

Multiperiod Portfolio Selection with Second-order Cone Constraints

Guidance

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Abstract

The multiperiod portfolio selection model has attracted much attention in the financial field, because a portfolio is usually rebalanced dynamically in order to get a desired return even if the situation changes in the future. The scenario tree model is a popular multiperiod model which consists of many “scenario paths”. However, this model has the disadvantage that scenario paths grow explosively if we try to describe future situations in detail. In this paper, in order to overcome this disadvantage, we propose a model that allows returns on assets at each node to vary in a certain region, rather than increasing the number of branchings at each node of the scenario tree. The model can deal with shortfall risks in the case where the worst situation occurs in the region. We formulate the proposed model as a second-order cone programming (SOCP) problem, which can be efficiently solved by using the interior point method. Finally, we present some numerical results to show the usefulness of the proposed model.

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1 Introduction

Recently, many regulations on the financial investment have been relaxed dramatically, especially in Japan. This movement has made it possible for investors to control the allocation of their fund in a more flexible manner. On the other hand, the investors are asked to actively take risks for getting certain returns, because the interest rates have been kept in a low level in Japan. Under such circumstances, the risk management in investments has become increasingly important. Furthermore, mathematical approaches have gradually been recognized as useful tools for the risk management, not only in the academic field but also in the practical field. The purpose of this paper is to propose a model which is useful in constructing a low risk portfolio. To this end, we aim to extend the scenario tree model, which is one of popular multiperiod models for portfolio selection, by introducing the second-order cone constraints.

Until now, various mathematical models have been proposed to formulate the portfolio selection problem. Those models may be classified into two categories, the single-period model and the multiperiod model, from the number of decision making opportunities in the investment period. The single-period model, such as Mean-Variance model by Markowitz [4] and CAPM model by Sharpe [12], have had significant effect on the real investment in business. On the other hand, numerous papers about multiperiod models have appeared since the framework of multiperiod models was introduced by Merton [5, 6] and Samuelson [11]. As to these two models, it is known that the optimal portfolio obtained by solving a single-period model repeatedly is the same as that obtained by solving a multiperiod model, provided that the distribution of the returns on assets at each period is independently and identically distributed (i.i.d) and the investor's utility function satisfies some assumptions [7]. However, if the distribution of the returns on assets varies with period, the optimal portfolio in the multiperiod model may differ from the one in the single-period model, because the optimal portfolio in the multiperiod model has a factor of hedging against changes in the state variables. In fact, the analysis using real market data shows that the optimal portfolio in the multiperiod model differs from that of the single-period model [1, 2].

Multiperiod decision making has become an important subject in the practical investment as well as in the academic research on finance. One of the typical examples is the investment for pension funds, of which investment period is so long that the economic situation may change drastically. The investors of the pension fund are asked to get a certain return under any situation, so that they have to take various scenarios into account in advance. Therefore, the investors should rebalance their asset allocation according to the change of the economic situation during the investment period. This practical requirement supports the significance of the multiperiod model.

If we know the distribution of the returns on assets in advance, then we can formulate

the multiperiod model as a stochastic programming problem [6]. However, this problem is difficult to deal with, because the wealth at each period is expressed as the product of random variables that represent the rate of return on asset. Moreover, we need strong assumptions on the distribution of the returns on assets for getting an optimal solution of this problem. Therefore, it is helpful to consider approximation to the multiperiod model. The scenario tree model approximates the distribution of the returns on assets at each period by a discrete distribution.

Although the scenario tree model has the advantage that it allows us to make decision according to the progress of situations, it has the disadvantage that we cannot describe situations in detail because the number of scenarios increases explosively as the number of branchings at each node increases. In this paper, we propose a model that does not suffer from this disadvantage of the multiperiod scenario tree model. The basic idea underlying the proposed model is as follows. Unlike the conventional scenario tree model, we regard the rate of return on each asset as a parameter which varies in a certain region. Furthermore, we are particularly interested in the case where the rate of return on the whole portfolio becomes worst when the return on each asset varies in such a region. To translate this idea into action, we add a special constraint to the conventional scenario tree model. The constraint requires the wealth to be greater than a certain constant, even if the worst case occurs. By using this constraint, we can cover the situation which cannot be dealt with in the conventional scenario tree model. We represent this constraint as a second-order cone constraint. Therefore, the proposed scenario tree model is formulated as a Second-Order Cone Programming (SOCP) problem, which may be written as follows:

$$\begin{aligned}
& \text{minimize} && f'x \\
& \text{subject to} && \|A_i x + b_i\| \leq c_i'x + d_i \quad i = 1, \dots, N \\
& && Fx = g,
\end{aligned} \tag{1.1}$$

where $x \in R^n$ is the decision variable, $f \in R^n$, $A_i \in R^{(n_i-1) \times n}$, $b_i \in R^{n_i-1}$, $c_i \in R^n$, $d_i \in R$, $F \in R^{m \times n}$, $g \in R^m$ are problem parameters and \prime denotes transpose. We can solve this problem efficiently by using the interior point method.

