm{I}} \nabla h_j^{\mathrm{I}}(x^{\mathrm{I}}) \\ \sum_{j=1}^{t_{\mathrm{II}}} \mu_j^{\mathrm{II}} \nabla h_j^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix},$$

where $\lambda = (\lambda^{\mathrm{I}}, \lambda^{\mathrm{II}}) \in \Re^{s_{\mathrm{I}}+s_{\mathrm{II}}}$ and $\mu = (\mu^{\mathrm{I}}, \mu^{\mathrm{II}}) \in \Re^{t_{\mathrm{I}}+t_{\mathrm{II}}}$. The Jacobin matrix $\nabla_x\hat{L}(x, \mu, \lambda)$ of the VI Lagrangian function with respect to x is written as

$$\begin{aligned} \nabla_x\hat{L}(x, \mu, \lambda) = & \begin{pmatrix} \nabla^2_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) & \nabla^2_{x^{\mathrm{II}}x^{\mathrm{I}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla^2_{x^{\mathrm{I}}x^{\mathrm{II}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) & \nabla^2_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} \\ & + \begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \end{pmatrix} \\ & + \begin{pmatrix} \sum_{i=1}^{s_{\mathrm{I}}} \lambda_i^{\mathrm{I}} \nabla^2 g_i^{\mathrm{I}}(x^{\mathrm{I}}) & 0 \\ 0 & \sum_{i=1}^{s_{\mathrm{II}}} \lambda_i^{\mathrm{II}} \nabla^2 g_i^{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix}. \end{aligned}$$

Lemma 3.3.1 *The matrix*

$$\begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \end{pmatrix}$$

is symmetric and positive semidefinite.

Proof. Set $Q = B^{-\frac{1}{2}}PB^{-\frac{1}{2}}$, which is symmetric and positive semidefinite. Then we can write

$$\begin{aligned} & \begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^\top & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^\top \end{pmatrix} \\ = & \begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^\top & D_{\mathrm{I}}Q(D_{\mathrm{II}})^\top \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^\top & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^\top \end{pmatrix}. \end{aligned}$$

Consider the following products of matrices:

$$
\begin{aligned}
&\begin{pmatrix} D_{\mathrm{I}} & \frac{1}{2}D_{\mathrm{I}} \\ \frac{1}{2}D_{\mathrm{II}} & D_{\mathrm{II}} \end{pmatrix} \begin{pmatrix} 2Q & 0 \\ 0 & 2Q \end{pmatrix} \begin{pmatrix} D_{\mathrm{I}} & \frac{1}{2}D_{\mathrm{I}} \\ \frac{1}{2}D_{\mathrm{II}} & D_{\mathrm{II}} \end{pmatrix}^{\top} \\
&= \begin{pmatrix} \frac{5}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^{\top} & 2D_{\mathrm{I}}Q(D_{\mathrm{II}})^{\top} \\ 2D_{\mathrm{II}}Q(D_{\mathrm{I}})^{\top} & \frac{5}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^{\top} \end{pmatrix} \\
&= 2\begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^{\top} & D_{\mathrm{I}}Q(D_{\mathrm{II}})^{\top} \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^{\top} & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^{\top} \end{pmatrix} - \begin{pmatrix} \frac{3}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^{\top} & 0 \\ 0 & \frac{3}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^{\top} \end{pmatrix}.
\end{aligned}
$$

Since the matrices $\begin{pmatrix} 2Q & 0 \\ 0 & 2Q \end{pmatrix}$ and $\begin{pmatrix} \frac{3}{2}D_{\mathrm{I}}Q(D_{\mathrm{I}})^{\top} & 0 \\ 0 & \frac{3}{2}D_{\mathrm{II}}Q(D_{\mathrm{II}})^{\top} \end{pmatrix}$ are both symmetric and positive semidefinite, the matrix

$$
\begin{pmatrix} 2D_{\mathrm{I}}Q(D_{\mathrm{I}})^{\top} & D_{\mathrm{I}}Q(D_{\mathrm{II}})^{\top} \\ D_{\mathrm{II}}Q(D_{\mathrm{I}})^{\top} & 2D_{\mathrm{II}}Q(D_{\mathrm{II}})^{\top} \end{pmatrix}
$$

is also symmetric and positive semidefinite. ■

Theorem 3.3.1 *If the function*

$$
F_0(x) = F_0(x^{\mathrm{I}}, x^{\mathrm{II}}) := \begin{pmatrix} \nabla_{x^{\mathrm{I}}}\omega_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \nabla_{x^{\mathrm{II}}}\omega_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \end{pmatrix} \tag{3.3.3}
$$

is strictly monotone, and the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has at least one solution, then the solution is unique.

Proof. By Lemma 3.3.1, the matrix

$$
\begin{pmatrix} 2D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^{\top} & D_{\mathrm{I}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^{\top} \\ D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{I}})^{\top} & 2D_{\mathrm{II}}B^{-\frac{1}{2}}PB^{-\frac{1}{2}}(D_{\mathrm{II}})^{\top} \end{pmatrix}
$$

is symmetric and positive semidefinite. It then follows from the definitions of matrices G, H and P that the function

$$
\tilde{F}(x) := \begin{pmatrix} D_{\mathrm{I}}r + 2D_{\mathrm{I}}Gx^{\mathrm{I}} + D_{\mathrm{I}}Hx^{\mathrm{II}} \\ D_{\mathrm{II}}r + D_{\mathrm{II}}Gx^{\mathrm{I}} + 2D_{\mathrm{II}}Hx^{\mathrm{II}} \end{pmatrix}
$$

is monotone. By the given assumption, the function $\hat{F} = F_0 + \tilde{F}$ is therefore strictly monotone. Moreover, by Proposition 2.2.8, if the VI$(X, \hat{F})$ has at least one solution, then the solution is unique. In view of (3.3.2), the definition of function $\hat{F}$, this in turn implies that the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a unique solution by Proposition 2.2.13. ■

Theorem 3.3.2 *If F_0 defined by* (3.3.3) *is strongly monotone, then the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a unique solution.*

Proof. In a similar manner to the proof of Theorem 3.3.1, we can deduce that function $\hat{F} = F_0 + \tilde{F}$ is strongly monotone. Then, by Proposition 2.2.9, the VI$(X, \hat{F})$ has a unique solution. Hence, by Proposition 2.2.13, the NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a unique solution. ∎

Remark 3.3.1 The NEP$(\Theta_\nu, X^\nu)_{\nu=\mathrm{I}}^{\mathrm{II}}$ has a nonempty, compact solution set when function $\hat{F}$ is coercive or X is bounded.

In the end of this section, we show that the L/F Nash equilibrium may be obtained as a stationary point of the natural merit function. For this purpose, by using the merit function, we consider the following constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & \theta_{\mathrm{FB}}(x, \mu, \lambda) \\ \text{subject to} & \lambda \geq 0, \end{array} \tag{3.3.4}$$

where

$$\theta_{\mathrm{FB}}(x, \mu, \lambda) := \frac{1}{2} \Phi_{\mathrm{FB}}(x, \mu, \lambda)^\top \Phi_{\mathrm{FB}}(x, \mu, \lambda),$$

and

$$\Phi_{\mathrm{FB}}(x, \mu, \lambda) := \begin{pmatrix} \hat{L}(x, \mu, \lambda) \\ -h^{\mathrm{I}}(x^{\mathrm{I}}) \\ -h^{\mathrm{II}}(x^{\mathrm{II}}) \\ \psi_{\mathrm{FB}}(-g_{\mathrm{I}}^{\mathrm{I}}(x^{\mathrm{I}}), \lambda_{\mathrm{I}}^{\mathrm{I}}) \\ \vdots \\ \psi_{\mathrm{FB}}(-g_{s_{\mathrm{I}}}^{\mathrm{I}}(x^{\mathrm{I}}), \lambda_{s_{\mathrm{I}}}^{\mathrm{I}}) \\ \psi_{\mathrm{FB}}(-g_{\mathrm{I}}^{\mathrm{II}}(x^{\mathrm{II}}), \lambda_{\mathrm{I}}^{\mathrm{II}}) \\ \vdots \\ \psi_{\mathrm{FB}}(-g_{s_{\mathrm{II}}}^{\mathrm{II}}(x^{\mathrm{II}}), \lambda_{s_{\mathrm{II}}}^{\mathrm{II}}) \end{pmatrix}.$$

By Proposition 2.3.2, we have the following result to guarantee that every stationary point of the natural merit function θ_{FB} is a solution of VI$(X, \hat{F})$, which is also a L/F Nash equilibrium of our multi-leader-follower game.

Theorem 3.3.3 *Let $(x^*, \mu^*, \lambda^*) \in \Re^{n+p+m}$ be a stationary point of* (3.3.4). *Then (x^*, μ^*, λ^*) is a solution of VI$(X, \hat{F})$* (3.3.1) *under the conditions* (a) *and* (b) *stated below:*

(a) *Function F_0 defined by* (3.3.3) *is monotone.*

(b) $\nabla_x L(x^*, \mu^*, \lambda^*)$ *is strictly copositive on the cone*

$$\begin{aligned}
C(x^*, \lambda^*) = {} & C(x^{*,\mathrm{I}}, x^{*,\mathrm{II}}, \lambda^{*,\mathrm{I}}, \lambda^{*,\mathrm{II}}) \\
:= {} & \{\nu \in \Re^n : \nabla h^{\mathrm{I}}(x^{*,\mathrm{I}})^\top \nu^{\mathrm{I}} = 0, \\
& \nabla h^{\mathrm{II}}(x^{*,\mathrm{II}})^\top \nu^{\mathrm{II}} = 0, \\
& \nabla g^{\mathrm{I}}_{i_{\mathrm{I}}}(x^{*,\mathrm{I}})^\top \nu^{\mathrm{I}} \geq 0, \quad \forall i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}}_{00} \cup \mathcal{I}^{\mathrm{I}}_{<}, \\
& \nabla g^{\mathrm{II}}_{i_{\mathrm{II}}}(x^{*,\mathrm{II}})^\top \nu^{\mathrm{II}} \geq 0, \quad \forall i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}}_{00} \cup \mathcal{I}^{\mathrm{II}}_{<}, \\
& \nabla g^{\mathrm{I}}_{i_{\mathrm{I}}}(x^{*,\mathrm{I}})^\top \nu^{\mathrm{I}} \geq 0, \quad \forall i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}}_{+} \cup \mathcal{I}^{\mathrm{I}}_{R}, \\
& \nabla g^{\mathrm{II}}_{i_{\mathrm{II}}}(x^{*,\mathrm{II}})^\top \nu^{\mathrm{II}} \geq 0, \quad \forall i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}}_{+} \cup \mathcal{I}^{\mathrm{II}}_{R}\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I} = \mathcal{I}^{\mathrm{I}} + \mathcal{I}^{\mathrm{II}} &:= \{1, ..., s_{\mathrm{I}}\} \cup \{s_{\mathrm{I}} + 1, ..., s_{\mathrm{I}} + s_{\mathrm{II}}\}, \\
\mathcal{I}^{\mathrm{I}}_0 &:= \{i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}} | g^{\mathrm{I}}_{i_{\mathrm{I}}}(x^{*,\mathrm{I}}) = 0 \leq \lambda^{\mathrm{I}}_{i_{\mathrm{I}}}\}, \\
\mathcal{I}^{\mathrm{II}}_0 &:= \{i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}} | g^{\mathrm{II}}_{i_{\mathrm{II}}}(x^{*,\mathrm{II}}) = 0 \leq \lambda^{\mathrm{II}}_{i_{\mathrm{II}}}\}, \\
\mathcal{I}^{\mathrm{I}}_< &:= \{i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}} | g^{\mathrm{I}}_{i_{\mathrm{I}}}(x^{*,\mathrm{I}}) < 0 = \lambda^{\mathrm{I}}_{i_{\mathrm{I}}}\}, \\
\mathcal{I}^{\mathrm{II}}_< &:= \{i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}} | g^{\mathrm{II}}_{i_{\mathrm{II}}}(x^{*,\mathrm{II}}) < 0 = \lambda^{\mathrm{II}}_{i_{\mathrm{II}}}\}, \\
\mathcal{I}^{\mathrm{I}}_R &:= \mathcal{I}^{\mathrm{I}} \setminus (\mathcal{I}^{\mathrm{I}}_0 \cup \mathcal{I}^{\mathrm{I}}_<), \\
\mathcal{I}^{\mathrm{II}}_R &:= I^{\mathrm{II}} \setminus (\mathcal{I}^{\mathrm{II}}_0 \cup \mathcal{I}^{\mathrm{II}}_<), \\
\mathcal{I}^{\mathrm{I}}_{00} &:= \{i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}}_0 | \lambda^{\mathrm{I}}_{i_{\mathrm{I}}} = 0\}, \\
\mathcal{I}^{\mathrm{II}}_{00} &:= \{i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}}_0 | \lambda^{\mathrm{II}}_{i_{\mathrm{II}}} = 0\}, \\
\mathcal{I}^{\mathrm{I}}_{+} &:= \{i_{\mathrm{I}} \in \mathcal{I}^{\mathrm{I}}_0 | \lambda^{\mathrm{I}}_{i_{\mathrm{I}}} > 0\}, \\
\mathcal{I}^{\mathrm{II}}_{+} &:= \{i_{\mathrm{II}} \in \mathcal{I}^{\mathrm{II}}_0 : \lambda^{\mathrm{II}}_{i_{\mathrm{II}}} > 0\}.
\end{aligned}$$

Proof. Similarly to the proof of Theorem 3.3.1, it is easy to see that the Jacobian matrix $\nabla_x \hat{L}(x, \mu, \lambda)$ is positive semidefinite on $\Re^n$. Note that h^{I} and h^{II} are both affine functions. Then by Proposition 2.3.2, the stationary point (x^*, μ^*, λ^*) is a KKT point of VI$(X, \hat{F})$ (3.3.1), which is also a solution of this VI by the convexity of each problem in the Nash equilibrium formulation of the multi-leader-follower game. Therefore, it is a L/F Nash equilibrium. ■

3.4 The Multi-Leader-Follower Game in Deregulated Electricity Market

Privatization and restructuring of the edregulated electricity markets have taken place in many countries, although an excessive free market also has a possibility to bring about some trouble. Under this situation, a lot of researchers have paid much attention to the noncooperative competition problems in this area, see [13, 54, 80]. In this section, we present a simple model of competitive bidding under some macroeconomic regulation. We will show that it can be formulated as the multi-leader-follower game that we have considered in the previous sections.

In this model, there are several firms and one market maker, called the independent system operator (ISO), who employs a market cleaning mechanism to collect the electricity from firms by paying the bid costs, determine the price of electricity and sell it to consumers. We omit the problem of consumers, which means any quantity of electricity power can be consumed. The structure of the model can be described as follows. Again, for simplicity, we assume there are only two firms I and II. The two firms are competing for market power in an electricity network with M nodes. We assume that firms I and II produce the electricity with fixed quantities a^{I} and a^{II}, respectively, and send it to all nodes. A firm receives its profit by dispatching electricity with the bid parameters to the ISO at each node. Let $\rho^\nu = (\rho_1^\nu, \cdots, \rho_M^\nu)^\top$, $\nu = \mathrm{I}, \mathrm{II}$, denote firm ν's bid parameter vectors, where the components ρ_i^ν are the bid parameters to nodes $i = 1, \cdots, M$. The vector $q = (q^{\mathrm{I}}, q^{\mathrm{II}}) \in \Re^{2M}$ with $q^\nu = (q_1^\nu, \cdots, q_M^\nu)^\top, \nu = \mathrm{I}, \mathrm{II}$, denotes the quantities supplied by the firms, or more specifically, q_i^{I} and q_i^{II} denote the quantities of electricity supplied by firm I and firm II at nodes $i = 1, \cdots, M$, respectively. Each firm ν will submit a bid function $b_\nu(q, \rho^\nu)$ to the ISO. The function b_ν represents how much revenue firm ν will receive by selling the electricity power. At the same time, each firm also needs to consider the transaction cost $\omega_\nu(\rho^\nu) = \frac{1}{2}(\rho^\nu)^\top \mathrm{Diag}(\zeta_1^\nu, \cdots, \zeta_M^\nu)\rho^\nu$, where $\zeta_i^\nu, i = 1, \cdots, M$, are given positive constants.

Each firm ν tries to determine its bid parameter vector ρ^ν by minimizing the difference between its transaction cost and revenue, i.e., by solving the following optimization problem with variables ρ^ν:

$$\begin{aligned} \underset{\rho^\nu}{\text{minimize}} \quad & \frac{1}{2}(\rho^\nu)^\top \mathrm{Diag}(\zeta_1^\nu, \cdots, \zeta_M^\nu)\rho^\nu - b_\nu(q, \rho^\nu) \\ \text{subject to} \quad & 0 \le \rho_i^\nu \le \xi_i^\nu, \ i = 1, \cdots, M, \end{aligned}$$

where ξ_i^ν, $i = 1, \cdots, M$, are positive empirical upper bounds set by the firm $\nu, \nu = \mathrm{I}, \mathrm{II}$. We further assume that the firms' bid functions are given by

$$b_\nu(q, \rho^\nu) := \rho_1^\nu q_1^\nu + \cdots + \rho_M^\nu q_M^\nu = -(\rho^\nu)^\top D_\nu q, \ \nu = \mathrm{I}, \mathrm{II},$$

where $D_\nu \in \Re^{M \times 2M}$ are bid matrices of firm ν, which are defined by $D_{\mathrm{I}} = (-I, \mathbf{0})$ and $D_{\mathrm{II}} = (\mathbf{0}, -I)$. We will write the constraints of the above problem as

$$g^\nu(\rho^\nu) \le 0,$$

where

$$g^\nu(\rho^\nu) = \Big(-\rho_1^\nu, \ \cdots, \ -\rho_M^\nu, \ \rho_1^\nu - \xi_1^\nu, \ \cdots, \ \rho_M^\nu - \xi_M^\nu \Big)^\top.$$

Remark 3.4.1 Pang and Fukushima [80] also considered a multi-leader-follower game with an application in a deregulated electricity market model where the firms are required to bid on their revenue functions. Here we extend their model by considering the transaction cost in each leader's objective function.

Moreover, we assume that some economic interventionism works in this electricity model in order to maintain some equilibrium between the quantities of electricity at each node from two firms, which is represented by some quadratic terms denoted by $\frac{1}{2}\varepsilon_i(\frac{q_i^{\rm I}}{a^{\rm I}} - \frac{q_i^{\rm II}}{a^{\rm II}})^2$, $i = 1, \cdots, M$, where $a^\nu, \nu = {\rm I}, {\rm II}$, are the quantities of electricity produced by firm I, II, respectively, and each economic interventionism parameter ε_i is positive and small. We assume $a^\nu, \nu = {\rm I}, {\rm II}$, are fixed. Under the economic interventionism, the ISO tries to let each firm ν supply the electricity in such a way that the quantity tends to be proportional to his total amount a^ν at each node. At the same time, the ISO employs a market mechanism to determine a set of nodal prices and electricity quantities from each firm at each node in order to maximize its profit (the revenue minus the bid costs), or minimize the negative profit. We further assume that, at each node, the affine demand curves determine the prices p_i as a function of the total quantity of electricity from firms I and II as follows.

$$p_i(q_i^{\rm I}, q_i^{\rm II}) := \alpha_i - \beta_i(q_i^{\rm I} + q_i^{\rm II}),\ i = 1, \cdots, M,$$

where α_i and β_i are given positive constants.

Then the ISO minimizes its negative profit by solving the following optimization problem with variables $q = (q^{\rm I}, q^{\rm II})$:

$$\begin{array}{ll} \underset{q}{\text{minimize}} & \displaystyle\sum_{i=1}^{M} \Big[\frac{\beta_i}{2}(q_i^{\rm I} + q_i^{\rm II})^2 - \alpha_i(q_i^{\rm I} + q_i^{\rm II})\Big] + \frac{1}{2}\sum_{i=1}^{M} \varepsilon_i\Big(\frac{q_i^{\rm I}}{a^{\rm I}} - \frac{q_i^{\rm II}}{a^{\rm II}}\Big)^2 + b_{\rm I}(q, \rho^{\rm I}) + b_{\rm II}(q, \rho^{\rm II}) \\ \text{subject to} & \displaystyle\sum_{i=1}^{M} q_i^{\rm I} - a^{\rm I} = 0, \\ & \displaystyle\sum_{i=1}^{M} q_i^{\rm II} - a^{\rm II} = 0. \end{array}$$

Note that the first two terms of the ISO's objective function is rewritten as

$$\sum_{i=1}^{M} \Big[\frac{\beta_i}{2}(q_i^{\rm I} + q_i^{\rm II})^2 - \alpha_i(q_i^{\rm I} + q_i^{\rm II})\Big] + \frac{1}{2}\sum_{i=1}^{M} \varepsilon_i\Big(\frac{q_i^{\rm I}}{a^{\rm I}} - \frac{q_i^{\rm II}}{a^{\rm II}}\Big)^2 = \frac{1}{2}q^\top Bq + c^\top q,$$

where

$$B = \begin{pmatrix} \beta_1 + \frac{2\varepsilon_1}{(a^{\mathrm{I}})^2} & \cdots & 0 & \beta_1 - \frac{\varepsilon_1}{a^{\mathrm{I}}a^{\mathrm{II}}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_M + \frac{2\varepsilon_M}{(a^{\mathrm{I}})^2} & 0 & \cdots & \beta_M - \frac{\varepsilon_M}{a^{\mathrm{I}}a^{\mathrm{II}}} \\ \beta_1 - \frac{\varepsilon_1}{a^{\mathrm{I}}a^{\mathrm{II}}} & \cdots & 0 & \beta_1 + \frac{2\varepsilon_1}{(a^{\mathrm{II}})^2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_M - \frac{\varepsilon_M}{a^{\mathrm{I}}a^{\mathrm{II}}} & 0 & \cdots & \beta_M + \frac{2\varepsilon_M}{(a^{\mathrm{II}})^2} \end{pmatrix}, \; c = \begin{pmatrix} -\alpha_1 \\ \vdots \\ -\alpha_M \\ -\alpha_1 \\ \vdots \\ -\alpha_M \end{pmatrix}.$$

Notice that matrix B is positive definite. Moreover, the constraints of the ISO can be rewritten as

$$Aq + a = 0,$$

where

$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix} \in \Re^{2\times 2M}, \; a = \begin{pmatrix} -a^{\mathrm{I}} \\ -a^{\mathrm{II}} \end{pmatrix}.$$

Therefore, the electricity market model under consideration can be formulated as the following multi-leader-follower game:

Firm (Leader) I's Problem.

$$\begin{aligned} \underset{\rho^{\mathrm{I}}}{\text{minimize}} \quad & \frac{1}{2}(\rho^{\mathrm{I}})^\top \mathrm{Diag}(\zeta_1^{\mathrm{I}}, \cdots, \zeta_M^{\mathrm{I}})\rho^{\mathrm{I}} + (\rho^{\mathrm{I}})^\top D_{\mathrm{I}} q \\ \text{subject to} \quad & g^{\mathrm{I}}(\rho^{\mathrm{I}}) \le 0. \end{aligned}$$

Firm (Leader) II's Problem.

$$\begin{aligned} \underset{\rho^{\mathrm{II}}}{\text{minimize}} \quad & \frac{1}{2}(\rho^{\mathrm{II}})^\top \mathrm{Diag}(\zeta_1^{\mathrm{II}}, \cdots, \zeta_M^{\mathrm{II}})\rho^{\mathrm{II}} + (\rho^{\mathrm{II}})^\top D_{\mathrm{II}} q \\ \text{subject to} \quad & g^{\mathrm{II}}(\rho^{\mathrm{II}}) \le 0. \end{aligned}$$

ISO (Follower)'s Problem.

$$\begin{aligned} \underset{q}{\text{minimize}} \quad & \frac{1}{2}q^\top Bq + c^\top q - (\rho^{\mathrm{I}})^\top D_{\mathrm{I}} q - (\rho^{\mathrm{II}})^\top D_{\mathrm{II}} q \\ \text{subject to} \quad & Aq + a = 0. \end{aligned}$$

In light of the analysis in the previous sections, we can further reformulate the multi-leader-follower game as the following VI:

Find a vector $\rho^* = (\rho^{*,\mathrm{I}}, \rho^{*,\mathrm{II}}) \in X = X^{\mathrm{I}} \times X^{\mathrm{II}}$ such that

$$\hat{F}(\rho^*)^\top(\rho - \rho^*) \ge 0 \quad \text{for all } \rho \in X, \tag{3.4.1}$$

where

$$\begin{aligned}
&\rho = (\rho^{\mathrm{I}}, \rho^{\mathrm{II}}) = (\rho_1^{\mathrm{I}}, \cdots, \rho_M^{\mathrm{I}}, \rho_1^{\mathrm{II}}, \cdots, \rho_M^{\mathrm{II}})^\top, \\
&X^\nu = \{\rho^\nu : g^\nu(\rho^\nu) \le 0\},\ \nu = \mathrm{I, II}, \\
&g^\nu(\rho^\nu) = (-\rho_1^\nu, \cdots, -\rho_M^\nu, \rho_1^\nu - \xi_1^\nu, \cdots, \rho_M^\nu - \xi_M^\nu)^\top,\ \nu = \mathrm{I, II}, \\
&\hat{F}(\rho) = \begin{pmatrix} \mathrm{Diag}(\zeta_1^{\mathrm{I}}, \cdots, \zeta_M^{\mathrm{I}})\rho^{\mathrm{I}} + D_{\mathrm{I}} r + 2D_{\mathrm{I}} G\rho^{\mathrm{I}} + D_{\mathrm{I}} H\rho^{\mathrm{II}} \\ \mathrm{Diag}(\zeta_1^{\mathrm{II}}, \cdots, \zeta_M^{\mathrm{II}})\rho^{\mathrm{II}} + D_{\mathrm{II}} r + D_{\mathrm{II}} G\rho^{\mathrm{I}} + 2D_{\mathrm{II}} H\rho^{\mathrm{II}} \end{pmatrix}.
\end{aligned}$$

Remark 3.4.2 The assumption that the follower's problem does not contain inequality constraints would narrow down the range of applications of the model. In fact, the approach presented in this paper cannot be extended directly to the inequality constrained case. Nevertheless we may still try to make some further efforts to deal with inequality constraints based on the current approach. Suppose that the follower solves the optimization problem

$$\begin{aligned}
&\underset{q}{\text{minimize}} \quad \frac{1}{2} q^\top B y + c^\top q - (\rho^{\mathrm{I}})^\top D_{\mathrm{I}} q - (\rho^{\mathrm{II}})^\top D_{\mathrm{II}} q \\
&\text{subject to} \quad Aq + a = 0,\ q \ge 0.
\end{aligned}$$

A possible idea to deal with the inequality constraint $q \ge 0$ is the following: First, we just ignore it, apply the above-mentioned VI formulation, and find a L/F Nash equilibrium. Next, we check the components of q in the equilibrium. If they are all nonnegative, then we accept the current equilibrium as a solution of the problem. Otherwise, we set the negative components of q to be 0, or in other words, we discard those components. Then we try to find a L/F Nash equilibrium of the reduced problem by ignoring the inequality constraint $q \ge 0$ again. Repeating this heuristic procedure, we will eventually obtain an approximate L/F Nash equilibrium of the original game.

3.5 Numerical Experiments

In this section, we show numerical results for the electricity market model described in the previous section.

Note that X in (3.4.1) can be represented as $X = \{\rho \in \Re^{2M} | 0 \le \rho_i^\nu \le \xi_i^\nu, i = 1, \cdots, M, \nu = \mathrm{I, II}\}$. Thus the VI is a box-constrained variational inequality (BVI), denoted by the BVI$(X, \hat{F}(\rho))$. To solve the BVI, Kanzow and Fukushima [63] present a nonsmooth Newton-type method applied to the nonlinear equation involving the natural residual of the BVI. The algorithm uses the D-gap function to ensure global convergence of the Newton-type method. We use the following parameter setting in the implementation of Algorithm

Table 3.1: The L/F Nash Equilibrium with Two Firms and Two Nodes

ε	(0.001, 0.001)	(0.005, 0.005)	(0.01, 0.01)	(0.05, 0.05)
ρ^*	(0.4139, 0.4137, 0.6429, 0.6428)	(0.4142, 0.4135, 0.6430, 0.6427)	(0.4146, 0.4132, 0.6432, 0.6426)	(0.4173, 0.4113, 0.6440, 0.6419)
valL1	-0.2483	-0.2483	-0.2483	-0.2485
valL2	-0.5786	-0.5786	-0.5786	-0.5786
q^*	(0.5331, 0.6669, 0.8515, 0.9485)	(0.5330, 0.6670, 0.8512, 0.9488)	(0.5330, 0.6670, 0.8510, 0.9490)	(0.5327, 0.6673, 0.8490, 0.9510)
valF	-1.8425	-1.8424	-1.8424	-1.8422
Iter	2	2	2	2

3.2 in [63]:

$$\alpha = 0.9, \quad \beta = 1.2, \quad \delta = 0.6, \ \omega = 10^{-6},$$
$$\sigma = 10^{-4}, \ p = 2.1, \quad \eta = 0.8, \ \tau = 10^{-6}.$$

In the following experiments shown in Table 3.1 and Table 3.2, valL1, valL2, valL3, and valF denote the optimal values of firm I, firm II, firm III, and the ISO, respectively, and Iter denotes the number of iterations. First we solve a model with two firms I, II, who dispatch the electricity to two nodes $i = 1, 2$. We set the problem data as follows:

$$\alpha_1 = 1.5, \alpha_2 = 1.8; \beta_1 = 0.6, \beta_2 = 0.7; a^{\mathrm{I}} = 1.2, a^{\mathrm{II}} = 1.8;$$
$$\xi_1^{\mathrm{I}} = 1, \xi_2^{\mathrm{I}} = 1, \xi_1^{\mathrm{II}} = 1, \xi_2^{\mathrm{II}} = 1; \zeta_1^{\mathrm{I}} = 1.2; \zeta_2^{\mathrm{I}} = 1, \zeta_1^{\mathrm{II}} = 1.3, \zeta_2^{\mathrm{II}} = 1.5.$$

We also set the problem data $\varepsilon = (\varepsilon_1, \varepsilon_2)$ as listed in Table 3.1, where the corresponding computational results $\rho^* = (\rho_1^{*,\mathrm{I}}, \rho_2^{*,\mathrm{I}}, \rho_1^{*,\mathrm{II}}, \rho_2^{*,\mathrm{II}})^\top$ and $q^* = (q_1^{*,\mathrm{I}}, q_2^{*,\mathrm{I}}, q_1^{*,\mathrm{II}}, q_2^{*,\mathrm{II}})^\top$ along with the objective values of the firms and the ISO are shown.

We may observe from the table that the economic interventionism terms play an important role in the distribution of electricity quantities at each node. As the economic interventionism parameters $\varepsilon_i, i = 1, 2$, become larger, the ratio of electricity quantities supplied by firm I and firm II gets closer to the ratio of the amount of electricity $a^{\mathrm{I}} : a^{\mathrm{II}} = 1 : 1.5$. For example, when the ε changes from $(0.001, 0.001)$ to $(0.05, 0.05)$, the ratio at two nodes changes from $q_1^{*,\mathrm{I}} : q_1^{*,\mathrm{II}} = 1 : 1.5973$ and $q_2^{*,\mathrm{I}} : q_2^{*,\mathrm{II}} = 1 : 1.4223$ to $1 : 1.5938$ and $1 : 1.4251$, respectively. Also in this procedure, both firms' optimal profits are nondecreasing, but the ISO's profit is decreasing.

We also solved a larger electricity market model with three firms and five nodes, where

Table 3.2: The L/F Nash Equilibrium with Three Firms and Five Nodes

ε	(0.001, 0.001, 0.001, 0.001, 0.001)	(0.005, 0.005, 0.005, 0.005, 0.005)	(0.01, 0.01, 0.01, 0.01, 0.01)	(0.05, 0.05, 0.05, 0.05, 0.05)
ρ^*	(0.1799, 0.1789, 0.1788, 0.1791, 0.1791; 0.3296, 0.3285, 0.3285, 0.3287, 0.3288; 0.2440, 0.2430, 0.2429, 0.2432, 0.2432)	(0.1831, 0.1780, 0.1776, 0.1790, 0.1790; 0.3327, 0.3277, 0.3273, 0.3285, 0.3287; 0.2469, 0.2421, 0.2418, 0.2432, 0.2431)	(0.1869, 0.177, 0.1762, 0.1789, 0.1788; 0.3364, 0.3267, 0.3259, 0.3283, 0.3286; 0.2505, 0.2411, 0.2404, 0.2431, 0.2430)	(0.2125, 0.1700, 0.1667, 0.1781, 0.1778; 0.3620, 0.3197, 0.3165, 0.3266, 0.3282; 0.2744, 0.2339, 0.2309, 0.2428, 0.2421)
valL1	-0.1078	-0.1091	-0.1107	-0.1202
valL2	-0.3952	-0.3978	-0.401	-0.4201
valL3	-0.2224	-0.2244	-0.2268	-0.2414
q^*	(0.5334, 0.1327, 0.1511, 0.1837, 0.1990; 1.0716, 0.2572, 0.2886, 0.4082, 0.3744; 0.8488, 0.1936, 0.2133, 0.2533, 0.2911)	(0.5317, 0.1334, 0.1516, 0.1841, 0.1993; 1.0668, 0.2590, 0.2905, 0.4087, 0.3750; 0.8450, 0.1949, 0.2146, 0.2539, 0.2916)	(0.5296, 0.1341, 0.1521, 0.1845, 0.1996; 1.0610, 0.2612, 0.2927, 0.4094, 0.3757; 0.8405, 0.1965, 0.2162, 0.2547, 0.2921)	(0.5154, 0.1394, 0.1559, 0.1875, 0.2018; 1.0217, 0.2756, 0.3079, 0.4141, 0.3808; 0.8098, 0.2072, 0.2267, 0.2602, 0.2960)
valF	-6.4479	-6.4419	-6.4346	-6.3871
Iter	4	23	3	14

the following problem data are used:

$$\alpha_1 = 1.2, \alpha_2 = 1.5, \alpha_3 = 1.6, \alpha_4 = 1.8, \alpha_5 = 1.9; a^{\mathrm{I}} = 1.2, a^{\mathrm{II}} = 2.4, a^{\mathrm{III}} = 1.8;$$
$$\beta_1 = 0.2, \beta_2 = 0.5, \beta_3 = 0.6, \beta_4 = 0.7, \beta_5 = 0.8; \xi_1^{\mathrm{I}} = 1, \xi_2^{\mathrm{I}} = 3, \xi_3^{\mathrm{I}} = 5, \xi_4^{\mathrm{I}} = 2, \xi_5^{\mathrm{I}} = 6;$$
$$\xi_1^{\mathrm{II}} = 1, \xi_2^{\mathrm{II}} = 1, \xi_3^{\mathrm{II}} = 4, \xi_4^{\mathrm{II}} = 1.5, \xi_5^{\mathrm{II}} = 5; \xi_1^{\mathrm{III}} = 1, \xi_2^{\mathrm{III}} = 1.7, \xi_3^{\mathrm{III}} = 1, \xi_4^{\mathrm{III}} = 1, \xi_5^{\mathrm{III}} = 2;$$
$$\zeta_1^{\mathrm{I}} = 1.1, \zeta_2^{\mathrm{I}} = 1.4, \zeta_3^{\mathrm{I}} = 1.7, \zeta_4^{\mathrm{I}} = 1.2, \zeta_5^{\mathrm{I}} = 1.3; \zeta_1^{\mathrm{II}} = 1.2, \zeta_2^{\mathrm{II}} = 1.5, \zeta_3^{\mathrm{II}} = 1.8, \zeta_4^{\mathrm{II}} = 1.5,$$
$$\zeta_5^{\mathrm{II}} = 1.3; \zeta_1^{\mathrm{III}} = 1.3, \zeta_2^{\mathrm{III}} = 1.6, \zeta_3^{\mathrm{III}} = 1.9, \zeta_4^{\mathrm{III}} = 1.2, \zeta_5^{\mathrm{III}} = 1.4.$$

We also set $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ as listed in Table 3.2, where the corresponding numerical results are shown. We may also observe similar properties to those for the pervious model from the table. When the economic interventionism parameters $\varepsilon_i, i = 1, \cdots, 5$, become larger, the ratio of electricity quantities supplied by firms I , II and III gets closer to the ratio of the amount of electricity $a^{\mathrm{I}} : a^{\mathrm{II}} : a^{\mathrm{III}} = 1 : 2 : 1.5$, and the optimal profit of each firm increases, while that of the ISO decreases.

Chapter 4

Smoothing Approach to Nash Equilibrium Formulations for a Class of Equilibrium Problems with Shared Complementarity Constraints

4.1 Introduction

In this chapter, we consider the EPEC, where the shared equilibrium constraints can be reformulated as a linear complementarity system.

As a generalization of the MPEC [69], the EPEC can be formulated as a problem that consists of several MPECs, for which one seeks an equilibrium point that is achieved when all MPECs are optimized simultaneously.

The EPEC can also be looked on as a formulation of a noncooperative multi-leader-follower game [80]. Several researchers have presented some practical applications of the EPEC, such as electricity markets [54, 58, 80]. However, only a few attempts have been made so far to develop numerical methods to solve EPECs, even for a special class of problems.

In the study on optimization, one of important subjects is the approximation of optimization problems. In particular, the notion of epiconvergence of functions plays a fundamental role [61, 83]. Recently, Gürkan and Pang [46] introduced a new notion of epiconvergence, which is called multi-epiconvergence, to study the approximation of Nash equilibrium problems. It is an extension of the notion of epiconvergence to a sequence of families of functions. By means of the new notion, the authors of [46] presented a sufficient condition to ensure the convergence of approximate global Nash equilibria.

In view of the difficulty in computing global Nash equilibria of EPECs that lack convexity, it is reasonable to consider the stationarity in the players' optimization problems.

For example, Hu and Ralph [58] propose two solution concepts called local Nash equilibrium and Nash stationary equilibrium, which are based on local optimality and stationarity, respectively, in each leader's MPEC.