This paper consists of five sections. In Section 2, we describe the conventional formulation of the multiperiod scenario tree model. In Section 3, we propose three models that involve second-order cone constraints. In Section 4, we report some numerical results for one of these models, and show the usefulness of the second-order cone programming formulation. In Section 5, we conclude this paper with some remarks.

2 Conventional Scenario Tree Model

In this section, we review the conventional formulation [14] of the scenario tree model. Throughout we suppose the following situations. The investor invests his/her wealth during T periods. That is, with the initial wealth W_0 , the investor first allocates W_0 to n assets, and then rebalances his/her assets at discrete time periods $t = 1, \dots, T - 1$. Finally he/she gets a terminal wealth at time $t = T$. We call the initial time $t = 0$ *the beginning period*, the intermediate times $t = 1, \dots, T - 1$ *the internal periods*, and the final time $t = T$ *the ending period*. We suppose that the investor knows possible future situations that may occur from the current situation, but does not know precisely which situation actually occurs. The situation changes from the beginning period to the ending period. We call the path of these situations a *scenario*. Note that some scenarios may have the same situations up to a certain period. These scenarios cannot be identified until we reach that period. Therefore, the investment strategy on these scenarios must be the same up to that period. From this reason, the progress of the situation can be expressed as a tree with height T (see Figure 1). In the scenario tree, nodes represent all possible situations that may occur in the future. We classify these nodes into three categories according to the height which corresponds to the time, that is, root node v_0 ($t = 0$), internal nodes ($t = 1, \dots, T - 1$) and leaf nodes ($t = T$). We use the following notations.

V	:	The set of all nodes
$L \subset V$:	The set of nodes at the ending period, that is, the set of leaf nodes
$U \subset V$:	The set of nodes at the internal periods
$v_0 \in V$:	The node at the beginning period, that is, the root node
$\pi_v \in R^n$:	The portfolio at node v
$W_v \in R$:	The wealth at node v
$\bar{R}_v \in R^n$:	The returns on assets when the node v occurs
$p_v \in [0, 1]$:	The probability that the node v occurs
$h(v)$:	The time when node v occurs
$w(v)$:	The parent node of node v
$I_v \in R$:	The additional fund (> 0) or the consumption (< 0) at node v
$\theta \in R$:	The target wealth at the ending period

Here, we assume

- The probability p_v is known for each $v \in V$;
- The return $\bar{R}_v \in R^n$ on assets is known for each $v \in V$;
- The target wealth at the ending period θ is given.