Inspired by their idea and the recent work of Chen and Fukushima [16], in this chapter, we consider a special class of EPECs with shared equilibrium constraints formulated as a linear complementarity system. Under some particular assumptions on the linear complementarity system, we show that the EPEC can be reformulated as an nonsmooth NEP. By means of a smoothing technique, we further construct a sequence of smoothed NEPs to approximate this NEP. We consider two solution concepts, global Nash equilibrium and stationary Nash equilibrium, and establish some results about the convergence of approximate Nash equilibria. Moreover, we present some numerical examples for this special class of EPECs.

The organization of the chapter is as follows. In Section 4.2, we introduce the particular EPEC considered in the chapter, and reformulate it as a NEP. We also introduce a sequence of NEPs by means of a smoothing technique to approximate this NEP. In Section 4.3, we show some conditions that ensure the convergence of approximate global Nash equilibria. We further consider the approximation of stationary Nash equilibria and show some corresponding results about convergence in Section 4.4. Finally, in Section 4.5, we present some numerical examples.

4.2 Equilibrium Problem with Shared P-matrix Linear Complementarity Constraints

In this chapter, we consider a particular class of EPECs, where each player's optimization problem contains a P-matrix linear complementarity constraint that is common to all players. Specifically, for $\nu = 1, \cdots, N$, given the other players' strategies $x^{-\nu}$, player ν solves the following MPEC:

$$\begin{aligned} \underset{x^\nu, y}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y) \\ \text{subject to} \quad & g^\nu(x^\nu) \le 0, \;\; h^\nu(x^\nu) = 0, \\ & \sum_{\mu=1}^{N} K_\mu x^\mu + My + q \ge 0, \;\; y \ge 0, \\ & (\sum_{\mu=1}^{N} K_\mu x^\mu + My + q)^\top y = 0, \end{aligned} \tag{4.2.1}$$

where functions $\theta_\nu : \Re^{n+m} \to \Re$ are continuously differentiable on $\Re^{n+m}$, $g^\nu : \Re^{n_\nu} \to \Re^{s_\nu}$ are continuously differentiable and convex functions, and $h^\nu : \Re^{n_\nu} \to \Re^{t_\nu}$ are affine functions. In the following, we let X^ν denote $X^\nu = \{x^\nu \in \Re^{n_\nu} \mid g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0\}$, which is convex under the given assumptions on g^ν and h^ν. The shared linear complementarity constraints

$$\sum_{\nu=1}^{N} K_\nu x^\nu + My + q \ge 0,\ y \ge 0,\ (\sum_{\nu=1}^{N} K_\nu x^\nu + My + q)^\top y = 0 \tag{4.2.2}$$

are denoted by LCP$(K_1, \cdots, K_N, M, q)$, where $K_\nu \in \Re^{m \times n_\nu}, \nu = 1, \cdots, N$, $M \in \Re^{m \times m}$, and $q \in \Re^m$. In particular, we assume that M is a P-matrix.

Now we define the concept of global Nash equilibria for the above EPEC, denoted by EPEC $\{(4.2.1)\}_{\nu=1}^N$.

Definition 4.2.1 *A tuple of strategies* $(x^*, y^*) = (x^{*,1}, \cdots, x^{*,N}, y^*)$ *is called a global Nash equilibrium of the EPEC* $\{(4.2.1)\}_{\nu=1}^N$*, if for each* ν*, the pair of strategies* $(x^{*,\nu}, y^*)$ *is a global optimal solution for the MPEC* (4.2.1) *with* $x^{-\nu} = x^{*,-\nu}$.

By Proposition 2.2.10, for any fixed $x = (x^\nu)_{\nu=1}^N$, the LCP$(K_1, \cdots, K_N, M, q)$ has a unique solution y, which is denoted $y(x)$ or $y(x^\nu, x^{-\nu})$, i.e., $S(x) = \{y(x)\}$. It is well known that the solution function $y : \Re^n \to \Re^m$ of the shared P-matrix linear complementarity problem (4.2.2) is piecewise linear with respect to the parameter x [69]. Since every piecewise linear function is globally Lipschitz, we have the following proposition.

Proposition 4.2.1 [16] *There is a positive number* γ *such that*

$$\|y(x) - y(x')\| \le \gamma \|x - x'\|, \quad \forall\ x, x' \in \Re^n.$$

We define the functions $\Theta_\nu : \Re^n \to \Re$ by

$$\Theta_\nu(x^\nu, x^{-\nu}) = \theta_\nu(x^\nu, x^{-\nu}, y(x^\nu, x^{-\nu})),\ \nu = 1, \cdots, N.$$

Further, we can reformulate the EPEC $\{(4.2.1)\}_{\nu=1}^N$ as the NEP, denoted by NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$, where each player ν solves the following problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \Theta_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} \quad & g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0. \end{aligned} \tag{4.2.3}$$

Since $y(\cdot)$ is not differentiable, the problem (4.2.3) is a nonsmooth optimization problem for each ν.

Recall that the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$ is equivalent to the EPEC $\{(4.2.1)\}_{\nu=1}^N$ thanks to the property of P-matrix. This is made precisely in the following proposition.

Proposition 4.2.2 *If* x^* *is a global Nash equilibrium of NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$*, then* $(x^*, y(x^*))$ *is a global Nash equilibrium of the EPEC* $\{(4.2.1)\}_{\nu=1}^N$*. Conversely, if* $(x^*, y(x^*))$ *is a global Nash equilibrium of the EPEC* $\{(4.2.1)\}_{\nu=1}^N$*, then* x^* *is a global Nash equilibrium of the NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$.

Proof. To prove the first half, suppose to the contrary that $(x^*, y(x^*))$ is not a global Nash equilibrium of the EPEC $\{(4.2.1)\}_{\nu=1}^N$. Then there exist an index ν as well as vectors $\tilde{x}^\nu$ and $\tilde{y}$ such that

$$\theta_\nu(\tilde{x}^\nu, x^{*,-\nu}, \tilde{y}) < \theta_\nu(x^{*,\nu}, x^{*,-\nu}, y(x^{*,\nu}, x^{*,-\nu})),$$

and $\tilde{x}^\nu \in X^\nu$, $\tilde{y} \in S(\tilde{x}^\nu, x^{*,-\nu})$. Since the solution set $S(\tilde{x}^\nu, x^{*,-\nu})$ is a singleton, we have $\tilde{y} = y(\tilde{x}^\nu, x^{*,-\nu})$, and hence

$$\theta_\nu(\tilde{x}^\nu, x^{*,-\nu}, y(\tilde{x}^\nu, x^{*,-\nu})) < \theta_\nu(x^{*,\nu}, x^{*,-\nu}, y(x^{*,\nu}, x^{*,-\nu})),$$

that is,

$$\Theta_\nu(\tilde{x}^\nu, x^{*,-\nu}) < \Theta_\nu(x^{*,\nu}, x^{*,-\nu}).$$

This contradicts the assumption that x^* is global Nash equilibrium of the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$. Thus the first half is proved.

The second half can be proved in a similar manner, and hence the proof is omitted. ■

It is well known that the complementarity problem (4.2.2) can be reformulated as the following system of nonsmooth equations:

$$\Phi(x, y) := \begin{pmatrix} \psi(y_1, (\sum_{\nu=1}^N K_\nu x^\nu + My + q)_1) \\ \vdots \\ \psi(y_m, (\sum_{\nu=1}^N K_\nu x^\nu + My + q)_m) \end{pmatrix} = 0,$$

where $\psi : \Re^2 \to \Re$ is the FB function that we introduced in Section 2.3.

Here, by the smoothing technique, we use a sequence of continuously differentiable functions $\{\Phi_\mu\}$ involving a scalar parameter $\mu > 0$ to approximate the nonsmooth function $\Phi : \Re^{n+m} \to \Re^m$, where $\Phi_\mu : \Re^{n+m} \to \Re^m$ is defined by

$$\Phi_\mu(x, y) := \begin{pmatrix} \psi_\mu(y_1, (\sum_{\nu=1}^N K_\nu x^\nu + My + q)_1) \\ \vdots \\ \psi_\mu(y_m, (\sum_{\nu=1}^N K_\nu x^\nu + My + q)_m) \end{pmatrix},$$

and $\psi_\mu : \Re^2 \to \Re$ is the SFB function that we introduced in Section 2.3.

By virtue of the P-matrix property of M, for any fixed x, the nonlinear equation

$$\Phi_\mu(x, y) = 0$$

has a unique solution y; see [16], which we denote $y_\mu(x)$. Compared with the solution function y of the original LCP, the function $y_\mu : \Re^n \to \Re^m$ has some desirable properties.

Proposition 4.2.3 [16] *There is a positive constant κ such that for any $x \in \Re^n$ and $\mu > \mu' \geq 0$, we have*

$$\|y_\mu(x) - y_{\mu'}(x)\| \leq \kappa(\mu - \mu').$$

Particularly, when $\mu' = 0$, we have

$$\|y_\mu(x) - y(x)\| \leq \kappa\mu.$$

Proposition 4.2.4 [16] *For any $\mu > 0$, the function y_μ is continuously differentiable on $\Re^n$, and there exists a bounded set $\Omega \subseteq \Re^{n\times m}$, which is independent of μ, such that $\nabla y_\mu(x) \in \Omega$ for all $x \in \Re^n$. Moreover, for any sequence $\{x^k\} \subseteq D_y$ converging to $\bar{x}$, where D_y denotes the set of points at which y is differentiable, there is a positive sequence $\{\mu^k\}$ tending to 0 such that*

$$\left\{\lim_{k\to\infty} \nabla y_{\mu^k}(x^k)\right\} \subseteq \left\{\lim_{\substack{x\to\bar{x}\\ x\in D_y}} \nabla y(x)\right\} \subseteq \partial y(\bar{x}), \tag{4.2.4}$$

where $\partial y(\bar{x})$ is the Clarke generalized Jacobian of $y(\cdot)$ at $\bar{x}$.

In view of this result, we assume that, for any ν and any sequence $\{x^k\} \subseteq D_y$ converging to $\bar{x}$,

$$\left\{\lim_{k\to\infty} \nabla_{x^\nu} y_{\mu^k}(x^k)\right\} \subseteq \partial_{x^\nu} y(\bar{x}), \tag{4.2.5}$$

where $\partial_{x^\nu} y(\bar{x})$ denotes the Clarke generalized partial Jacobian with respect to x^ν of y at $\bar{x}$.

For any $\mu > 0$, we consider a family of functions $\{\hat{\Theta}_\nu\}_{\nu=1}^N$, where the functions $\hat{\Theta}_\nu(\cdot;\mu) : \Re^n \to \Re$ are defined by

$$\hat{\Theta}_\nu(x^\nu, x^{-\nu}; \mu) = \theta_\nu(x^\nu, x^{-\nu}, y_\mu(x^\nu, x^{-\nu})),\ \nu = 1, \cdots, N.$$

By Proposition 4.2.4, the functions $\hat{\Theta}_\nu(\cdot;\mu)$ are continuously differentiable for any $\mu > 0$. Let $\{\mu^k\}_{k\in\mathcal{I}} \subseteq \Re$ be an arbitrary positive sequence such that $\mu^k \to 0$. Then we define a sequence of families of functions $\{\{\Theta_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$ by

$$\Theta_{\nu,k}(x^\nu, x^{-\nu}) := \hat{\Theta}_\nu(x^\nu, x^{-\nu}; \mu^k),\ \nu = 1, \cdots, N. \tag{4.2.6}$$

As an approximation to the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$, we may consider the sequence of NEPs, denoted by $\{\text{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N\}_{k\in\mathcal{I}}$, where each player ν solves the following smooth optimization problem:

$$\begin{aligned} &\underset{x^\nu}{\text{minimize}} && \Theta_{\nu,k}(x^\nu, x^{-\nu}) \\ &\text{subject to} && g^\nu(x^\nu) \leq 0, h^\nu(x^\nu) = 0. \end{aligned} \tag{4.2.7}$$

4.3 Convergence of Approximate Global Nash Equilibria

In this section, we focus on the properties about the convergence of global Nash equilibria for the sequence of NEPs, $\{\mathrm{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N\}_{k\in\mathcal{I}}$, where functions $\Theta_{\nu,k}$ are defined by (4.2.6) for a positive sequence $\{\mu^k\}$ such that $\mu^k \to 0$. We start with the property of the sequence of functions $\{y_{\mu^k}\}$.

Lemma 4.3.1 *The sequence of functions $\{y_{\mu^k}\}_{k\in\mathcal{I}}$ converges continuously to the function y on the set X.*

Proof. By the definition of continuous convergence, we need to show that $y_{\mu^k}(x^k) \to y(x)$ for any sequence $\{x^k\}_{k\in\mathcal{I}} \subseteq X$ converging to $x \in X$. By Proposition 4.2.4 and Proposition 4.2.5,

$$\|y_{\mu^k}(x^k) - y(x)\| \le \|y_{\mu^k}(x^k) - y(x^k)\| + \|y(x^k) - y(x)\| \le \kappa\mu^k + \gamma\|x^k - x\|,$$

and hence we have $y_{\mu^k}(x^k) \to y(x)$ as $k \to \infty$. ∎

Next we show that the sequence of families of functions $\{\{\Theta_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$ multi-epiconverges to $\{\Theta_\nu\}_{\nu=1}^N$ on X. To this end, let $x \in X$ be an arbitrary point and let $\{x^k\} \subseteq X$ be an arbitrary sequence converging to x. Then, for each ν, we define the functions $\Psi_{\nu,k} : \Re^{n_\nu} \to \Re$ and $\Psi_\nu : \Re^{n_\nu} \to \Re$ as follows:

$$\Psi_{\nu,k} := \Theta_{\nu,k}(\cdot, x^{-\nu,k}) \tag{4.3.1}$$

and

$$\Psi_\nu := \Theta_\nu(\cdot, x^{-\nu}). \tag{4.3.2}$$

Then we have the following lemmas about functions $\Psi_{\nu,k}$ and Ψ_ν.

Lemma 4.3.2 *For each ν and k, the functions Ψ_ν and $\Psi_{\nu,k}$ are continuous on the set X.*

Proof. Since $\Psi_{\nu,k} = \Theta_{\nu,k}(\cdot, x^{-\nu,k}) = \theta_\nu(\cdot, x^{-\nu,k}, y_{\mu^k}(\cdot, x^{-\nu,k}))$ and $\Psi_\nu = \Theta_\nu(\cdot, x^{-\nu}) = \theta_\nu(\cdot, x^{-\nu}, y(\cdot, x^{-\nu}))$, the continuity of these functions follows from the continuity of functions θ_ν, y_{μ^k} and y. ∎

Lemma 4.3.3 *For each ν, the sequence of functions $\{\Psi_{\nu,k}\}_{k\in\mathcal{I}}$ converges continuously to the function Ψ_ν on the set X^ν.*

Proof. By the definition of continuous convergence, we need to show that for any $\tilde{x}^\nu \in X^\nu$ and any sequence $\{\tilde{x}^{\nu,k}\}_{k\in\mathcal{I}} \in X^\nu$ such that $\tilde{x}^{\nu,k} \to \tilde{x}^\nu$, we have $\Psi_{\nu,k}(\tilde{x}^{\nu,k}) \to \Psi_\nu(\tilde{x}^\nu)$, i.e., $\theta_\nu(\tilde{x}^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(\tilde{x}^{\nu,k}, x^{-\nu,k})) \to \theta_\nu(\tilde{x}^\nu, x^{-\nu}, y(\tilde{x}^\nu, x^{-\nu}))$. Since $x^{-\nu,k} \to x^{-\nu}$ and $\mu^k \to 0$, it follows from Lemma 4.3.1 that $y_{\mu^k}(\tilde{x}^{\nu,k}, x^{-\nu,k}) \to y(\tilde{x}^\nu, x^{-\nu})$, and hence $\theta_\nu(\tilde{x}^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(\tilde{x}^{\nu,k}, x^{-\nu,k})) \to \theta_\nu(\tilde{x}^\nu, x^{-\nu}, y(\tilde{x}^\nu, x^{-\nu}))$ by the continuity of the function θ_ν. ■

By Proposition 2.2.6, Lemma 4.3.3 means that the sequence of functions $\{\Psi_{\nu,k}\}_{k\in\mathcal{I}}$ epiconverges to the function Ψ_ν. Thus we have the following lemma.

Lemma 4.3.4 *The sequence of families of functions $\{\{\Theta_{\nu,k}\}_{\nu=1}^N\}_{k\in\mathcal{I}}$ multi-epiconverges to the family of functions $\{\Theta_\nu\}_{\nu=1}^N$.*

Proof. By Proposition 2.2.7, it is sufficient to show that for every $\nu = 1, \cdots, N$ and every sequence $\{x^{-\nu,k}\}_{k\in\mathcal{I}} \subseteq X^{-\nu}$ converging to some $x^{-\nu} \in X^{-\nu}$, the sequence of functions $\{\Psi_{\nu,k}\}_{k\in\mathcal{I}}$ defined by (4.3.1) epiconverges to the function Ψ_ν defined by (4.3.2). It is true as mentioned just after Lemma 4.3.3. ■

Based on this result, we establish convergence of the approximate global Nash equilibria.

Theorem 4.3.1 *Let $\{x^k\}_{k\in\mathcal{I}} \subseteq X$ be a sequence which satisfies that each x^k is a global Nash equilibrium of the NEP$(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$. If $\{x^k\}_{k\in\mathcal{I}}$ converges to x^*, then x^* is a global Nash equilibrium of the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$.*

Proof. Follows immediately from Lemma 4.3.4 and Proposition 2.2.14. ■

4.4 Convergence of Approximate Stationary Nash Equilibria

In this section, we further investigate the behavior of a sequence of stationary Nash equilibria for $\{\text{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N\}_{k\in\mathcal{I}}$. This is important from the numerical point of view, since in practice we may only expect to compute a stationary point of each player's optimization problem.

To this end, we need to introduce some stationary concepts which are associated with the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$. First, we notice that the function y is directionally differentiable everywhere [20, Theorem 7.4.2].

Based on the concepts of B-stationary point and C-stationary point for a nonsmooth optimization problem, we introduce the corresponding concepts for a nonsmooth NEP as follows.

We call $x \in X$ a *Bouligand stationary (B-stationary) Nash equilibrium* [16] of $\mathrm{NEP}(\Theta_\nu, X^\nu)_{\nu=1}^N$ if for each $\nu = 1, \cdots, N$,

$$\begin{aligned}\Theta'_{\nu,x^\nu}(x^\nu, x^{-\nu}; d^\nu) &= \nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu}, y(x^\nu, x^{-\nu}))^\top d^\nu \\ &\quad + \nabla_y \theta_\nu(x^\nu, x^{-\nu}, y(x^\nu, x^{-\nu}))^\top y'_{x^\nu}(x^\nu, x^{-\nu}; d^\nu) \geq 0,\end{aligned} \tag{4.4.1}$$

for all $d^\nu \in \mathcal{T}(x^\nu; X^\nu)$.

Associated with the players' optimization problems, we introduce the following Mangasarian-Fromovitz constraint qualification (MFCQ): For each $\nu = 1, \cdots, N$, $\{\nabla h_j^\nu(x^\nu) \mid j = 1, \cdots, s_\nu\}$ is a linearly independent set and there exists a vector $d^\nu \in \Re^{n_\nu}$ such that

$$\nabla g_i^\nu(x^\nu)^\top d^\nu < 0, i \in \mathcal{I}(x^\nu),\ \nabla h_j^\nu(x^\nu)^\top d^\nu = 0, j = 1, \cdots, s_\nu,$$

where $\mathcal{I}(x^\nu) = \{i \mid g_i^\nu(x^\nu) = 0\}$.

Under the MFCQ, the tangent cone $\mathcal{T}(x^\nu; X^\nu)$ can be represented precisely as follows:

$$\mathcal{T}(x^\nu; X^\nu) = \{d^\nu \mid \nabla g_i^\nu(x^\nu)^\top d^\nu \leq 0, i \in \mathcal{I}(x^\nu),\ \nabla h_j^\nu(x^\nu)^\top d^\nu = 0, j = 1, \cdots, s_\nu\}.$$

Then, $x = (x^\nu)_{\nu=1}^N \in \Re^n$ is said to be a *Clarke stationary (C-stationary) Nash equilibrium* for $\mathrm{NEP}(\Theta_\nu, X^\nu)_{\nu=1}^N$ if for each ν, $x^\nu \in \Re^{n_\nu}$ together with some Lagrange multipliers $(\lambda^\nu, \eta^\nu) \in \Re^{s_\nu} \times \Re^{t_\nu}$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned}\partial_{x^\nu}\Theta_\nu(x^\nu, x^{-\nu}) + \nabla g^\nu(x^\nu)\lambda^\nu + \nabla h^\nu(x^\nu)\eta^\nu &\ni 0, \\ g^\nu(x^\nu) \leq 0, \lambda^\nu \geq 0, (\lambda^\nu)^T g^\nu(x^\nu) &= 0, \\ h^\nu(x^\nu) &= 0,\end{aligned} \tag{4.4.2}$$

where $\partial_{x^\nu}\Theta_\nu(x^\nu, x^{-\nu})$ is the Clarke generalized gradient of $\Theta_\nu(\cdot, x^{-\nu})$ at x^ν.

Since function y_μ is continuously differentiable on $\Re^n$ for any $\mu > 0$, the B-stationarity condition for the $\mathrm{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$ can be written as follows:

$$\begin{aligned}\nabla_{x^\nu}\Theta_{\nu,k}(x^\nu, x^{-\nu})^T d^\nu =& \nabla_{x^\nu}\theta_{\nu,k}(x^\nu, x^{-\nu}, y_{\mu^k}(x^\nu, x^{-\nu}))^T d^\nu \\ &+ \nabla_y\theta_{\nu,k}(x^\nu, x^{-\nu}, y_{\mu^k}(x^\nu, x^{-\nu}))^\top \nabla_{x^\nu} y_{\mu^k}(x^\nu, x^{-\nu})^T d^\nu \geq 0,\end{aligned} \tag{4.4.3}$$

for all $d^\nu \in \mathcal{T}(x^\nu; X^\nu)$.

Moreover, for each k, we say $x = (x^\nu)_{\nu=1}^N \in R^n$ is a KKT point of $\mathrm{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$, if for each ν, $x^\nu \in R^{n_\nu}$ together with some $(\lambda^\nu, \eta^\nu) \in R^{s_\nu} \times R^{t_\nu}$ satisfies the following system:

$$\begin{aligned}\nabla_{x^\nu}\Theta_{\nu,k}(x^\nu, x^{-\nu}) + \nabla g^\nu(x^\nu)\lambda^\nu + \nabla h^\nu(x^\nu)\eta^\nu &= 0, \\ g^\nu(x^\nu) \leq 0, \lambda^\nu \geq 0, (\lambda^\nu)^T g^\nu(x^\nu) &= 0, \\ h^\nu(x^\nu) &= 0.\end{aligned} \tag{4.4.4}$$

Note that under the MFCQ, conditions (4.4.3) and (4.4.4) are equivalent.

Now we establish the following result about the convergence of a sequence of KKT points of smoothed problems $\{\mathrm{NEP}(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N\}_{k\in\mathcal{I}}$ to a C-stationary Nash equilibrium of $\mathrm{NEP}(\Theta_\nu, X^\nu)_{\nu=1}^N$.

Theorem 4.4.1 *Suppose that the feasible set* $X = \{x = (x^\nu)_{\nu=1}^N \in \Re^n | g^\nu(x^\nu) \le 0, h^\nu(x^\nu) = 0, \nu = 1, \cdots, N\}$ *of the NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$ *is bounded and the MFCQ holds at any* $x \in X$. *Let* $\{x^k\}_{k\in\mathcal{I}}$ *be a sequence of B-stationary Nash equilibria of the family of smoothed problems* $\{NEP(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N\}_{k\in\mathcal{I}}$. *Moreover, we assume that* (4.2.5) *holds. Then every accumulation point* $\bar{x}$ *of the sequence* $\{x^k\}_{k\in\mathcal{I}}$ *is a C-stationary Nash equilibrium of the NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$.

Proof. Since under the MFCQ, a B-stationary Nash equilibrium and a KKT point are equivalent for the NEP$(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$. Therefore, for each k and ν, along with some corresponding Lagrange multipliers pair $(\lambda^{\nu,k}, \eta^{\nu,k})$, we have from (4.4.4)

$$\begin{aligned}
&\nabla_{x^\nu} y_{\mu^k}(x^{\nu,k}, x^{-\nu,k}) \nabla_y \theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k})) \\
&+\nabla_{x^\nu} \theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k})) + \nabla g^\nu(x^{\nu,k})\lambda^{\nu,k} + \nabla h^\nu(x^{\nu,k})\eta^{\nu,k} = 0, \\
&\qquad g^\nu(x^{\nu,k}) \le 0, \lambda^{\nu,k} \ge 0, (\lambda^{\nu,k})^\top g^\nu(x^{\nu,k}) = 0, \\
&\qquad h^\nu(x^{\nu,k}) = 0.
\end{aligned} \tag{4.4.5}$$

Note that the sequence $\{x^k\}$ is bounded since the feasible set is bounded. Let $\bar{x}$ be an accumulation point of $\{x^k\}$. For each $\nu = 1, \cdots, N$, by the continuity of functions $\nabla_{x^\nu}\theta_\nu$, $\nabla_y \theta_\nu$, ∇g^ν, ∇h^ν and y along with Proposition 4.2.3, we have

$$\begin{aligned}
\lim_{k\to\infty} \nabla_{x^\nu} \theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k})) &= \nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})), \\
\lim_{k\to\infty} \nabla_y \theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k})) &= \nabla_y \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})), \\
\lim_{k\to\infty} \nabla g^\nu(x^{\nu,k}) &= \nabla g^\nu(\bar{x}^\nu), \\
\lim_{k\to\infty} \nabla h^\nu(x^{\nu,k}) &= \nabla h^\nu(\bar{x}^\nu).
\end{aligned}$$

By (4.2.5), for each ν there exist a matrix $V^\nu \in \partial_{x^\nu} y(\bar{x})$ and a subsequence of $\{x^k\}$, still denoted by $\{x^k\}$ for simplicity, such that $x^k \to \bar{x}$ and

$$\lim_{k\to\infty} \nabla_{x^\nu} y_{\mu^k}(x^{\nu,k}, x^{-\nu,k}) = V^\nu \in \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu}).$$

Since the MFCQ holds at $\bar{x}$, it is not difficult to show that the sequence $\{(\lambda^k, \mu^k)\}_{k\in\mathcal{I}}$ is bounded. Without loss of generality, we assume that $(x^k, \lambda^k, \mu^k) \to (\bar{x}, \bar{\lambda}, \bar{\mu})$.

Then, for each $\nu = 1, \cdots, N$, it follows from (4.4.5) that

$$\begin{aligned}
&V^\nu \nabla_y \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) + \nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) \\
&\qquad + \nabla g^\nu(\bar{x}^\nu)\bar{\lambda}^\nu + \nabla h^\nu(\bar{x}^\nu)\bar{\eta}^\nu = 0, \\
&\qquad g^\nu(\bar{x}^\nu) \le 0, \bar{\lambda}^\nu \ge 0, (\bar{\lambda}^\nu)^T g^\nu(\bar{x}^\nu) = 0, \\
&\qquad h^\nu(\bar{x}^\nu) = 0.
\end{aligned} \tag{4.4.6}$$

By the continuous differentiability of function θ_ν for each ν, the Jacobian chain rule [18, Theorem 2.6.6] yields

$$\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) + \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu})\nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) = \partial_{x^\nu}\Theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}).$$

This along with (4.4.6) means that (4.4.2) holds for all $\nu = 1, \cdots, N$ with $(x, \lambda, \mu) = (\bar{x}, \bar{\lambda}, \bar{\mu})$. Therefore, $\bar{x}$ is a C-stationary Nash equilibrium of NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$. ∎

Now we consider some properties about the B-stationary Nash equilibria.

Lemma 4.4.1 *Let $\bar{x} \in X$ be an accumulation point of the sequence $\{x^k\}$, where each x^k is a B-stationary Nash equilibrium of the NEP$(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$. Assume that* (4.2.5) *holds and so does the MFCQ at $\bar{x}$. Then for each $\nu = 1, \cdots, N$, there exists a matrix $V^\nu \in \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu})$ such that*

$$\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^T d^\nu + \nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^\top (V^\nu)^\top d^\nu \geq 0,$$

for all $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$.

Proof. Since each x^k is a B-stationary Nash equilibrium of the NEP$(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$, we have

$$\begin{aligned}\nabla_{x^\nu}\Theta_{\nu,k}(x^{\nu,k}, x^{-\nu,k})^T d^{\nu,k} &= \nabla_{x^\nu}\theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k}))^\top d^{\nu,k} \\ &\quad + \nabla_y\theta_\nu(x^{\nu,k}, x^{-\nu,k}, y_{\mu^k}(x^{\nu,k}, x^{-\nu,k}))^\top \nabla_{x^\nu} y_{\mu^k}(x^{\nu,k}, x^{-\nu,k})^\top d^{\nu,k} \\ &\geq 0\end{aligned}$$

for all $d^{\nu,k} \in \mathcal{T}(x^{\nu,k}; X^\nu)$.

For each $\nu = 1, \cdots, N$, since $\bar{x}$ is an accumulation point of $\{x^k\}$, by (4.2.5), there exist a matrix $V^\nu \in \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu})$ and a subsequence of $\{x^{\nu,k}\}$, still denoted by $\{x^{\nu,k}\}$ for simplicity, which converges to $\bar{x}^\nu$, such that

$$\lim_{k\to\infty} \nabla_{x^\nu} y_{\mu^k}(x^{\nu,k}, x^{-\nu,k}) = V^\nu \in \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu}).$$

Further, for any $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$, by the MFCQ, there exists a sequence $\{d^{\nu,k}\}$ such that $d^{\nu,k} \to d^\nu$ as $k \to \infty$. Then, by passing to the limit $k \to \infty$ in (4.4.3), we can deduce that

$$\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^\top d^\nu + \nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^\top (V^\nu)^\top d^\nu \geq 0.$$

Since this holds for all $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$, the desired result follows. ∎

In the light of Lemma 4.4.1, we show a convergence result about B-stationary Nash equilibria.

Theorem 4.4.2 *Suppose that the feasible set* $X = \{x = (x^\nu)_{\nu=1}^N \in R^n | g^\nu(x^\nu) \leq 0, h^\nu(x^\nu) = 0, \nu = 1, \cdots, N\}$ *of the NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$ *is bounded, the MFCQ holds at any* $x \in X$, *and each objective function* Θ_ν *is regular with respect to its own variable* $x^\nu, \nu = 1, \cdots, N$. *Let* $\{x^k\}$ *be a sequence of B-stationary Nash equilibria of the NEP*$(\Theta_{\nu,k}, X^\nu)_{\nu=1}^N$, *and assume that* (4.2.5) *holds. Then every accumulation point of* $\{x^k\}$ *is a B-stationary Nash equilibrium of the NEP*$(\Theta_\nu, X^\nu)_{\nu=1}^N$.

Proof. The sequence $\{x^k\}$ is bounded since X is bounded. Let $\bar{x}$ be an arbitrary accumulation point of $\{x^k\}$. Then, by Lemma 4.4.1, for each $\nu = 1, \cdots, N$, there exists a matrix $V^\nu \in \partial_{x^\nu} y(\bar{x}^\nu, \bar{x}^{-\nu})$ such that

$$\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^\top d^\nu + \nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^\top (V^\nu)^\top d^\nu \geq 0 \tag{4.4.7}$$

for all $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$. Moreover, by the Jacobian chain rule [18, Theorem 2.6.6], we have

$$\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) + V^\nu \nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu})) \in \partial_{x^\nu}\Theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}). \tag{4.4.8}$$

Further, by the regularity of Θ_ν and [18, Proposition 2.1.2 (b)], we also have

$$\Theta'_{\nu,x^\nu}(\bar{x}^\nu, \bar{x}^{-\nu}; d^\nu) = \max\{(\xi^\nu)^T d^\nu : \xi^\nu \in \partial_{x^\nu}\Theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu})\}$$

for all $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$.

Therefore, in view of (4.4.7) and (4.4.8), we can deduce that

$$\begin{aligned}\Theta'_{\nu,x^\nu}(\bar{x}^\nu, \bar{x}^{-\nu}; d^\nu) \geq &\nabla_{x^\nu}\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^T d^\nu \\ &+ \nabla_y\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}, y(\bar{x}^\nu, \bar{x}^{-\nu}))^T (V^\nu)^T d^\nu \geq 0\end{aligned}$$

for all $d^\nu \in \mathcal{T}(\bar{x}^\nu; X^\nu)$. Since this holds for all $\nu = 1, \cdots, N$, $\bar{x}$ is a B-stationary Nash equilibrium of the NEP$(\Theta_\nu, X^\nu)_{\nu=1}^N$. ■

4.5 Numerical Experiments

In this section, we show some numerical results for the class of EPECs described in Section 4.2. Specifically, we consider the following EPEC with two players, which contains a shared P-matrix linear complementarity constraint parameterized by the upper level variables $x = (x^{\mathrm{I}}, x^{\mathrm{II}}) \in \Re^n$:

Player I's problem:

$$\begin{aligned}\underset{x^{\mathrm{I}},y}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top G_{\mathrm{I}} x^{\mathrm{II}} + (c^{\mathrm{I}})^\top y \\ \text{subject to} \quad & A_{\mathrm{I}} x^{\mathrm{I}} \le b^{\mathrm{I}}, \\ & My + N_{\mathrm{I}} x^{\mathrm{I}} + N_{\mathrm{II}} x^{\mathrm{II}} + q \ge 0, \ y \ge 0, \\ & y^\top (My + N_{\mathrm{I}} x^{\mathrm{I}} + N_{\mathrm{II}} x^{\mathrm{II}} + q) = 0.\end{aligned}$$

Player II's problem:

$$\begin{aligned}\underset{x^{\mathrm{II}},y}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top G_{\mathrm{II}} x^{\mathrm{I}} + (c^{\mathrm{II}})^\top y \\ \text{subject to} \quad & A_{\mathrm{II}} x^{\mathrm{II}} \le b^{\mathrm{II}}, \\ & My + N_{\mathrm{I}} x^{\mathrm{I}} + N_{\mathrm{II}} x^{\mathrm{II}} + q \ge 0, \ y \ge 0, \\ & y^\top (My + N_{\mathrm{I}} x^{\mathrm{I}} + N_{\mathrm{II}} x^{\mathrm{II}} + q) = 0.\end{aligned}$$

We assume that $M \in \Re^{m\times m}$ is a P-matrix, $H_\nu \in \mathcal{S}^{n_\nu}_{++}$, and $G_\nu \in \Re^{n_\nu \times n_{-\nu}}, q \in \Re^m, c^\nu \in \Re^m, A_\nu \in \Re^{s_\nu \times n_\nu}, b^\nu \in \Re^{s_\nu}, N_\nu \in \Re^{m\times n_\nu}$, $\nu = \mathrm{I}, \mathrm{II}$.

In light of the analysis about the P-matrix LCP in the previous sections, after substituting $y(x) = y(x^{\mathrm{I}}, x^{\mathrm{II}})$ for y in the objective functions of two players, we can further reformulate this EPEC as the following nonsmooth NEP:

Player I's problem:

$$\begin{aligned}\underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top G_{\mathrm{I}} x^{\mathrm{II}} + (c^{\mathrm{I}})^\top y(x) \\ \text{subject to} \quad & A_{\mathrm{I}} x^{\mathrm{I}} \le b^{\mathrm{I}}.\end{aligned}$$

Player II's problem:

$$\begin{aligned}\underset{x^{\mathrm{II}}}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top G_{\mathrm{II}} x^{\mathrm{I}} + (c^{\mathrm{II}})^\top y(x) \\ \text{subject to} \quad & A_{\mathrm{II}} x^{\mathrm{II}} \le b^{\mathrm{II}}.\end{aligned}$$

Remark 4.5.1 Since we replace y by $y(x)$ in the objective functions of two players, the two vector variables (x^ν, y) of player ν in the above EPEC reduce to one vector variable x^ν in the nonsmooth NEP.

Moreover, we have the following sequence of smoothed approximations to the above NEP:

Player I's problem:

$$\begin{aligned}\underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top G_{\mathrm{I}} x^{\mathrm{II}} + (c^{\mathrm{I}})^\top y_{\mu^k}(x) \\ \text{subject to} \quad & A_{\mathrm{I}} x^{\mathrm{I}} \le b^{\mathrm{I}}.\end{aligned}$$

Player II's problem:

$$\begin{array}{ll} \underset{x^{\mathrm{II}}}{\text{minimize}} & \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top G_{\mathrm{II}} x^{\mathrm{I}} + (c^{\mathrm{II}})^\top y_{\mu^k}(x) \\ \text{subject to} & A_{\mathrm{II}} x^{\mathrm{II}} \le b^{\mathrm{II}}. \end{array}$$

By concatenating their KKT systems, we have the following mixed CP for each k:

$$\begin{aligned} H_{\mathrm{I}} x^{\mathrm{I}} + G_{\mathrm{I}} x^{\mathrm{II}} + \nabla_{x^{\mathrm{I}}} y_{\mu^k}(x) c^{\mathrm{I}} + A_{\mathrm{I}}^\top \lambda^{\mathrm{I}} = 0, \\ H_{\mathrm{II}} x^{\mathrm{II}} + G_{\mathrm{II}} x^{\mathrm{I}} + \nabla_{x^{\mathrm{II}}} y_{\mu^k}(x) c^{\mathrm{II}} + A_{\mathrm{II}}^\top \lambda^{\mathrm{II}} = 0, \\ -A_{\mathrm{I}} x^{\mathrm{I}} + b^{\mathrm{I}} \ge 0, \lambda^{\mathrm{I}} \ge 0, (-A_{\mathrm{I}} x^{\mathrm{I}} + b^{\mathrm{I}})^\top \lambda^{\mathrm{I}} = 0, \\ -A_{\mathrm{II}} x^{\mathrm{II}} + b^{\mathrm{II}} \ge 0, \lambda^{\mathrm{II}} \ge 0, (-A_{\mathrm{II}} x^{\mathrm{II}} + b^{\mathrm{II}})^\top \lambda^{\mathrm{II}} = 0, \end{aligned}$$

where $\lambda^\nu \in \Re^{s_\nu}, \nu = \mathrm{I}, \mathrm{II}$.