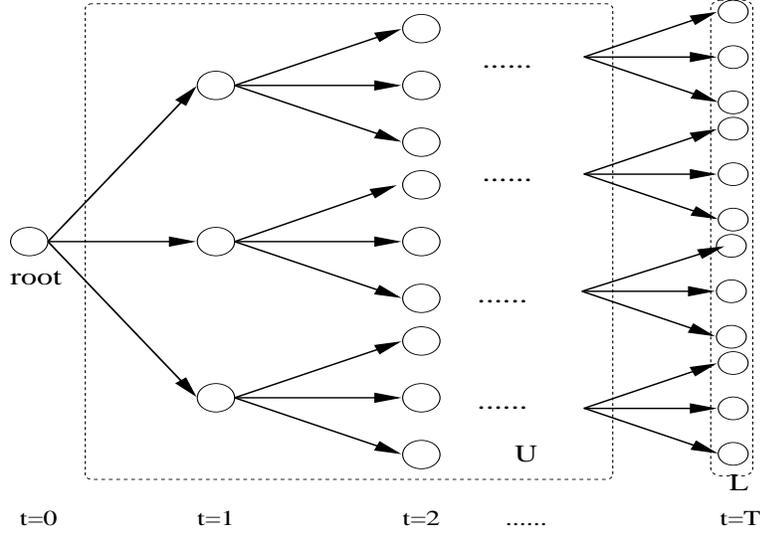


Figure 1: Scenario Tree

Under these assumptions, we consider the risk given by

$$\sum_{v \in L} p_v (\max \{0, \theta - W_v\})^k, \quad (2.1)$$

where k is a positive integer. This risk measure is called *the lower partial moment of dimension k* . The model based on this risk measure is called the Mean Lower Partial Moment model (MLPM model). Moreover, when the target wealth θ is replaced with $\sum_{v \in L} p_v W_v$ and $k = 2$, *the lower partial moment* reduces to *the lower semi-variance*. Recent studies [3, 9] prove that the mean standard semideviation (square root of semi-variance) model has a higher consistency with the expected utility theory than mean deviation model.

Now, we consider the problem of finding a portfolio $\pi_v (v \in \{v_0\} \cup U)$ which minimizes the lower partial moment with $k = 2$

$$\sum_{v \in L} p_v (\max \{0, \theta - W_v\})^2, \quad (2.2)$$

under the following two constraints: (i) The expected wealth in the ending period must be no less than a certain amount α . (ii) The investment in each node must satisfy the budget constraints on funds. In the multiperiod model, the budget constraints at each node depend on the investment action in the previous period. Therefore, the budget constraints at each node depend on the time $h(v)$. We consider the budget constraints for each node in detail. Let $\phi : R^n \rightarrow R$ be a transaction cost function. That is, $\phi(\pi_v - \pi_{w(v)})$ is the transaction cost when the investor rebalances his/her portfolio from $\pi_{w(v)}$ to π_v . To

simplify the discussion, we assume that the transaction cost function is separable, that is, it can be expressed as $\phi(\pi_v - \pi_{w(v)}) = \sum_{i=1}^n \phi_i((\pi_v)_i - (\pi_{w(v)})_i)$, where $\phi_i : R \rightarrow R$, $i = 1, \dots, n$. Moreover, we assume that the transaction cost $\phi_i((\pi_v)_i - (\pi_{w(v)})_i)$ on each asset i is linear. Therefore, it is expressed as $\phi(\pi_v - \pi_{w(v)}) = a'(\pi_v - \pi_{w(v)})$, where the vector $a \in R^n$ denotes the transaction cost of each asset. Then we may specify the budget constraints for each period as follows:

Asset allocation at the beginning period : The investor allocates his/her initial wealth W_0 on the root node v_0 . So the budget constraint is written as

$$e' \pi_{v_0} = W_0, \quad (2.3)$$

where e is the n -dimensional vector whose entries are all 1.

The wealth and rebalancing at each internal period : At node $v \in U$, the investor obtains the wealth W_v which is determined by the portfolio at the parent node $w(v)$. The wealth at node v therefore can be expressed as

$$W_v = \bar{R}'_v \pi_{w(v)}. \quad (2.4)$$

Furthermore, when the investor allocates his/her wealth on assets at node $v \in U$, he/she has to pay the transaction cost $\phi(\pi_v - \pi_{w(v)})$ incurred by the rebalance. The investor may also add new fund $I_v > 0$ or consume a part of his/her wealth $I_v < 0$. Therefore, the budget constraint at node $v \in U$ is given by

$$e' \pi_v + \phi(\pi_v - \pi_{w(v)}) = W_v + I_v. \quad (2.5)$$

The wealth at the ending period : The investor no longer rebalances his/her portfolio at node $v \in L$. So, the constraint at the ending period is simply written as

$$W_v = \bar{R}'_v \pi_{w(v)}. \quad (2.6)$$

Now we formulate the problem of minimizing the risk given by (2.2) under the constraints (2.3), (2.4), (2.5) and (2.6):

$$\begin{aligned} & \text{minimize} && \sum_{v \in L} \left[p_v (\max \{0, \theta - W_v\})^2 \right] \\ & \text{subject to} && \sum_{v \in L} p_v W_v \geq \alpha \\ & && \left. \begin{aligned} e' \pi_{v_0} &= W_0 && \text{root} \\ e' \pi_v + \phi(\pi_v - \pi_{w(v)}) &= W_v + I_v \\ W_v &= \bar{R}'_v \pi_{w(v)} \end{aligned} \right\} && v \in U \\ & && W_v = \bar{R}'_v \pi_{w(v)} && v \in L, \end{aligned}$$

where the decision variables are the portfolios π_v at nodes $v \in \{v_0\} \cup U$, and the wealths W_v at nodes $v \in U \cup L$ are dependent on π_v and the additional funds I_v at nodes $v \in U$ are given constants. Since it is not convenient to deal with the function $\max\{0, \theta - W_v\}$, we transform this problem into an equivalent problem by using the artificial variables $y_v, z_v \in R^{|L|}$ such that

$$\theta - W_v = y_v - z_v, \quad y_v \geq 0, z_v \geq 0. \quad (2.7)$$

Then, we can rewrite the above problem as follows:

[**Problem 1**]

$$\begin{aligned} & \text{minimize} && \sum_{v \in L} p_v y_v^2 \\ & \text{subject to} && \sum_{v \in L} p_v W_v \geq \alpha \\ & && \left. \begin{aligned} e' \pi_{v_0} &= W_0 && \text{root} \\ \left. \begin{aligned} e' \pi_v + \phi(\pi_v - \pi_{w(v)}) &= W_v + I_v \\ W_v &= \bar{R}'_v \pi_{w(v)} \end{aligned} \right\} && v \in U \\ \left. \begin{aligned} W_v &= \bar{R}'_v \pi_{w(v)} \\ \theta - W_v &= y_v - z_v, \quad y_v \geq 0, z_v \geq 0 \end{aligned} \right\} && v \in L \end{aligned} \right\} \end{aligned}$$

Note that this problem is a convex quadratic programming problem. If we use the risk measure $\max\{0, \theta - W_v\}$, instead of $\max\{0, \theta - W_v\}^2$, we can formulate the portfolio selection problem as a liner programming problem by means of a similar procedure. The liner programming problem is easier to solve than the quadratic programming problem. Therefore, when we deal with many assets, we may employ this risk measure $\max\{0, \theta - W_v\}$.

In **Problem 1**, the number of nodes and hence the number of decision variables increase exponentially with the number of periods or branchings of situations. Until now, various methods, such as the interior point method [15] and the decomposition method using parallel computing [8, 10], have been proposed for solving **Problem 1**. However, it is reported that those methods hardly reach the level of business use [14].

3 The Proposed Models

In this section, we present three models that cover the disadvantage of the conventional scenario tree model.

Since the scenario tree model can reflect the economical trends, it provides a rough strategy of the investment under varying situations. This is the very advantage of the scenario tree model. On the other hand, as mentioned in Section 2, the scenario tree model has the serious disadvantage that it cannot describe future situations in full detail.

To overcome this difficulty, we regard the rate of return on each node as a parameter which varies in a certain region. For each node v , let a set $\Gamma_v \subseteq R^n$ be defined by

$$\Gamma_v := \{\bar{R}_v + H_v u \mid \|u\| \leq 1\}, \quad (3.1)$$

where $H_v \in R^{n \times n}$ is given matrix. We call this set a return set for the node v . We suppose that the rate of return R_v at node v should belong to the return set Γ_v .

Remark 1 *If matrix H_v in (3.1) is nonsingular, then $R_v = \bar{R}_v + H_v u \in \Gamma_v$ if and only if*

$$\|u\|^2 = (R_v - \bar{R}_v)' H_v^{-2} (R_v - \bar{R}_v) \leq 1, \quad (3.2)$$

and hence R_v belongs to an ellipsoid centered at \bar{R}_v .

Remark 2 *If H_v is a diagonal matrix with positive diagonal entries σ_i , ($i = 1, \dots, n$), then, $R_v \in \Gamma_v$ satisfies $\sum_{i=1}^n \sigma_i^{-2} (R_v - \bar{R}_v)_i^2 \leq 1$, which particularly implies a return on asset i lies in between $(\bar{R}_v)_i - \sigma_i$ and $(\bar{R}_v)_i + \sigma_i$.*

Now, we consider the case where the rate of return R_v on the whole portfolio becomes worst when the return R_v varies in the return set Γ_v . In the following, we will use the same notations as in the previous section.

3.1 The model securing floors at each node

We consider the requirement that the wealth at each node be no less than a certain constant β_v . This β_v means the lowest acceptable wealth of the investor at node v , and it is called a *floor* at node v . This requirement may be represented as follows.

$$\text{[floor constraints]} \quad \min_{R_v \in \Gamma_v} R'_v \pi_{w(v)} \geq \beta_v \quad v \in U \cup L. \quad (3.3)$$

It is easy to understand the meaning of the constraints, but it is not convenient to deal with. So, we transform the constraints into second-order cone constraints. Since

$$\begin{aligned} \min_{R_v \in \Gamma_v} R'_v \pi_{w(v)} &= \min_{\|u\| \leq 1} (\bar{R}'_v \pi_{w(v)} + u' H'_v \pi_{w(v)}) \\ &= \bar{R}'_v \pi_{w(v)} + \min_{\|u\| \leq 1} u' (H_v \pi_{w(v)}) \\ &= \bar{R}'_v \pi_{w(v)} - \|H_v \pi_{w(v)}\|, \end{aligned} \quad (3.4)$$

the constraint (3.3) is rewritten as

$$\bar{R}'_v \pi_{w(v)} - \|H_v \pi_{w(v)}\| \geq \beta_v. \quad (3.5)$$

These constraints are second-order cone constraints.

Adding the constraints (3.5) to **Problem 1**, we get the problem securing the floors at each node:

[**Problem 2**]

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & s \geq \|(\text{diag}(p))^{1/2} y\| \\ & \sum_{v \in L} p_v W_v \geq \alpha \\ & \left. \begin{array}{l} e' \pi_{v_0} = W_0 \\ e' \pi_v + \phi(\pi_v - \pi_{w(v)}) = W_v + I_v \\ W_v = \bar{R}'_v \pi_{w(v)} \end{array} \right\} \begin{array}{l} \text{root} \\ v \in U \end{array} \\ & \left. \begin{array}{l} \theta - W_v = y_v - z_v, \quad y_v \geq 0, z_v \geq 0 \\ W_v = \bar{R}'_v \pi_{w(v)} \end{array} \right\} v \in L \\ & \|H_v \pi_{w(v)}\| \leq \bar{R}'_v \pi_{w(v)} - \beta_v \quad v \in U \cup L, \end{array}$$

where y is the vector whose entries are y_v ($v \in L$) and $\text{diag}(p)$ is the diagonal matrix whose diagonal entries are p_v ($v \in L$). This problem is SOCP, and hence it can be efficiently solved by the interior point method.

3.2 The model securing floors on each scenario

In this subsection, we are interested in securing a desired return at the ending period even if the worst case occurs. Let us pay attention to a scenario itself. Since a scenario is a path of situations changing from the beginning period to the ending period, a scenario i can be represented as

$$\tau^i = \{v_0, v(1, i), v(2, i), \dots, v(T, i)\}, \quad (3.6)$$

where $v(t, i) \in V$ ($t = 1, \dots, T$). We consider the extreme case where the return of the portfolio $\pi_{v(t, i)}$ becomes worst at every node $v(t, i) \in \tau^i$ ($t = 1, \dots, T$). In other words, we suppose that the wealth $W_{v(t, i)}$ at each node $v(t, i) \in \tau^i$ is no greater than $\min_{R_{v(t, i)} \in \Gamma_{v(t, i)}} R'_{v(t, i)} \pi_{w(v(t, i))}$. Moreover, the budget in the next period is estimated by the lowest possible wealth. Consequently, the ending wealth $W_{v(T, i)}$ becomes the worst among all possible wealths yielded by the scenario τ^i . Based on this idea, we give the

following budget constraints.

$$\begin{aligned}
& e' \pi_{v_0} = W_0 && \text{root} \\
& \left. \begin{aligned} e' \pi_v + \phi(\pi_v - \pi_{w(v)}) &= W_v + I_v \\ W_v &\leq \min_{R_v \in \Gamma_v} R'_v \pi_{w(v)} \end{aligned} \right\} && v \in U \\
& W_v \leq \min_{R_v \in \Gamma_v} R'_v \pi_{w(v)} && v \in L.
\end{aligned} \tag{3.7}$$

Using the equation (3.4), the constraint $W_v \leq \min_{R_v \in \Gamma_v} R'_v \pi_{w(v)}$ can be rewritten as

$$W_v \leq \bar{R}'_v \pi_{w(v)} - \|H_v \pi_{w(v)}\|. \tag{3.8}$$

Replacing the constraints of **Problem 1** with (3.7) and using (3.8), we get the following problem.