It is well-known that a mixed CP is equivalent to a box constrained variational inequality problem (BVIP) [32]. Consequently, to deal with the above mixed CP, we consider the following BVIP: Find a vector $z = (x, \lambda) \in \mathbb{B}$ such that

$$F_{\mu^k}(z)^\top (z' - z) \ge 0 \quad \text{for all } z' = (x', \lambda') \in \mathbb{B}, \tag{4.5.1}$$

where $\mathbb{B} = \{z \in \Re^{n_{\mathrm{I}}+n_{\mathrm{II}}+s_{\mathrm{I}}+s_{\mathrm{II}}} | l_i \le z_i \le u_i, l_i = -\infty, i = 1, \cdots, n_{\mathrm{I}} + n_{\mathrm{II}}; l_i = 0, i = n_{\mathrm{I}} + n_{\mathrm{II}} + 1, \cdots, n_{\mathrm{I}} + n_{\mathrm{II}} + s_{\mathrm{I}} + s_{\mathrm{II}}; u_i = +\infty, i = 1, \cdots, n_{\mathrm{I}} + n_{\mathrm{II}} + s_{\mathrm{I}} + s_{\mathrm{II}}\}$, and

$$F_{\mu^k}(z) = \begin{pmatrix} H_{\mathrm{I}} x^{\mathrm{I}} + G_{\mathrm{I}} x^{\mathrm{II}} + \nabla_{x^{\mathrm{I}}} y_{\mu^k}(x) c^{\mathrm{I}} + A_{\mathrm{I}}^\top \lambda^{\mathrm{I}} \\ H_{\mathrm{II}} x^{\mathrm{II}} + G_{\mathrm{II}} x^{\mathrm{I}} + \nabla_{x^{\mathrm{II}}} y_{\mu^k}(x) c^{\mathrm{II}} + A_{\mathrm{II}}^\top \lambda^{\mathrm{II}} \\ -A_{\mathrm{I}} x^{\mathrm{I}} + b^{\mathrm{I}} \\ -A_{\mathrm{II}} x^{\mathrm{II}} + b^{\mathrm{II}} \end{pmatrix}.$$

The Jacobian matrix of function F_{μ^k} can be written as

$$\nabla F_{\mu^k}(z) = \begin{pmatrix} \begin{pmatrix} H_{\mathrm{I}} & G_{\mathrm{II}}^\top \\ G_{\mathrm{I}}^\top & H_{\mathrm{II}} \end{pmatrix} + \nabla_x \begin{pmatrix} \nabla_{x^{\mathrm{I}}} y_{\mu^k}(x) c^{\mathrm{I}} \\ \nabla_{x^{\mathrm{II}}} y_{\mu^k}(x) c^{\mathrm{II}} \end{pmatrix}, & \begin{pmatrix} -A_{\mathrm{I}}^\top & 0 \\ 0 & -A_{\mathrm{II}}^\top \end{pmatrix} \\ \begin{pmatrix} A_{\mathrm{I}} & 0 \\ 0 & A_{\mathrm{II}} \end{pmatrix}, & 0 \end{pmatrix}.$$

To solve the BVIP, Kanzow and Fukushima [63] present a Newton-type method applied to the nonsmooth equation involving the natural residual of the BVIP. The algorithm uses the D-gap function to ensure global convergence of the Newton-type method. To solve the BVIP (22), we use Algorithm 3.2 in [63] with the following parameter setting:

$$\begin{aligned} \alpha = 0.9, \quad \beta = 1.1, \quad \delta = 0.6, \ \rho = 10^{-7}, \\ \sigma = 10^{-5}, \ p = 2.1, \quad \eta = 0.9, \ \epsilon = 10^{-6}. \end{aligned}$$

Table 4.1: Computational Results of Approximate Stationary Nash Equilibria for Example 4.5.1

μ^k	0.01	0.001	0.0001	0.00001
x^k	$\begin{pmatrix} -0.26175308 \\ 0.806760257 \\ 2.694216811 \\ -0.364020746 \end{pmatrix}$	$\begin{pmatrix} -0.261753277 \\ 0.806760076 \\ 2.694217254 \\ -0.364019171 \end{pmatrix}$	$\begin{pmatrix} -0.261752853 \\ 0.806760137 \\ 2.694217618 \\ -0.36401958 \end{pmatrix}$	$\begin{pmatrix} -0.261753229 \\ 0.806760144 \\ 2.694217171 \\ -0.364018987 \end{pmatrix}$
y^k	$\begin{pmatrix} 7.153482977 \\ 8.519043099 \end{pmatrix}$	$\begin{pmatrix} 7.153476511 \\ 8.519035193 \end{pmatrix}$	$\begin{pmatrix} 7.153477701 \\ 8.519037369 \end{pmatrix}$	$\begin{pmatrix} 7.153476063 \\ 8.519034592 \end{pmatrix}$
Valgap	3.43e-11	4.25e-11	3.34e-11	4.58e-11

Example 4.5.1 The problem data are given as follows:

$$H_{\mathrm{I}} = \begin{pmatrix} 3.6 & -1.2 \\ -1.5 & 2.8 \end{pmatrix}, H_{\mathrm{II}} = \begin{pmatrix} 7.5 & -2.6 \\ -2.6 & 5.7 \end{pmatrix}, G_{\mathrm{I}} = \begin{pmatrix} 1.1 & -1.3 \\ -2.4 & 1.6 \end{pmatrix},$$

$$G_{\mathrm{II}} = \begin{pmatrix} -1.2 & 2.3 \\ 1.4 & -2.5 \end{pmatrix}, M = \begin{pmatrix} 3.6 & -1.2 \\ -1.2 & 2.8 \end{pmatrix}, N_{\mathrm{I}} = \begin{pmatrix} 2.1 & -1.3 \\ -3.4 & 2.3 \end{pmatrix},$$

$$N_{\mathrm{II}} = \begin{pmatrix} -5.4 & 1.6 \\ -6.2 & 2.1 \end{pmatrix}, q = \begin{pmatrix} 1.2 \\ 1.6 \end{pmatrix}, c^{\mathrm{I}} = \begin{pmatrix} -2.3 \\ -3.2 \end{pmatrix}, c^{\mathrm{II}} = \begin{pmatrix} -2.5 \\ -2.4 \end{pmatrix},$$

$$A_{\mathrm{I}} = \begin{pmatrix} 3.3 & -2.4 \end{pmatrix}, A_{\mathrm{II}} = \begin{pmatrix} -2.5 & 2.1 \end{pmatrix}, b^{\mathrm{I}} = -2.8, b^{\mathrm{II}} = -7.5.$$

The computed solutions $x^k = (x_1^{k,\mathrm{I}}, x_2^{k,\mathrm{I}}, x_1^{k,\mathrm{II}}, x_2^{k,\mathrm{II}})^\top$ and $y^k = (y_{\mu^k,1}(x^k), y_{\mu^k,2}(x^k))^\top$ of the sequence of smoothed NEPs with $\mu^k = 10^{-k-1}, k = 1,2,3,4$ along with the corresponding values of the D-gap functions denoted by Valgap are shown in Table 4.1.

We confirm that these approximate stationary Nash equilibria of smoothed NEPs converge to a B-stationary Nash equilibrium of the original NEP as μ^k tends to 0. In fact, since $y(x)$ is differentiable at x^k with $\mu^k = 0.00001$, we can check the KKT conditions for each player's problem directly at $x = x^k$ as follows:

For Player I.

$$H_{\mathrm{I}}x^{\mathrm{I}} + G_{\mathrm{I}}x^{\mathrm{II}} + \nabla_{x^{\mathrm{I}}}y(x)c^{\mathrm{I}} + A_{\mathrm{I}}^\top \lambda^{\mathrm{I}} = \begin{pmatrix} 4.56\text{e-}006 \\ -8.14\text{e-}007 \end{pmatrix}, \text{with } \lambda^{\mathrm{I}} = 0.597171017,$$

$$A_{\mathrm{I}}x^{\mathrm{I}} - b^{\mathrm{I}} = -1.00\text{e-}005.$$

Table 4.2: Computational Results of approximate stationary Nash equilibria for Example 4.5.2

μ^k	0.01	0.001	0.0001	0.00001
x^k	$\begin{pmatrix} -0.692667223 \\ 1.016579205 \\ -0.108166204 \\ -0.464648787 \\ 0.034905477 \\ 0.619646979 \end{pmatrix}$	$\begin{pmatrix} -0.692694868 \\ 1.016540797 \\ -0.108194512 \\ -0.455051037 \\ 0.033601554 \\ 0.631711506 \end{pmatrix}$	$\begin{pmatrix} -0.692701651 \\ 1.016545116 \\ -0.108193163 \\ -0.455051025 \\ 0.033580781 \\ 0.631712791 \end{pmatrix}$	$\begin{pmatrix} -0.692831978 \\ 1.016420082 \\ -0.108237243 \\ -0.450723891 \\ 0.032985916 \\ 0.637152985 \end{pmatrix}$
y^k	$\begin{pmatrix} 0.361580945 \\ 1.098110671 \\ 0.348600029 \end{pmatrix}$	$\begin{pmatrix} 0.385396590 \\ 1.252027604 \\ 0.341167025 \end{pmatrix}$	$\begin{pmatrix} 0.385401922 \\ 1.252031927 \\ 0.341160304 \end{pmatrix}$	$\begin{pmatrix} 0.469066735 \\ 1.642687333 \\ 0.259482587 \end{pmatrix}$
Valgap	1.52e-10	9.78e-11	9.34e-11	8.82e-11

For Player II.

$$H_{\rm II}x^{\rm II} + G_{\rm II}x^{\rm I} + \nabla_{x^{\rm II}}y(x)c^{\rm II} + A_{\rm II}^{\top}\lambda^{\rm II} = \begin{pmatrix} 5.07\text{e-}006 \\ -3.25\text{e-}006 \end{pmatrix}, \text{with } \lambda^{\rm II} = 3.077495909,$$

$$A_{\rm II}x^{\rm II} - b^{\rm II} = 1.72\text{e-}005.$$

This indicates that we can look on x^k as an approximate B-stationary Nash equilibrium of the original NEP.

Next, we solve two EPECs where both the upper level variables and the lower level variable are three dimensional.

Table 4.3: Computational Results of approximate stationary Nash equilibria for Example 4.5.3

μ^k	0.01	0.001	0.0001	0.00001
x^k	$\begin{pmatrix} -0.716520923 \\ 0.994069232 \\ -0.115604893 \\ -0.583389678 \\ 0.051365820 \\ 0.470423193 \end{pmatrix}$	$\begin{pmatrix} -0.71659095 \\ 0.994004031 \\ -0.115627191 \\ -0.583377728 \\ 0.051365611 \\ 0.470437974 \end{pmatrix}$	$\begin{pmatrix} -0.716591707 \\ 0.994003398 \\ -0.115627574 \\ -0.583377443 \\ 0.051364984 \\ 0.470437871 \end{pmatrix}$	$\begin{pmatrix} -0.716591204 \\ 0.994002798 \\ -0.115626892 \\ -0.583378032 \\ 0.051366123 \\ 0.470438807 \end{pmatrix}$
y^k	$\begin{pmatrix} 0.063229798 \\ 0.066149112 \\ 0.781141113 \end{pmatrix}$	$\begin{pmatrix} 0.063221059 \\ 0.06612975 \\ 0.781112525 \end{pmatrix}$	$\begin{pmatrix} 0.063221067 \\ 0.066129587 \\ 0.781112180 \end{pmatrix}$	$\begin{pmatrix} 0.063221092 \\ 0.066129569 \\ 0.781112210 \end{pmatrix}$
Valgap	2.52e-10	2.46e-10	2.60e-10	2.19e-10

Example 4.5.2 The problem data are given as follows:

$$H_{\mathrm{I}} = \begin{pmatrix} 10.0 & 3.6 & 2.7 \\ 3.6 & 12.0 & -1.9 \\ 2.7 & -1.9 & 15.0 \end{pmatrix}, H_{\mathrm{II}} = \begin{pmatrix} 12.0 & -1.2 & 3.1 \\ -1.2 & 10.0 & 2.5 \\ 3.1 & 2.5 & 8.0 \end{pmatrix}, G_{\mathrm{I}} = \begin{pmatrix} 1.2 & 0.0 & -1.6 \\ 1.3 & -2.1 & 0.0 \\ -1.2 & 1.5 & 0.3 \end{pmatrix},$$

$$G_{\mathrm{II}} = \begin{pmatrix} 1.2 & 0.0 & -1.5 \\ 1.5 & 1.4 & 0.0 \\ -1.2 & 1.1 & -1.4 \end{pmatrix}, M = \begin{pmatrix} 5.6 & -1.2 & 1.5 \\ 3.2 & 7.2 & -2.4 \\ -1.8 & 2.5 & 6.4 \end{pmatrix}, N_{\mathrm{I}} = \begin{pmatrix} -1.1 & 0.0 & -1.2 \\ 1.5 & -1.0 & -0.3 \\ -1.4 & 0.0 & 1.3 \end{pmatrix},$$

$$N_{\mathrm{II}} = \begin{pmatrix} -1.3 & 0.9 & -0.6 \\ -1.4 & 1.2 & 0.0 \\ 1.5 & -0.7 & 1.4 \end{pmatrix}, q = \begin{pmatrix} -3.2 \\ -2.5 \\ -4.8 \end{pmatrix}, c^{\mathrm{I}} = \begin{pmatrix} -3.6 \\ -2.7 \\ -4.8 \end{pmatrix}, c^{\mathrm{II}} = \begin{pmatrix} -3.2 \\ -2.4 \\ -4.5 \end{pmatrix},$$

$$A_{\mathrm{I}} = \begin{pmatrix} 1.6 & -1.3 & -1.2 \\ 1.2 & -1.7 & 1.3 \end{pmatrix}, A_{\mathrm{II}} = \begin{pmatrix} 1.3 & -1.5 & -1.2 \\ 1.8 & 1.2 & -1.3 \end{pmatrix}, b^{\mathrm{I}} = \begin{pmatrix} -2.3 \\ -2.7 \end{pmatrix}, b^{\mathrm{II}} = \begin{pmatrix} -1.4 \\ -1.6 \end{pmatrix}.$$

Example 4.5.3 The problem data are the same as those in Example 6.3 except that c^{I}, c^{II} and q are given as follows:

$$c^{\mathrm{I}} = \begin{pmatrix} -3.6 \\ 2.7 \\ -4.8 \end{pmatrix}, c^{\mathrm{II}} = \begin{pmatrix} 3.2 \\ -2.4 \\ 4.5 \end{pmatrix}, q = \begin{pmatrix} -3.2 \\ 2.5 \\ -4.8 \end{pmatrix}.$$

The computed solutions $x^k = (x_1^{k,\mathrm{I}}, x_2^{k,\mathrm{I}}, x_3^{k,\mathrm{I}}, x_1^{k,\mathrm{II}}, x_2^{k,\mathrm{II}}, x_3^{k,\mathrm{II}})^\top$ and $y^k = (y_{\mu^k,1}(x^k), y_{\mu^k,2}(x^k), y_{\mu^k,3}(x^k))^\top$ of the sequence of smoothed NEPs with $\mu^k = 10^{-k-1}, k = 1, 2, 3, 4$ along with the values of D-gap functions are shown in Table 4.2 and Table 4.3.

For these two examples, we may observe the similar properties to those of Example 4.5.2. As μ^k tends to 0, we confirm that these approximate stationary Nash equilibria of smoothed NEPs also converge to a B-stationary Nash equilibrium of the original NEP. In fact, in Example 4.5.3, since the solution function y is differentiable at x^k with $\mu^k = 0.00001$, we can check the KKT conditions for each player's problem directly at $x = x^k$ as follows:

For Player I.

$$H_{\mathrm{I}} x^{\mathrm{I}} + G_{\mathrm{I}} x^{\mathrm{II}} + \nabla_{x^{\mathrm{I}}} y(x) c^{\mathrm{I}} + A_{\mathrm{I}}^\top \lambda^{\mathrm{I}} = \begin{pmatrix} 1.76\text{e-}006 \\ -1.49\text{e-}006 \\ -1.49\text{e-}007 \end{pmatrix}, \text{with } \lambda^{\mathrm{I}} = \begin{pmatrix} 0.316295946 \\ 5.198888467 \end{pmatrix},$$

$$A_{\mathrm{I}} x^{\mathrm{I}} - b^{\mathrm{I}} = \begin{pmatrix} 7.42\text{e-}006 \\ 2.09\text{e-}005 \end{pmatrix}.$$

For Player II.

$$H_{\mathrm{II}} x^{\mathrm{II}} + G_{\mathrm{II}} x^{\mathrm{I}} + \nabla_{x^{\mathrm{II}}} y(x) c^{\mathrm{II}} + A_{\mathrm{II}}^\top \lambda^{\mathrm{II}} = \begin{pmatrix} 2.65\text{e-}006 \\ -1.68\text{e-}008 \\ -1.51\text{e-}006 \end{pmatrix}, \text{with } \lambda^{\mathrm{II}} = \begin{pmatrix} 3.355974139 \\ 0.829159978 \end{pmatrix},$$

$$A_{\mathrm{II}} x^{\mathrm{II}} - b^{\mathrm{II}} = \begin{pmatrix} -3.51\text{e-}006 \\ -1.88\text{e-}005 \end{pmatrix}.$$

This indicates that we can look on x^k as an approximate B-stationary Nash equilibrium of the original NEP.

Chapter 5

Existence, Uniqueness, and Computation of Robust Nash Equilibrium in a Class of Multi-Leader-Follower Games

5.1 Introduction

In this chapter, we consider a class of multi-leader-follower games with uncertain data. There have been some fundamental work about the games with uncertain data. For example, under the assumption on probability distributions called Bayesian hypothesis, Harsanyi [49, 50, 51] considered a game with incomplete information, where the players have no complete information about some important parameters of the game. Further assuming all players shared some common knowledge about those probability distributions, the game was finally reformulated as a game with complete information essentially, called the Bayes-equivalent of the original game.

Besides the probability distribution models, the distribution-free models based on the worst case scenario have received attention in recent years [2, 52, 76]. In the latter models, each player makes a decision according to the concept of robust optimization [7, 8, 9, 24]. Basically, in robust optimization, uncertain data are assumed to belong to some set called an uncertainty set, and then a solution is sought by taking into account the worst case in terms of the objective function value and/or the constraint violation. In a NEP containing some uncertain parameters, we may also define an equilibrium called robust Nash equilibrium. Namely, if each player has chosen a strategy pessimistically and no player can obtain more benefit by changing his/her own current strategy unilaterally (i.e., the other players hold their current strategies), then the tuple of the current strategies of all players is defined as a robust

Nash equilibrium, and the problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem. Such an equilibrium problem was studied by Hayashi, Yamashita and Fukushima [52], where the authors considered the bimatrix game with uncertain data and proposed a new concept of equilibrium called robust Nash equilibrium. Under some assumptions on the uncertainty sets, they presented some existence results about robust Nash equilibria. Furthermore, the authors showed that such a robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) by converting each player's problem into a second-order cone program (SOCP). Aghassi and Bertsimas [2] considered a robust Nash equilibrium in an N-person NEP with bounded polyhedral uncertainty sets, where each player solves a linear programming problem. They also proposed a method of computing robust Nash equilibria. Note that both of these models [2, 52] particularly deal with linear objective functions in players' optimization problems.