[**Problem 3**]

$$\begin{aligned}
& \text{minimize} && s \\
& \text{subject to} && s \geq \|(\text{diag}(p))^{1/2} y\| \\
& && \sum_{v \in L} p_v W_v \geq \alpha \\
& && e' \pi_{v_0} = W_0 && \text{root} \\
& && \left. \begin{aligned} e' \pi_v + \phi(\pi_v - \pi_{w(v)}) &= W_v + I_v \\ \|H_v \pi_{w(v)}\| &\leq \bar{R}'_v \pi_{w(v)} - W_v \end{aligned} \right\} && v \in U \\
& && \left. \begin{aligned} \|H_v \pi_{w(v)}\| &\leq \bar{R}'_v \pi_{w(v)} - W_v \\ \theta - W_v = y_v - z_v, \quad y_v \geq 0, z_v \geq 0 \end{aligned} \right\} && v \in L
\end{aligned}$$

Similarly to **Problem 2**, we may also add the constraint (3.5) to **Problem 3**. Then the constraints are given by

$$\begin{aligned}
& e' \pi_0 = W_0 && \text{root} \\
& \left. \begin{aligned} e' \pi_v + \phi(\pi_v - \pi_{w(v)}) &= W_v + I_v \\ \|H_v \pi_{w(v)}\| &\leq \bar{R}'_v \pi_{w(v)} - W_v \end{aligned} \right\} && v \in U \\
& \|H_v \pi_{w(v)}\| \leq \bar{R}'_v \pi_{w(v)} - W_v && v \in L \\
& \|H_v \pi_{w(v)}\| \leq \bar{R}'_v \pi_{w(v)} - \beta_v && v \in U \cup L.
\end{aligned}$$

The combined problem is also SOCP, so this problem can be also solved efficiently [AppendixA].

3.3 The model treating asymmetric returns

In the previous sections, we defined a return set Γ_v by using matrix H_v . This implies that the return R_v belongs to a symmetric region. However, the distribution of the returns on

assets is generally asymmetric, especially in the case of the derivatives such as options. Note that an option is the right to buy or sell an asset at the price (strike price) which is determined in advance. So an option holder exercises his/her option only if he/she can profit from it, and he/she abandons it otherwise. That is, the return on the option depends on whether the holder exercises or not. This implies that the distribution of the return on an option is asymmetric.

In such cases, the equality

$$\min_{\bar{R}_v \in \Gamma_v} \bar{R}'_v \pi_{w(v)} = \bar{R}'_v \pi_{w(v)} - \|H_v \pi_{w(v)}\|$$

is not satisfied in general. Therefore, the wealth in the worst case cannot be expressed as a second-order cone constraint as in the previous subsections. In the following, we describe a return set Γ_v by using two diagonal matrices H_v^+ and H_v^- , where H_v^+ are used for holding assets and H_v^- for shorting assets. Moreover we define H_v as

$$H_v = \begin{pmatrix} H_v^+ & 0 \\ 0 & H_v^- \end{pmatrix}. \quad (3.9)$$

Using this matrix, we define the region Γ_v as

$$\Gamma_v = \left\{ \left(\begin{array}{c} \bar{R}_v \\ -\bar{R}_v \end{array} \right) + H_v \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \mid u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \quad u^+ \in R^n, u^- \in R^n, \quad \|u\| \leq 1 \right\}. \quad (3.10)$$

Using this return set, we may derive an equation similar to (3.4) for asymmetric returns. To this end, we represent the portfolio π_v on node v as the difference of two nonnegative vectors π_v^+ and π_v^- , that is,

$$\pi_v = \pi_v^+ - \pi_v^-, \quad \pi_v^+ \geq 0, \pi_v^- \geq 0. \quad (3.11)$$

Note that π_v^+ corresponds to the holding portfolio and π_v^- corresponds the shorting portfolio.

Let

$$\Pi_{w(v)} = \begin{pmatrix} \pi_{w(v)}^+ \\ \pi_{w(v)}^- \end{pmatrix}.$$

Then we have

$$\begin{aligned} \min_{\bar{R}_v \in \Gamma_v} \bar{R}'_v \Pi_{w(v)} &= \min_{\|u\| \leq 1} \left(\begin{pmatrix} \bar{R}_v \\ -\bar{R}_v \end{pmatrix}' \Pi_{w(v)} + u' H'_v \Pi_{w(v)} \right) \\ &= \begin{pmatrix} \bar{R}_v \\ -\bar{R}_v \end{pmatrix}' \Pi_{w(v)} + \min_{\|u\| \leq 1} u' (H_v \Pi_{w(v)}) \\ &= \begin{pmatrix} \bar{R}_v \\ -\bar{R}_v \end{pmatrix}' \Pi_{w(v)} - \|H_v \Pi_{w(v)}\| \end{aligned}$$

$$\begin{aligned}
&= \bar{R}'_v \pi_{w(v)}^+ - \bar{R}'_v \pi_{w(v)}^- - \left\| \begin{pmatrix} H_v^+ \pi_{p(v)}^+ \\ H_v^- \pi_{p(v)}^- \end{pmatrix} \right\| \\
&= \bar{R}'_v \pi_{w(v)} - \left\| \begin{pmatrix} H_v^+ \pi_{p(v)}^+ \\ H_v^- \pi_{p(v)}^- \end{pmatrix} \right\|. \tag{3.12}
\end{aligned}$$

Therefore, we can deal with an asymmetric return distribution by means of second-order cone constraints.

The obtained solution for this case seems to suggest that the investor should buy and sell the same asset at the same time. However, we can show that the portfolio π_v obtained from (3.11) is valid even for such assets. To this end, suppose that π_v^+ and π_v^- , satisfy the following inequality for a given $\eta > 0$:

$$\left\| \begin{pmatrix} H_v^+ \pi_v^+ \\ H_v^- \pi_v^- \end{pmatrix} \right\| \leq \eta. \tag{3.13}$$

Let $\tilde{\pi}_v^+, \tilde{\pi}_v^-$ satisfy (3.11) with $(\tilde{\pi}_v^+)_i = 0$ or $(\tilde{\pi}_v^-)_i = 0$ for each i . Here, without loss of generality, we consider $\tilde{\pi}_v^- = 0$. Then, we have $\pi_v = \tilde{\pi}_v^+, \tilde{\pi}_v^+ \geq 0$ and $\tilde{\pi}_v^- = 0$ from (3.11). Since H_v^+ and H_v^- are diagonal matrices, and since $0 \leq \tilde{\pi}_v^+ \leq \pi_v^+$, we have

$$\begin{aligned}
\|H_v^+ \tilde{\pi}_v^+\|^2 &\leq \|H_v^+ \pi_v^+\|^2 \\
\|H_v^- \tilde{\pi}_v^-\|^2 = 0 &\leq \|H_v^- \pi_v^-\|^2.
\end{aligned}$$

By adding these inequalities, we obtain

$$\left\| \begin{pmatrix} H_v^+ \tilde{\pi}_v^+ \\ 0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} H_v^+ \pi_v^+ \\ H_v^- \pi_v^- \end{pmatrix} \right\| \leq \eta, \tag{3.14}$$

where the last inequality follows from (3.13). This implies that $(\tilde{\pi}_v^+, 0)$ is also a solution of the problem, indicating the validity of the representation (3.11).

4 Numerical Experiments

In this section, we report some numerical experiments for the proposed model with real market data. First, we will explain how to construct a scenario tree from the real market data. Next we will solve **Problem 1** and **Problem 2**, and compare the obtained solutions from the viewpoint of risk.

4.1 The model used in the experiments

In this subsection, we describe how to construct the scenario tree model used in the numerical experiments from real market data. Moreover, we modify **Problem 1** and **Problem 2**, so that the problems involve *the short-sale constraints*. The reason is that the optimal solutions of the original **Problem 1** and **Problem 2** may not be realistic, since the portfolio may include a large amount of short selling.

The scenario tree : In our experiments, we consider three assets; stock, bond and cash.

We assume that the returns on these assets follow a multivariate normal distribution. We estimate the mean vector $\mu \in R^3$ of the three assets, and the variance-covariance matrix $\Sigma \in R^{3 \times 3}$ from the historical data. The used data are annual returns from 1990 to 2001 in Japanese market. We determine the return on each node of scenario tree by using the multivariate normal distribution with μ and Σ . That is, the price $S_i(t)$ ($i = 1, 2, 3$) of each asset i at each discrete time t ($t = 0, 1, \dots, T$) is governed by

$$\Delta S(t) = D(t) \left(\mu + (\Sigma)^{1/2} \varepsilon \right), \quad (4.1)$$

where

$$\Delta S(t) = \begin{pmatrix} \Delta S_1(t) \\ \Delta S_2(t) \\ \Delta S_3(t) \end{pmatrix} \equiv \begin{pmatrix} S_1(t+1) - S_1(t) \\ S_2(t+1) - S_2(t) \\ S_3(t+1) - S_3(t) \end{pmatrix}, \quad D(t) = \begin{pmatrix} S_1(t) & 0 & 0 \\ 0 & S_2(t) & 0 \\ 0 & 0 & S_3(t) \end{pmatrix}$$

with the normal random variable $\varepsilon \in R^3$. We generate a scenario tree with $T = 4$. Each node of the scenario tree has 5 branchings. Therefore, the whole scenario tree has 781 nodes. The rate of the return of each scenario is determined by (4.1).