More recently, Nishimura, Hayashi and Fukushima [76] considered a more general NEP with uncertain data, where each player solves an optimization problem with a nonlinear objective function. Under some mild assumptions on the uncertainty sets, the authors p-resented some results about the existence and uniqueness of the robust Nash equilibrium. They also proposed to compute a robust Nash equilibrium by reformulating the problem as an SOCCP.

In this paper, inspired by the previous work on the robust Nash equilibrium problem, we extend the idea of robust optimization for the NEP to the multi-leader-follower game. We propose a new concept of equilibrium for the multi-leader-follower game with uncertain data, called robust L/F Nash equilibrium. In particular, we show some results about the existence and uniqueness of the robust L/F Nash equilibrium. We also consider the computation of the equilibrium by reformulating the problem as a GVI problem. It may be mentioned here that the idea of this paper also comes from Hu and Fukushima [56], where the authors considered a class of multi-leader-follower games with complete information and showed some existence and uniqueness results for the L/F Nash equilibrium by way of the VI formulation. A remarkable feature of the multi-leader-follower game studied in this paper is that the leaders anticipate the follower's response under their respective uncertain circumstances, and hence the follower's responses estimated by the leaders are generally different from each other.

The organization of this chapter is as follows. In Section 5.2, we consider a special multi-leader-follower game with uncertainty, and reformulate it as an NEP with uncertainty, moreover, a standard NEP with complete information. In Section 5.3, we show sufficient conditions to guarantee the existence of a robust L/F Nash equilibrium by reformulating it by its Nash equilibrium formulation. In Section 5.4, we consider a particular class of robust multi-leader-follower games with uncertain data, and discuss the uniqueness of the robust Nash equilibrium by way of the GVI formulation. Finally, we show results of numerical experiments where the GVI formulation is solved by the forward-backward splitting method in Section 5.5.

5.2 A Special Multi-Leader-Follower Game with Uncertainty

5.2.1 Robust Nash Equilibrium Problem

In this subsection, we describe the robust Nash equilibrium problem and its solution concept, robust Nash equilibrium. First, we introduce the NEP and Nash equilibrium.

In the NEP with complete information, all players are in the equal position. Nash equilibrium is well-defined when all players seek their own optimal strategies simultaneously by observing and estimating the opponents' strategies, as well as the values of their own objective functions, exactly. However, in many real-world models, such information may contain some uncertain parameters, because of observation errors and/or estimation errors.

To deal with some uncertainty in the NEP, Nishimura, Hayashi and Fukushima [76] considered a robust Nash equilibrium problem and defined the corresponding equilibrium called robust Nash equilibrium. Here we briefly explain it under the following assumption:

> A parameter $u^\nu \in \Re^{l_\nu}$ is involved in player ν's objective function, which is now expressed as $\theta_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \times \Re^{l_\nu} \to \Re$. Although the player ν does not know the exact value of parameter u^ν, yet he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \Re^{l_\nu}$.

Then, player ν solves the following optimization problem with parameter u^ν for his/her own variable x^ν:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, u^\nu) \\ \text{subject to} \quad & x^\nu \in X^\nu, \end{aligned} \tag{5.2.1}$$

where $u^\nu \in U^\nu$. According to the RO paradigm, we assume that each player ν tries to minimize the worst value of his/her objective function. Under this assumption, each player $\nu = 1, \cdots, N$, considers the worst cost function $\tilde{\theta}_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \to (-\infty, +\infty]$ defined by

$$\tilde{\theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\theta_\nu(x^\nu, x^{-\nu}, u^\nu) \mid u^\nu \in U^\nu\}$$

and solves the following optimization problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \tilde{\theta}_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned} \tag{5.2.2}$$

Since it is regarded as an NEP with complete information, denoted by $\text{NEP}(\tilde{\theta}_\nu, X^\nu)_{\nu=1}^N$, we can define the equilibrium of the NEP with uncertain parameters as follows.

Definition 5.2.1 *A strategy tuple* $x = (x^\nu)_{\nu=1}^N$ *is called a robust Nash equilibrium of the non-cooperative game comprised of problems* (5.2.1)*, if* x *is a Nash equilibrium of the NEP* $(\tilde{\theta}_\nu, X^\nu)_{\nu=1}^N$ *comprised of problems* (5.2.2)*. The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.*

5.2.2 Robust Multi-Leader-Follower Game with One Follower

In this subsection, we describe a robust multi-leader-follower game with one follower, and then define the corresponding robust L/F Nash equilibrium based on the above discussions about the robust Nash equilibrium.

For the multi-leader-follower game comprised of problems (**??**) and (**??**) defined in Section 2.2.5, we can define an equilibrium called L/F Nash equilibrium [56], under the assumption that all the leaders can anticipate the follower's responses, observe and estimate their opponents' strategies, and evaluate their own objective functions exactly. However, in many real-world models, the information may contain uncertainty, due to some observation errors and/or estimation errors. In this paper, we particularly consider a robust multi-leader-follower game with N leaders and one follower, where leader ν tries to solve the following uncertain optimization problem for his/her own variable x^ν:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) \\ \text{subject to} \quad & x^\nu \in X^\nu, \end{aligned} \tag{5.2.3}$$

where an uncertain parameter $u^\nu \in \Re^{l_\nu}$ appears in the objective function $\theta_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \times \Re^m \times \Re^{l_\nu} \to \Re$. We assume that although leader ν does not know the exact value of parameter u^ν, yet he/she can confirm that it must belong to a given nonempty set $U^\nu \subseteq \Re^{l_\nu}$.

On the other hand, we also assume that although the follower responds to the leaders' strategies with his/her optimal strategy, each leader cannot anticipate the response of the follower exactly because of some observation errors and/or estimation errors. Consequently, each leader ν estimates that the follower solves the following uncertain optimization problem for variable y:

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \gamma^\nu(x, y, v^\nu) \\ \text{subject to} \quad & y \in K(x), \end{aligned} \tag{5.2.4}$$

where an uncertain parameter $v^\nu \in \Re^{k_\nu}$ appears in the objective function $\gamma^\nu : \Re^n \times \Re^m \times \Re^{k_\nu} \to \Re$ conceived by leader ν. We assume that although leader ν cannot know the exact value of v^ν, yet he/she can estimate that it belongs to a given nonempty set $V^\nu \subseteq \Re^{k_\nu}$. It should be emphasized that the uncertain parameter v^ν is associated with leader ν, which means the leaders may estimate the follower's problem differently. Hence, the follower's response anticipated by a leader may be different from the one anticipated by another leader.

In the follower's problem (5.2.4) anticipated by leader ν, we assume that for any fixed $x \in X$ and $v^\nu \in V^\nu$, $\gamma^\nu(x, \cdot, v^\nu)$ is a strictly convex function and $K(x)$ is a nonempty, closed, convex set. That is, problem (5.2.4) is a strictly convex optimization problem parameterized by x and v^ν. We denote its unique optimal solution by $y^\nu(x, v^\nu)$, which we assume to exist.

Therefore, the above multi-leader-follower game with uncertain data can be reformulated as a robust Nash equilibrium problem where each player ν solves the following uncertain optimization problem with his/her own variable x^ν:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu) \\ \text{subject to} \quad & x^\nu \in X^\nu, \end{aligned} \tag{5.2.5}$$

where uncertain parameters $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$.

By means of the RO technique, we define the worst cost function $\tilde{\Theta}_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \to (-\infty, +\infty]$ for each player ν as follows:

$$\tilde{\Theta}_\nu(x^\nu, x^{-\nu}) := \sup\{\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) |\ u^\nu \in U^\nu, v^\nu \in V^\nu\}, \tag{5.2.6}$$

where $\Theta_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \times \Re^{k_\nu} \times \Re^{l_\nu} \to \Re$ is defined by $\Theta_\nu(x^\nu, x^{-\nu}, v^\nu, u^\nu) := \theta_\nu(x^\nu, x^{-\nu}, y^\nu(x^\nu, x^{-\nu}, v^\nu), u^\nu)$.

Thus, we obtain a NEP with complete information, denoted by NEP$(\tilde{\Theta}_\nu, X^\nu)_{\nu=1}^N$, where each player ν solves the following optimization problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \tilde{\Theta}_\nu(x^\nu, x^{-\nu}) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned} \tag{5.2.7}$$

Moreover, we can define an equilibrium for the multi-leader-follower game comprised of problems (5.2.3) and (5.2.4) as follows.

Definition 5.2.2 *A strategy tuple $x = (x^\nu)_{\nu=1}^N \in X$ is called a robust L/F Nash equilibrium of the multi-leader-follower game with uncertain data comprised of problems* (5.2.3) *and* (5.2.4)*, if x is a robust Nash equilibrium of the robust NEP comprised of problems* (5.2.5)*, i.e., a Nash equilibrium of the NEP$(\tilde{\Theta}_\nu, X^\nu)_{\nu=1}^N$ comprised of problems* (5.2.7)*. The problem of finding such an equilibrium is called a robust multi-leader-follower game.*

5.3 Existence of Robust L/F Nash Equilibrium

In this section, we discuss the existence of a robust L/F Nash equilibrium for a robust multi-leader-follower game with one follower.

Assumption 5.3.1 *For each leader $\nu = 1, \cdots, N$, the following conditions hold:*

(a) *The functions* $\theta_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \times \Re^m \times \Re^{l_\nu} \to \Re$ *and* $y^\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \times \Re^{k_\nu} \to \Re^m$ *are both continuous.*

(b) *The uncertainty sets* $U^\nu \subseteq \Re^{l_\nu}$ *and* $V^\nu \subseteq \Re^{k_\nu}$ *are both nonempty and compact.*

(c) *The strategy set* X^ν *is nonempty, compact and convex.*

(d) *The function* $\Theta_\nu(\cdot, x^{-\nu}, v^\nu, u^\nu) : \Re^{n_\nu} \to \Re$ *is convex for any fixed* $x^{-\nu}$, v^ν, *and* u^ν.

Under Assumption 5.3.1, we have the following property for function $\tilde{\Theta}_\nu$ defined by (5.2.6):

Proposition 5.3.1 *For each leader* $\nu = 1, \cdots, N$, *under Assumption* 5.3.1, *we have*

(a) $\tilde{\Theta}_\nu(x)$ *is finite for any* $x \in X$, *and the function* $\tilde{\Theta}_\nu : \Re^{n_\nu} \times \Re^{n_{-\nu}} \to \Re$ *is continuous;*

(b) *the function* $\tilde{\Theta}_\nu(\cdot, x^{-\nu}) : \Re^{n_\nu} \to \Re$ *is convex on* X^ν *for any fixed* $x^{-\nu} \in X^{-\nu}$.

Proof. The results follow from Theorem 1.4.16 in [5] and Proposition 1.2.4(c) in [10] directly. ■

Now, we establish the existence of a robust L/F Nash equilibrium.

Theorem 5.3.1 *If Assumption* 5.3.1 *holds, then the robust multi-leader-follower game with one follower comprised of problems* (5.2.3) *and* (5.2.4) *has at least one robust L/F Nash equilibrium.*

Proof. For each leader ν, since Assumption 5.3.1 holds, the function Θ_ν is continuous and finite at any $x \in X$ and it is also convex with respect to x^ν on X^ν from Proposition 5.3.1. Therefore, from Proposition 2.2.12, the NEP$(\tilde{\Theta}_\nu, X^\nu)_{\nu=1}^N$ comprised of problems (5.2.7) has at least one Nash equilibrium, that is to say, the robust Nash equilibrium problem comprised of problems (5.2.5) has at least one robust Nash equilibrium. This means, the robust multi-leader-follower game with one follower comprised of problems (5.2.3) and (5.2.4) has at least one robust L/F Nash equilibrium. ■

5.4 Uniqueness of Robust L/F Nash Equilibrium

In this section, we discuss the uniqueness of a robust L/F Nash equilibrium for the robust multi-leader-follower game with the following special structure:

Leader ν's Problem ($\nu = 1, \cdots, N$).

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta_\nu(x^\nu, x^{-\nu}, y, u^\nu) := \omega_\nu(x^\nu, x^{-\nu}, u^\nu) + \varphi_\nu(x^\nu, y) \\ \text{subject to} \quad & x^\nu \in X^\nu. \end{aligned}$$

Follower's Problem.

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \gamma(x, y) := \psi(y) - \sum_{\nu=1}^{N} \varphi_\nu(x^\nu, y) \\ \text{subject to} \quad & y \in \mathcal{Y}. \end{aligned}$$

In this robust multi-leader-follower game, the objective functions of N leaders and the follower contain some related terms. In partular, the last term of each leader's objective function appears in the follower's objective function in the negated form. Therefore, the game partly contains a kind of zero-sum structure between each leader and the follower. An application of such special multi-leader-follower games with complete information has been presented with some illustrative numerical examples in [**?**]. Here, in each leader ν's problem, we assume that the strategy set X^ν is nonempty, compact and convex. Due to some estimation errors, leader ν cannot evaluate his/her objective function exactly, but only knows that it contains some uncertain parameter u^ν belonging to a fixed uncertainty set $U^\nu \subseteq \Re^{l_\nu}$. We further assume that functions ω_ν, φ_ν, ψ and the set $\mathcal{Y}$ have the following explicit representations:

$$\begin{aligned} \omega_\nu(x^\nu, x^{-\nu}, u^\nu) &:= \frac{1}{2}(x^\nu)^\top H_\nu x^\nu + \sum_{\nu'=1, \nu' \neq \nu}^{N} (x^\nu)^\top E_{\nu\nu'} x^{\nu'} + (x^\nu)^\top R_\nu u^\nu, \\ \varphi_\nu(x^\nu, y) &:= (x^\nu)^\top D_\nu y, \\ \psi(y) &:= \frac{1}{2} y^\top B y + c^\top y, \\ \mathcal{Y} &:= \{y \in \Re^m | Ay + a = 0\}, \end{aligned}$$

where $H_\nu \in \Re^{n_\nu} \times \Re^{n_\nu}$, $D_\nu \in \Re^{n_\nu} \times \Re^m$, $R_\nu \in \Re^{n_\nu} \times \Re^{l_\nu}$, $E_{\nu\nu'} \in \Re^{n_\nu} \times \Re^{n_{\nu'}}$, $\nu, \nu' = 1, \cdots, N$, and $c \in \Re^m$. Matrix $B \in \Re^m \times \Re^m$ is assumed to be symmetric and positive definite. Moreover, $A \in \Re^{p_0} \times \Re^m$, $a \in \Re^{p_0}$, and A has full row rank.

We assume that although the follower can respond to all leaders' strategies exactly, yet each leader ν cannot exactly know the follower's problem, but can only estimate it as follows:

Follower's Problem Estimated by Leader ν.

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \gamma^\nu(x, y, v^\nu) := \frac{1}{2} y^\top B y + (c + v^\nu)^\top y - \sum_{\nu=1}^{N} \varphi_\nu(x^\nu, y) \\ \text{subject to} \quad & y \in \mathcal{Y}. \end{aligned}$$

Here, the uncertain parameter v^ν belongs to some fixed uncertainty set $V^\nu \subseteq \Re^m$.

In the remainder of the paper, for simplicity, we will mainly consider the following game with two leaders, labelled I and II. The results presented below can be extended to the case of more than two leaders in a straightforward manner.

Leader I's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{I}}}{\text{minimize}} & \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top E_{\mathrm{I}} x^{\mathrm{II}} + (x^{\mathrm{I}})^\top R_{\mathrm{I}} u^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} y \\ \text{subject to} & x^{\mathrm{I}} \in X^{\mathrm{I}}. \end{array} \tag{5.4.1}$$

Leader II's Problem.

$$\begin{array}{ll} \underset{x^{\mathrm{II}}}{\text{minimize}} & \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top E_{\mathrm{II}} x^{\mathrm{I}} + (x^{\mathrm{II}})^\top R_{\mathrm{II}} u^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ \text{subject to} & x^{\mathrm{II}} \in X^{\mathrm{II}}. \end{array} \tag{5.4.2}$$

Follower's Problem Estimated by Leader ν.

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \frac{1}{2} y^\top B y + (c + v^\nu)^\top y - (x^{\mathrm{I}})^\top D_{\mathrm{I}} y - (x^{\mathrm{II}})^\top D_{\mathrm{II}} y \\ \text{subject to} & Ay + a = 0. \end{array} \tag{5.4.3}$$

Here, $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$.

Since the follower's problems estimated by two leaders are both strictly convex quadratic programming problems with equality constraints, each of them is equivalent to finding a pair $(y, \lambda) \in \Re^m \times \Re^{p_0}$ satisfying the following KKT system of linear equations for $\nu = \mathrm{I}, \mathrm{II}$:

$$\begin{aligned} By + c + v^\nu - (D_{\mathrm{I}})^\top x^{\mathrm{I}} - (D_{\mathrm{II}})^\top x^{\mathrm{II}} + A^\top \lambda &= 0, \\ Ay + a &= 0. \end{aligned}$$

Note that, under the given assumptions, a KKT pair (y, λ) exists uniquely for each $(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ and is denoted by $(y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu), \lambda^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu))$. For each $\nu = \mathrm{I}, \mathrm{II}$, by direct calculations, we have

$$\begin{aligned} y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu) = &- B^{-1}(c + v^\nu) - B^{-1}A^\top (AB^{-1}A^\top)^{-1}(a - AB^{-1}(c + v^\nu)) \\ &+ [B^{-1}(D_{\mathrm{I}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{I}})^\top] x^{\mathrm{I}} \\ &+ [B^{-1}(D_{\mathrm{II}})^\top - B^{-1}A^\top (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{II}})^\top] x^{\mathrm{II}}, \\ \lambda^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu) = &\ (AB^{-1}A^\top)^{-1}(a - AB^{-1}(c + v^\nu)) + (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{I}})^\top x^{\mathrm{I}} \\ &+ (AB^{-1}A^\top)^{-1}AB^{-1}(D_{\mathrm{II}})^\top x^{\mathrm{II}}. \end{aligned}$$

Let $P = I - B^{-\frac{1}{2}}A^\top(AB^{-1}A^\top)^{-1}AB^{-\frac{1}{2}}$. Then, by substituting each $y^\nu(x^{\mathrm{I}}, x^{\mathrm{II}}, v^\nu)$ for y in the respective leader's problem, the leaders' objective functions can be rewritten as

$$\begin{aligned}\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{I}}, u^{\mathrm{I}}) &:= \theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{I}}), u^{\mathrm{I}}) \\ &= \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} G_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top R_{\mathrm{I}} u^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} r \\ &\quad + (x^{\mathrm{I}})^\top (D_{\mathrm{I}} G_{\mathrm{II}} + E_{\mathrm{I}}) x^{\mathrm{II}} - (x^{\mathrm{I}})^\top D_{\mathrm{I}} B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^{\mathrm{I}},\end{aligned} \tag{5.4.4}$$

$$\begin{aligned}\Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{II}}, u^{\mathrm{II}}) &:= \theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, y^{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{II}}), u^{\mathrm{II}}) \\ &= \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} G_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top R_{\mathrm{II}} u^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} r \\ &\quad + (x^{\mathrm{II}})^\top (D_{\mathrm{II}} G_{\mathrm{I}} + E_{\mathrm{II}}) x^{\mathrm{I}} - (x^{\mathrm{II}})^\top D_{\mathrm{II}} B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^{\mathrm{II}},\end{aligned} \tag{5.4.5}$$

where $G_{\mathrm{I}} \in \Re^{m \times n_{\mathrm{I}}}$, $G_{\mathrm{II}} \in \Re^{m \times n_{\mathrm{II}}}$, and $r \in \Re^m$ are given by

$$\begin{aligned}G_{\mathrm{I}} &= B^{-\frac{1}{2}} P B^{-\frac{1}{2}} (D_{\mathrm{I}})^T, \\ G_{\mathrm{II}} &= B^{-\frac{1}{2}} P B^{-\frac{1}{2}} (D_{\mathrm{II}})^T, \\ r &= -B^{-\frac{1}{2}} P B^{-\frac{1}{2}} c - B^{-1} A^T (AB^{-1}A^T)^{-1} a.\end{aligned}$$

With the functions $\Theta_{\mathrm{I}} : \Re^{n_{\mathrm{I}}} \times \Re^{n_{\mathrm{II}}} \times \Re^m \times \Re^{l_{\mathrm{I}}} \to \Re$ and $\Theta_{\mathrm{II}} : \Re^{n_{\mathrm{I}}} \times \Re^{n_{\mathrm{II}}} \times \Re^m \times \Re^{l_{\mathrm{II}}} \to \Re$ defined by (5.4.4) and (5.4.5), respectively, we can formulate the above robust multi-leader-follower game as the following robust Nash equilibrium problem:

Leader I's Problem.

$$\begin{aligned}\underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{I}}, u^{\mathrm{I}}) \\ \text{subject to} \quad & x^{\mathrm{I}} \in X^{\mathrm{I}}.\end{aligned}$$

Leader II's Problem.

$$\begin{aligned}\underset{x^{\mathrm{II}}}{\text{minimize}} \quad & \Theta_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{II}}, u^{\mathrm{II}}) \\ \text{subject to} \quad & x^{\mathrm{II}} \in X^{\mathrm{II}}.\end{aligned}$$

Here, $u^\nu \in U^\nu$ and $v^\nu \in V^\nu$, $\nu = \mathrm{I}, \mathrm{II}$.

By means of the RO technique, we construct the following robust counterpart of the above robust Nash equilibrium problem, which is a NEP with complete information:

Leader I's Problem.

$$\begin{aligned}\underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \tilde{\Theta}_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) \\ \text{subject to} \quad & x^{\mathrm{I}} \in X^{\mathrm{I}}.\end{aligned} \tag{5.4.6}$$

Leader II's Problem.

$$\begin{aligned}&\underset{x^{\mathrm{II}}}{\text{minimize}} && \tilde{\Theta}_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}})\\ &\text{subject to} && x^{\mathrm{II}} \in X^{\mathrm{II}}.\end{aligned} \tag{5.4.7}$$

Here, functions $\tilde{\Theta}_{\mathrm{I}} : \Re^{n_{\mathrm{I}}} \times \Re^{n_{\mathrm{II}}} \to \Re$ and $\tilde{\Theta}_{\mathrm{II}} : \Re^{n_{\mathrm{I}}} \times \Re^{n_{\mathrm{II}}} \to \Re$ are defined by

$$\begin{aligned}\tilde{\Theta}_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}) :=& \sup\{\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{I}}, u^{\mathrm{I}}) | u^{\mathrm{I}} \in U^{\mathrm{I}}, v^{\mathrm{I}} \in V^{\mathrm{I}}\}\\ =& \frac{1}{2}(x^{\mathrm{I}})^\top H_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} G_{\mathrm{I}} x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} r\\ &+ (x^{\mathrm{I}})^\top (D_{\mathrm{I}} G_{\mathrm{II}} + E_{\mathrm{I}}) x^{\mathrm{II}} + \phi_{\mathrm{I}}(x^{\mathrm{I}}),\\ \tilde{\Theta}_{\mathrm{II}}(x^{\mathrm{I}}, x^{\mathrm{II}}) :=& \sup\{\Theta_{\mathrm{I}}(x^{\mathrm{I}}, x^{\mathrm{II}}, v^{\mathrm{II}}, u^{\mathrm{II}}) | u^{\mathrm{II}} \in U^{\mathrm{II}}, v^{\mathrm{II}} \in V^{\mathrm{II}}\}\\ =& \frac{1}{2}(x^{\mathrm{II}})^\top H_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} G_{\mathrm{II}} x^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} r\\ &+ (x^{\mathrm{II}})^\top (D_{\mathrm{II}} G_{\mathrm{I}} + E_{\mathrm{II}}) x^{\mathrm{I}} + \phi_{\mathrm{II}}(x^{\mathrm{II}}),\end{aligned}$$

where $\phi_{\mathrm{I}} : \Re^{n_{\mathrm{I}}} \to \Re$ and $\phi_{\mathrm{II}} : \Re^{n_{\mathrm{II}}} \to \Re$ are defined by

$$\begin{aligned}\phi_{\mathrm{I}}(x^{\mathrm{I}}) :=& \sup\{(x^{\mathrm{I}})^\top R_{\mathrm{I}} u^{\mathrm{I}} | u^{\mathrm{I}} \in U^{\mathrm{I}}\}\\ &+ \sup\{-(x^{\mathrm{I}})^\top D_{\mathrm{I}} B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^{\mathrm{I}} | v^{\mathrm{I}} \in V^{\mathrm{I}}\},\\ \phi_{\mathrm{II}}(x^{\mathrm{II}}) :=& \sup\{(x^{\mathrm{II}})^\top R_{\mathrm{II}} u^{\mathrm{II}} | u^{\mathrm{II}} \in U^{\mathrm{II}}\}\\ &+ \sup\{-(x^{\mathrm{II}})^\top D_{\mathrm{II}} B^{-\frac{1}{2}} P B^{-\frac{1}{2}} v^{\mathrm{II}} | v^{\mathrm{II}} \in V^{\mathrm{II}}\}.\end{aligned} \tag{5.4.8}$$

In what follows, based on the analysis of the previous section, we first show the existence of a robust L/F Nash equilibrium.

Theorem 5.4.1 *Suppose that for each* $\nu = \mathrm{I}, \mathrm{II}$, *the strategy set* X^ν *is nonempty, compact and convex, the matrix* $H_\nu \in \Re^{n_\nu} \times \Re^{n_\nu}$ *is symmetric and positive semidefinite, and the uncertainty sets* U^ν *and* V^ν *are nonempty and compact. Then, the robust multi-leader-follower game comprised of problems* (5.4.1), (5.4.2) *and* (5.4.3) *has at least one robust L/F Nash equilibrium.*

Proof. We will show that the conditions in Assumption 5.3.1 hold. Since conditions (a)−(c) clearly hold, we only confirm that condition (d) holds. In fact, it is easy to see that $D_{\mathrm{I}} G_{\mathrm{I}}$ and $D_{\mathrm{II}} G_{\mathrm{II}}$ are both positive semidefinite. Since H_{I} and H_{II} are also positive semidefinite, the functions Θ_{I} and Θ_{II} are convex with respect to x^{I} and x^{II}, respectively. Therefore, Assumption 5.3.1 holds. Hence, by Theorem 5.3.1, the proof is complete. ■

In order to investigate the uniqueness of a robust L/F Nash equilibrium, we reformulate the robust Nash equilibrium counterpart comprised of problems (5.4.6) and (5.4.7) as a GVI problem.

Notice that the functions $\tilde{\Theta}_\nu$ are convex with respect to x^ν. Let us define the mappings $T_{\rm I} : \Re^{n_{\rm I}} \times \Re^{n_{\rm II}} \to \Re^{n_{\rm I}}$ and $T_{\rm II} : \Re^{n_{\rm I}} \times \Re^{n_{\rm II}} \to \Re^{n_{\rm II}}$ as

$$T_{\rm I}(x^{\rm I}, x^{\rm II}) := H_{\rm I}x^{\rm I} + D_{\rm I}r + 2D_{\rm I}G_{\rm I}x^{\rm I} + (D_{\rm I}G_{\rm II} + E_{\rm I})x^{\rm II},$$
$$T_{\rm II}(x^{\rm I}, x^{\rm II}) := H_{\rm II}x^{\rm II} + D_{\rm II}r + (D_{\rm II}G_{\rm I} + E_{\rm II})x^{\rm I} + 2D_{\rm II}G_{\rm II}x^{\rm II}.$$

Then, the subdifferentials of $\tilde{\Theta}_\nu$ with respect to x^ν can be written as

$$\partial_{x^{\rm I}}\tilde{\Theta}_{\rm I}(x^{\rm I}, x^{\rm II}) = T_{\rm I}(x^{\rm I}, x^{\rm II}) + \partial\phi_{\rm I}(x^{\rm I}),$$
$$\partial_{x^{\rm II}}\tilde{\Theta}_{\rm II}(x^{\rm I}, x^{\rm II}) = T_{\rm II}(x^{\rm I}, x^{\rm II}) + \partial\phi_{\rm II}(x^{\rm II}),$$

where $\partial\phi_\nu$ denotes the subdifferentials of ϕ_ν, $\nu = {\rm I, II}$. By [10, Proposition B.24(f)], for each $\nu = {\rm I, II}$, $x^{*,\nu}$ solves problem (5.4.6) or (5.4.7) if and only if there exists a subgradient $\xi^\nu \in \partial_{x^\nu}\tilde{\Theta}_\nu(x^{*,\nu}, x^{-\nu})$ such that

$$(\xi^\nu)^\top(x^\nu - x^{*,\nu}) \geq 0 \quad \forall x^\nu \in X^\nu. \tag{5.4.9}$$

Therefore, we can investigate the uniqueness of a robust L/F Nash equilibrium by considering the following GVI problem which is formulated by concatenating the above first-order optimality conditions (5.4.9) of all leaders' problems: Find a vector $x^* = (x^{*,{\rm I}}, x^{*,{\rm II}}) \in X =: X^{\rm I} \times X^{\rm II}$ such that

$$\exists\, \xi \in \tilde{\mathcal{F}}(x^*), \quad \xi^\top(x - x^*) \geq 0 \quad \text{for all } x \in X,$$

where $\xi = (\xi^{\rm I}, \xi^{\rm II}) \in \Re^n$, $x = (x^{\rm I}, x^{\rm II}) \in \Re^n$, and the set-valued mapping $\tilde{\mathcal{F}} : \Re^n \to \mathcal{P}(\Re^n)$ is defined by $\tilde{\mathcal{F}}(x) := \partial_{x^{\rm I}}\tilde{\Theta}_{\rm I}(x^{\rm I}, x^{\rm II}) \times \partial_{x^{\rm II}}\tilde{\Theta}_{\rm II}(x^{\rm I}, x^{\rm II})$.

In what follows, we show that $\tilde{\mathcal{F}}$ is strictly monotone under suitable conditions. Then, by Proposition 2.2.11, we can ensure the uniqueness of a robust L/F Nash equilibrium. Since the subdifferentials $\partial\phi_{\rm I}$ and $\partial\phi_{\rm II}$ are monotone, we only need to establish the strict monotonicity of mapping $T : \Re^{n_{\rm I}+n_{\rm II}} \to \Re^{n_{\rm I}+n_{\rm II}}$ defined by

$$T(x) := \begin{pmatrix} T_{\rm I}(x^{\rm I}, x^{\rm II}) \\ T_{\rm II}(x^{\rm I}, x^{\rm II}) \end{pmatrix}. \tag{5.4.10}$$

For this purpose, we assume that the matrix

$$\mathcal{J} := \begin{pmatrix} H_{\rm I} & E_{\rm I} \\ E_{\rm II} & H_{\rm II} \end{pmatrix} \tag{5.4.11}$$

is positive definite. Note that the transpose of matrix $\mathcal{J}$ is the Jacobian of the so-called pseudo gradient of the first two terms $\frac{1}{2}(x^\nu)^\top H_\nu x^\nu + (x^\nu)^\top E_\nu x^{\nu'}$ in the objective functions of problems (5.4.1) and (5.4.2). The positive definiteness of such a matrix often assumed in the study on NEP and GNEP; see [64, 65, 84].

Proposition 5.4.1 *Suppose that matrix $\mathcal{J}$ defined by* (5.4.11) *is positive definite. Then, the mapping T defined by* (5.4.10) *is strictly monotone.*

Proof. For any $x = (x^{\mathrm{I}}, x^{\mathrm{II}}), \tilde{x} = (\tilde{x}^{\mathrm{I}}, \tilde{x}^{\mathrm{II}}) \in X$ such that $x \neq \tilde{x}$, we have

$$\begin{aligned}
&(x-\tilde{x})^\top (T(x) - T(\tilde{x}))\\
&= (x-\tilde{x})^\top \begin{pmatrix} H_{\mathrm{I}} & E_{\mathrm{I}} \\ E_{\mathrm{II}} & H_{\mathrm{II}} \end{pmatrix} (x-\tilde{x}) + (x-\tilde{x})^\top \begin{pmatrix} 2D_{\mathrm{I}}G_{\mathrm{I}} & D_{\mathrm{I}}G_{\mathrm{II}} \\ D_{\mathrm{II}}G_{\mathrm{I}} & 2D_{\mathrm{II}}G_{\mathrm{II}} \end{pmatrix} (x-\tilde{x}).
\end{aligned}$$

It can be shown [56, Lemma 4.1] that matrix $\begin{pmatrix} 2D_{\mathrm{I}}G_{\mathrm{I}} & D_{\mathrm{I}}G_{\mathrm{II}} \\ D_{\mathrm{II}}G_{\mathrm{I}} & 2D_{\mathrm{II}}G_{\mathrm{II}} \end{pmatrix}$ is positive semidefinite. Hence, the mapping T is strictly monotone since matrix $\mathcal{J}$ is positive definite by assumption. The proof is complete. ■

Now, we are ready to establish the uniqueness of a robust L/F Nash equilibrium.

Theorem 5.4.2 *Suppose that matrix $\mathcal{J}$ defined by* (5.4.11) *is positive definite, and the uncertainty sets U^ν and V^ν are nonempty and compact. Then the robust multi-leader-follower game comprised of problems* (5.4.1), (5.4.2) *and* (5.4.3) *has a unique robust L/F Nash equilibrium.*

Proof. It follows directly from Theorem 5.4.1, Proposition 2.2.11 and Proposition 5.4.1. We omit the details. ■

5.5 Numerical Experiments on Robust L/F Nash Equilibria

In this section, we present some numerical results for robust L/F Nash equilibrium problems. For this purpose, we use a splitting method for finding a zero of the sum of two maximal monotone mappings $\mathcal{A}$ and $\mathcal{B}$. The splitting method solves a sequence of subproblems, each of which involves only one of the two mappings $\mathcal{A}$ and $\mathcal{B}$. In particular, the forward-backward splitting method [43] may be regarded as a generalization of the gradient projection method for constrained convex optimization problems and monotone variational inequality problems. In the case where $\mathcal{B}$ is vector-valued, the forward-backward splitting method for finding a zero of the mapping $\mathcal{A} + \mathcal{B}$ uses the recursion

$$\begin{aligned}
x^{k+1} &= (I + \lambda\mathcal{A})^{-1}(I - \lambda\mathcal{B})(x^k)\\
&:= J_{\lambda\mathcal{A}}((I - \lambda\mathcal{B})(x^k)) \quad k = 0, 1, \cdots,
\end{aligned} \tag{5.5.1}$$

where the mapping $J_{\lambda\mathcal{A}} := (I + \lambda\mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ (with constant $\lambda > 0$), which is a vector-valued mapping from $\Re^n$ to dom$\mathcal{A}$.

In what follows, we assume that, in the robust multi-leader-follower game comprised of problems (5.4.1), (5.4.2) and (5.4.3), for each leader $\nu = \mathrm{I}, \mathrm{II}$, the uncertainty sets $U^\nu \in \Re^{l_\nu}$ and $V^\nu \in \Re^m$ are given by

$$U^\nu := \{u^\nu \in \Re^{l_\nu} | \|u^\nu\| \le \rho^\nu\},$$
$$V^\nu := \{v^\nu \in \Re^{m} | \|v^\nu\| \le \sigma^\nu\},$$

with given constants $\rho^\nu > 0$ and $\sigma^\nu > 0$. Further we assume that the constraints $x^\nu \in X^\nu$ are explicitly written as $g^\nu(x^\nu) := A_\nu x^\nu + b_\nu \le 0$, where $A_\nu \in \Re^{l_\nu} \times \Re^{n_\nu}$ and $b_\nu \in \Re^{l_\nu}, \nu = \mathrm{I}, \mathrm{II}$.

Under these assumptions, the functions $\phi_\nu, \nu = \mathrm{I}, \mathrm{II}$, defined by (5.4.8) can be written explicitly as

$$\phi_\nu(x^\nu) := \rho^\nu \|R_\nu^\top x^\nu\| + \sigma^\nu \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_\nu^\top x^\nu\|, \quad \nu = \mathrm{I}, \mathrm{II}.$$

Hence, we can rewrite problems (5.4.6) and (5.4.7) as follows:

Leader I's Problem.

$$\begin{aligned} \underset{x^{\mathrm{I}}}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{I}})^\top (H_{\mathrm{I}} + 2D_{\mathrm{I}}G_{\mathrm{I}})x^{\mathrm{I}} + (x^{\mathrm{I}})^\top D_{\mathrm{I}} r + (x^{\mathrm{I}})^\top (D_{\mathrm{I}}G_{\mathrm{II}} + E_{\mathrm{I}})x^{\mathrm{II}} \\ & + \rho^{\mathrm{I}} \|R_{\mathrm{I}}^\top x^{\mathrm{I}}\| + \sigma^{\mathrm{I}} \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_{\mathrm{I}}^\top x^{\mathrm{I}}\| \\ \text{subject to} \quad & A_{\mathrm{I}} x^{\mathrm{I}} + b_{\mathrm{I}} \le 0. \end{aligned} \tag{5.5.2}$$

Leader II's Problem.

$$\begin{aligned} \underset{x^{\mathrm{II}}}{\text{minimize}} \quad & \frac{1}{2}(x^{\mathrm{II}})^\top (H_{\mathrm{II}} + 2D_{\mathrm{II}}G_{\mathrm{II}})x^{\mathrm{II}} + (x^{\mathrm{II}})^\top D_{\mathrm{II}} r + (x^{\mathrm{II}})^\top (D_{\mathrm{II}}G_{\mathrm{I}} + E_{\mathrm{II}})x^{\mathrm{I}} \\ & + \rho^{\mathrm{II}} \|R_{\mathrm{II}}^\top x^{\mathrm{II}}\| + \sigma^{\mathrm{II}} \|B^{-\frac{1}{2}} P B^{-\frac{1}{2}} D_{\mathrm{II}}^\top x^{\mathrm{II}}\| \\ \text{subject to} \quad & A_{\mathrm{II}} x^{\mathrm{II}} + b_{\mathrm{II}} \le 0. \end{aligned} \tag{5.5.3}$$

To apply the forward-backward splitting method to the above NEP, we let the mappings $\mathcal{A}$ and $\mathcal{B}$ be specified by

$$\mathcal{A}(x) := \begin{pmatrix} \partial\phi_{\mathrm{I}}(x^{\mathrm{I}}) \\ \partial\phi_{\mathrm{II}}(x^{\mathrm{II}}) \end{pmatrix} + N_X(x),$$
$$\mathcal{B}(x) := T(x),$$

where $T(x)$ is given by (5.4.10). Note that the mapping $\mathcal{A}$ is set-valued, while $\mathcal{B}$ is vector-valued.

In order to evaluate the iterative point $x^{k+1} := (x^{\mathrm{I},k+1}, x^{\mathrm{II},k+1})$ in (5.5.1), we first compute $z^{\mathrm{I},k} := x^{\mathrm{I},k} - \lambda T_{\mathrm{I}}(x^k)$ and $z^{\mathrm{II},k} := x^{\mathrm{II},k} - \lambda T_{\mathrm{II}}(x^k)$. Then $x^{\mathrm{I},k+1}$ and $x^{\mathrm{II},k+1}$ are evaluated by solving the following two optimization problems separately:

$$
\begin{array}{ll}
\underset{x^{\mathrm{I}}}{\text{minimize}} & \dfrac{1}{2\lambda}\|x^{\mathrm{I}} - z^{\mathrm{I},k}\|^2 + \rho^{\mathrm{I}}\|R_{\mathrm{I}}^\top x^{\mathrm{I}}\| + \sigma^{\mathrm{I}}\|B^{-\frac{1}{2}}PB^{-\frac{1}{2}}D_{\mathrm{I}}^\top x^{\mathrm{I}}\| \\
\text{subject to} & A_{\mathrm{I}}x^{\mathrm{I}} + b_{\mathrm{I}} \le 0, \\
\underset{x^{\mathrm{II}}}{\text{minimize}} & \dfrac{1}{2\lambda}\|x^{\mathrm{II}} - z^{\mathrm{II},k}\|^2 + \rho^{\mathrm{II}}\|R_{\mathrm{II}}^\top x^{\mathrm{II}}\| + \sigma^{\mathrm{II}}\|B^{-\frac{1}{2}}PB^{-\frac{1}{2}}D_{\mathrm{II}}^\top x^{\mathrm{II}}\| \\
\text{subject to} & A_{\mathrm{II}}x^{\mathrm{II}} + b_{\mathrm{II}} \le 0.
\end{array}
$$

Note that these problems can be rewritten as linear second-order cone programming problems, for which efficient solvers are available [91, 85]. In what follows, we show some numerical results to observe the behavior of robust L/F Nash equilibria with different uncertainty bounds. To compute those equilibria, we use the forward-backward splitting method with $\lambda = 0.2$.

Example 5.5.1 The problem data are given as follows:

$$
H_{\mathrm{I}} = \begin{pmatrix} 2.5 & 1.6 \\ 1.6 & 3.8 \end{pmatrix}, H_{\mathrm{II}} = \begin{pmatrix} 2.9 & 1.3 \\ 1.3 & 1.8 \end{pmatrix}, D_{\mathrm{I}} = \begin{pmatrix} 0.8 & 2.1 & 1.3 \\ 1.5 & 2.3 & 0.7 \end{pmatrix},
$$
$$
D_{\mathrm{II}} = \begin{pmatrix} 1.5 & 0.9 & 2.4 \\ 1.8 & 2.3 & 3.6 \end{pmatrix}, E_{\mathrm{I}} = \begin{pmatrix} 0.8 & 0.4 \\ 1.5 & 0.7 \end{pmatrix}, E_{\mathrm{II}} = \begin{pmatrix} 1.2 & 1.5 \\ 0.4 & 1.3 \end{pmatrix},
$$
$$
R_{\mathrm{I}} = \begin{pmatrix} 1.7 & 2.8 \\ 0.6 & 0.7 \end{pmatrix}, R_{\mathrm{II}} = \begin{pmatrix} 1.8 & 2.7 \\ 1.9 & 1.4 \end{pmatrix}, B = \begin{pmatrix} 2.7 & 1.4 & 1.2 \\ 1.4 & 4.6 & 2.1 \\ 1.2 & 2.1 & 5.7 \end{pmatrix},
$$
$$
A_{\mathrm{I}} = \begin{pmatrix} 1.4 & 1.8 & 1.7 \\ 2.4 & 1.2 & 0.7 \end{pmatrix}, A_{\mathrm{II}} = \begin{pmatrix} 1.8 & 0.9 & 1.6 \\ 2.3 & 1.2 & 0.7 \end{pmatrix}, c = \begin{pmatrix} 0.4 \\ 1.6 \\ 3.1 \end{pmatrix},
$$
$$
A = \begin{pmatrix} 2.3 & 1.4 & 1.5 \end{pmatrix}, a = 1.9, b_{\mathrm{I}} = \begin{pmatrix} 2.1 \\ 1.8 \\ 0.4 \end{pmatrix}, b_{\mathrm{II}} = \begin{pmatrix} 1.3 \\ 2.3 \\ 1.6 \end{pmatrix}.
$$

Table 5.1 shows the computational results. In the table, $(x^{*,\mathrm{I}}, x^{*,\mathrm{II}})$ denotes the leaders' optimal strategies and $(y^{*,\mathrm{I}}, y^{*,\mathrm{II}})$ denotes the follower's responses estimated respectively by the two leaders, at the computed equilibria for various values of the uncertainty bounds $\rho = (\rho^{\mathrm{I}}, \rho^{\mathrm{II}})$ and $\sigma = (\sigma^{\mathrm{I}}, \sigma^{\mathrm{II}})$. In particular, when there is no uncertainty ($\rho = 0, \sigma = 0$), the follower's response anticipated by the two leaders naturally coincide, i.e., $y^{*,\mathrm{I}} = y^{*,\mathrm{II}}$, which is denoted $\bar{y}^*$ in the table. ValL1 and ValL2 denote the optimal objective values of the two

Table 5.1: Computational Results of Robust Nash Equilibria for Example 5.5.1

$(\rho;\sigma)$	$(0.0, 0.0; 0.0, 0.0)$	$(1.2, 1.2; 1.2, 1.2)$	$(1.6, 1.6; 1.6, 1.6)$	$(2.0, 2.0; 2.0, 2.0)$
$x^{*,\mathrm{I}}$	$\begin{pmatrix} -0.681818183 \\ -0.477272729 \end{pmatrix}$	$\begin{pmatrix} -0.681818179 \\ -0.477272734 \end{pmatrix}$	$\begin{pmatrix} -0.681818178 \\ -0.477272733 \end{pmatrix}$	$\begin{pmatrix} -0.681818172 \\ -0.477272743 \end{pmatrix}$
$x^{*,\mathrm{II}}$	$\begin{pmatrix} -0.562992330 \\ -1.494422418 \end{pmatrix}$	$\begin{pmatrix} -0.240310089 \\ -1.736434101 \end{pmatrix}$	$\begin{pmatrix} -0.240310123 \\ -1.736434078 \end{pmatrix}$	$\begin{pmatrix} -0.240310086 \\ -1.736434103 \end{pmatrix}$
ValL1	6.633481993	10.35851406	11.64260329	12.92669250
ValL2	16.97846333	24.17709160	26.53508709	28.89308242
$y^{*,\mathrm{I}}$	$\begin{pmatrix} 0.761176796 \\ -1.058444166 \\ -1.445923198 \end{pmatrix}$	$\begin{pmatrix} 0.997358753 \\ -1.51790086 \\ -1.379242618 \end{pmatrix}$	$\begin{pmatrix} 1.060323511 \\ -1.646101658 \\ -1.356134503 \end{pmatrix}$	$\begin{pmatrix} 1.123288278 \\ -1.774302473 \\ -1.333026384 \end{pmatrix}$
$\|y^{*,\mathrm{I}} - \bar{y}^*\|$	0	0.520892187	0.665501386	0.810137971
$y^{*,\mathrm{II}}$	$\begin{pmatrix} 0.761176796 \\ -1.058444166 \\ -1.445923198 \end{pmatrix}$	$\begin{pmatrix} 1.017282702 \\ -1.221407425 \\ -1.686519880 \end{pmatrix}$	$\begin{pmatrix} 1.086888777 \\ -1.250777074 \\ -1.765837522 \end{pmatrix}$	$\begin{pmatrix} 1.156494862 \\ -1.280146748 \\ -1.845155157 \end{pmatrix}$
$\|y^{*,\mathrm{II}} - \bar{y}^*\|$	0	0.387342255	0.495404296	0.603998812
$\|y^{*,\mathrm{I}} - y^{*,\mathrm{II}}\|$	0	0.427463024	0.569950703	0.712438375
Iter	47	5	4	5

leaders' respective optimization problems. Iter denotes the number of iterations required by the forward-backward splitting method to compute each equilibrium.

Both ValL1 and ValL2 increase as the uncertainty increases, indicating that the leaders have to pay additional costs that compensate for the loss of information.

Moreover, the two leaders' estimates of the follower's response tend not only to deviate from the estimate under complete information but to have a larger gap between them. It may be interesting to notice that optimal strategies of leader I do not change for all values of $(\rho;\sigma)$ and those of leader II also remain unchanged for all values of $(\rho;\sigma)$ except $(\rho;\sigma) = (0;0)$. This is because those solutions $x^{*,\mathrm{I}}$ and $x^{*,\mathrm{II}}$ lie at vertices of the polyhedral feasible sets X^{I} and X^{II}, respectively. This may also explain why, in those cases, the forward-backward splitting method was able to find a solution in a small number of iterations.

Table 5.2: Computational Results of Robust Nash Equilibria for Example 5.5.2

$(\rho;\sigma)$	$(0.0, 0.0; 0.0, 0.0)$	$(1.2, 1.2; 1.2, 1.2)$	$(1.6, 1.6; 1.6, 1.6)$	$(2.0, 2.0; 2.0, 2.0)$
$x^{*,\mathrm{I}}$	$\begin{pmatrix} -1.050189184 \\ -0.784382543 \end{pmatrix}$	$\begin{pmatrix} -0.906766095 \\ -0.960903266 \end{pmatrix}$	$\begin{pmatrix} -0.868067094 \\ -1.008532807 \end{pmatrix}$	$\begin{pmatrix} -0.833594894 \\ -1.050960131 \end{pmatrix}$
$x^{*,\mathrm{II}}$	$\begin{pmatrix} -0.330559946 \\ -0.988108829 \end{pmatrix}$	$\begin{pmatrix} -0.347881870 \\ -0.979101429 \end{pmatrix}$	$\begin{pmatrix} -0.355719780 \\ -0.975025716 \end{pmatrix}$	$\begin{pmatrix} -0.364290973 \\ -0.970568695 \end{pmatrix}$
ValL1	16.30334723	24.94426033	27.76848493	30.57141528
ValL2	14.18144686	20.60317568	22.73752011	24.86848672
$y^{*,\mathrm{I}}$	$\begin{pmatrix} -0.967381871 \\ -1.226057898 \\ 0.064205530 \\ 0.320365142 \end{pmatrix}$	$\begin{pmatrix} -1.122672963 \\ -1.599713048 \\ 0.427238704 \\ 0.359015063 \end{pmatrix}$	$\begin{pmatrix} -1.187649770 \\ -1.702411464 \\ 0.545865274 \\ 0.361767923 \end{pmatrix}$	$\begin{pmatrix} -1.258071455 \\ -1.793515720 \\ 0.662019680 \\ 0.359573224 \end{pmatrix}$
$\lVert y^{*,\mathrm{I}} - \bar{y}^{*} \rVert$	0	0.544995777	0.713541145	0.874887333
$y^{*,\mathrm{II}}$	$\begin{pmatrix} -0.967381871 \\ -1.226057898 \\ 0.064205530 \\ 0.320365142 \end{pmatrix}$	$\begin{pmatrix} -1.039383356 \\ -1.705205521 \\ 0.461673225 \\ 0.372236676 \end{pmatrix}$	$\begin{pmatrix} -1.062968562 \\ -1.873373862 \\ 0.594388183 \\ 0.396827110 \end{pmatrix}$	$\begin{pmatrix} -1.086598765 \\ -2.045504958 \\ 0.727104441 \\ 0.425087073 \end{pmatrix}$
$\lVert y^{*,\mathrm{II}} - \bar{y}^{*} \rVert$	0	0.628838520	0.845632811	1.065883579
$\lVert y^{*,\mathrm{I}} - y^{*,\mathrm{II}} \rVert$	0	0.139378505	0.219902626	0.318480063
Iter	39	30	26	26

Example 5.5.2 The problem data are given as follows:

$$H_{\mathrm{I}} = \begin{pmatrix} 3.6 & 1.2 \\ 1.2 & 4.5 \end{pmatrix}, H_{\mathrm{II}} = \begin{pmatrix} 2.7 & 1.5 \\ 1.5 & 4.8 \end{pmatrix}, D_{\mathrm{I}} = \begin{pmatrix} 1.9 & 3.2 & 1.6 & 1.5 \\ 2.4 & 1.4 & 1.8 & 2.3 \end{pmatrix},$$

$$D_{\mathrm{II}} = \begin{pmatrix} 2.6 & 1.7 & 3.5 & 1.6 \\ 2.3 & 3.8 & 1.6 & 2.8 \end{pmatrix}, E_{\mathrm{I}} = \begin{pmatrix} 1.7 & 1.3 \\ 2.6 & 1.8 \end{pmatrix}, E_{\mathrm{II}} = \begin{pmatrix} 0.8 & 2.6 \\ 1.4 & 2.2 \end{pmatrix},$$

$$R_{\mathrm{I}} = \begin{pmatrix} 2.4 & 3.3 \\ 2.5 & 1.2 \end{pmatrix}, R_{\mathrm{II}} = \begin{pmatrix} 2.5 & 1.3 \\ 1.4 & 3.2 \end{pmatrix}, B = \begin{pmatrix} 4.8 & 1.3 & 0.7 & 1.2 \\ 1.3 & 3.4 & 1.6 & 2.1 \\ 0.7 & 1.6 & 2.5 & 1.3 \\ 1.2 & 2.1 & 1.3 & 4.6 \end{pmatrix},$$

$$A_{\mathrm{I}} = \begin{pmatrix} 1.2 & 1.6 & 3.2 \\ 1.5 & 1.3 & 1.1 \end{pmatrix}, A_{\mathrm{II}} = \begin{pmatrix} 2.3 & 1.5 & 3.4 \\ 2.5 & 1.3 & 1.4 \end{pmatrix}, c = \begin{pmatrix} 2.1 \\ 1.3 \\ 2.6 \\ 1.4 \end{pmatrix},$$

$$A = \begin{pmatrix} 1.4 & 2.3 & 2.7 & 2.5 \end{pmatrix}, a = 3.2, b_{\mathrm{I}} = \begin{pmatrix} 1.3 \\ 1.7 \\ 2.4 \end{pmatrix}, b_{\mathrm{II}} = \begin{pmatrix} 2.9 \\ 2.1 \\ 0.4 \end{pmatrix}.$$

The computational results are shown in Table 5.2, where we can observe the computed solutions' behavior similar to that in the previous example. A difference is that the leaders' strategies change as the uncertainty bounds increase, since they are not vertex solutions in all cases.

Chapter 6

Conclusion

In this thesis, we have considered some multi-leader-follower games and equilibrium problems with equilibrium constraints which possess some special structures. We show some results on the existence and uniqueness of equilibria. We have also introduced the concept of robust L/F Nash equilibrium for a class of multi-leader-follower games under uncertainty, and discussed the properties of the robust L/F Nash equilibrium. The main idea to deal with these problems with two levels is to reformulate them as NEPs. We summarize the results obtained in this thesis as follows:

(a) In Chapter 3, we considered a class of multi-leader-follower games that satisfy some particular, but still reasonable assumptions, and show that these games can be formulated as ordinary Nash equilibrium problems, and then as variational inequalities. We established some results on the existence and uniqueness of a leader-follower Nash equilibrium. We also presented illustrative numerical examples from an electricity power market model.

(b) In Chapter 4, we considered a special class of EPECs where a common parametric P-matrix linear complementarity system is contained in all players' strategy sets. After reformulating the EPEC as an equivalent nonsmooth NEP, we used a smoothing method to construct a sequence of smoothed NEPs that approximate the original problem. We considered two solution concepts, global Nash equilibrium and stationary Nash equilibrium, and established some results about the convergence of approximate Nash equilibria. Moreover we showed some illustrative numerical examples.

(c) In Chapter 5, we focused on a class of multi-leader-follower games under uncertainty with some special structure. We particularly assumed that the follower's problem only contains equality constraints. By means of the robust optimization technique, we first formulated the games as robust Nash equilibrium problems, and then GVI problems. We then established some results on the existence and uniqueness of a robust leader-follower Nash equilibrium. We also applied the forward-backward splitting method to solve the GVI formulation of the problem and presented some numerical examples to illustrate the behavior of robust Nash equilibria.

As we summarized above, we have made several contributions on the multi-leader-follower games and the EPECs. However, due to some inherent difficulties, such as the lack of convexity and the failure of the constraint qualifications, this vigorous research field still has many problems unsolved. In what follows, we mention some future issues based on our current achievements.

(a) In Chapter 3, we restricted ourselves to a class of multi-leader-follower games, where only one follower solves a strictly convex quadratic programming problem with only equality constraints. Moreover, we also assumed that the second term of each leader's objective function appears in the follower's objective function in the negated form. However, these conditions may be restrictive from the practical point of view. It is worthwhile to improve the results under some weaker assumptions. For example, it seems more reasonable if there are inequality constraints in the follower problem.

(b) In Chapter 3, based on the special assumptions on the payoff functions of all leaders and follower, as well as the constraints in the follower's problem, we reformulated the multi-leader-follower game as an NEP by solving the KKT system of the follower problem explicitly to eliminate the follower's variable in the payoff functions of all leaders. Alternatively, instead of solving the KKT system explicitly, we may simply leave it as constraints of each leader's problem and deal with a GNEP with some shared constraints. We may also obtain some results on the properties of the L/F Nash equilibrium under other assumptions.

(c) In Chapter 4, we particularly considered a class of EPECs where each player's strategy set contains a shared parametric P-matrix linear complementarity system. We can further reformulate it as a standard NEP equivalently, due to some properties of P-matrix. However, from the practical point of view, such conditions are a little restrictive. It is worthwhile to improve the results under some weaker assumptions. For example, we may consider an EPEC where each player's strategy set contains a shared parametric P_0-matrix linear complementarity system.

(d) In Chapter 5, we considered a class of multi-leader-follower games under the Euclidean uncertainty assumption. It is also meaningful to consider the multi-leader-follower game under other uncertainty assumptions, such as the ellipsoidal or polyhedral uncertainty.

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