The return sets Γ_v : We give the return sets Γ_v used in the proposed model as

$$\Gamma_v := \{ \bar{R}_v + \delta (\Sigma)^{1/2} u \mid \|u\| \leq 1 \},$$

where Σ is a variance-covariance matrix obtained from the historical data and δ is a positive constant. The reason why we choose Γ_v as the return set is as follows. If the return on assets at node v is given by (4.1), then the return on assets is expressed as

$$\bar{R}_v = \mu + (\Sigma)^{1/2} \varepsilon_v,$$

where ε_v denotes a constant vector randomly generated from normal distribution. So, it is reasonable to expect that R_v belongs to the set Γ_v .

The short-sale constraints : The short-sale constraints are defined by

$$\pi_v \geq -\nu_v \quad \forall v \in v_0 \cup U,$$

where $\nu_v \in R^n$ is a given vector whose components are nonnegative. This constraint means that the amount of short selling should not exceed ν_v . Without these constraints, the optimal portfolio may include a large amount of short selling, especially in the model that used the historical data in Japan. To avoid unrealistic shorting of assets, we add the short-sale constraints to the original problems. Note that the constraints are liner, and hence adding the constraints to a SOCP does not increase the complexity of the problem.

4.2 Numerical results

To see the usefulness of the proposed model, we made some numerical experiments with the scenario tree constructed as described in the previous subsection, and compared it with the conventional model.

We coded programs with MATLAB Version 5. We solved **Problem 2**, which is the second order cone problem, by using SeDuMi [13]. Note that SeDuMi is a solver for convex minimization problems on a self-dual cone and it uses the interior point method.

Simulations

First we explain our simulations. In the practical investment, the obtained strategy is used only in the beginning period, and will not be in the future periods. This is because the actual changes of situations are usually different from those predicted initially, and hence it does not make sense to follow the strategy through the remaining periods. In practice, whatever the investor allocates at the beginning period, he/she would consider a new scenario tree in the next period and update the strategy using the revised information. This procedure is repeated recursively from the beginning period $t = 0$ to the ending period $t = T$, and the resulting final wealth is evaluated.

The procedure is as follows.

Step.0 Solve the problem with the scenario tree constructed in the manner explained in the previous subsection, and obtain the portfolio π_0^* . Set $t := 1$.

Step.1 Generate a sample return R by

$$R = \mu + (\Sigma)^{1/2}\varepsilon,$$

where ε is a vector randomly generated from normal distribution, and revise the wealth by $R'\pi_{t-1}^*$.

Step.2 If $t = T$, then terminate with the final wealth $R'\pi_{t-1}^*$. Otherwise, let the initial wealth be $R'\pi_{t-1}^*$ and solve the problem with $T - t$ investment periods to obtain the updated portfolio π_t^* . Set $t := t + 1$ and go to **Step.1**.

In our experiments, we did the above simulations J times for **Problem 1** and **Problem 2** with the same scenario tree. Let the investor's final wealth obtained in the j th simulation be W_T^j and define

$$\bar{W} := \frac{1}{J} \sum_{j=1}^J \max(\theta - W_T^j, 0)^2. \quad (4.2)$$

This value means the average of the risks obtained through the simulations. We did the simulations with various values of the parameter α (see below). Note that when the problem was infeasible at a certain period, we did not rebalance the portfolio at that period, and kept holding the portfolio of the previous period.

We used the following values of the parameters.

Investment period T	$T = 4$
The number of assets n	$n = 3$
The number of branchings in each node	5
The initial wealth W_0	$W_0 = 100$
The matrix H_v at each node v	$H_v = 0.5 \Sigma^{1/2}$
The floor β_v at node v	$\beta_v = W_0$
The target wealth θ	$\theta = 1.055^4 W_0$
The number of simulation paths J	$J = 100$
The required expected return α	$\alpha = (1.03 + 0.0025k)^4 W_0 \quad (k = 1, \dots, 30)$
The additional fund I_v at node v	$I_v = 0$
The vector of transaction costs a	$a = (0.01, 0.005, 0.001)'$

Results of Simulations

We present the results of simulations in Figures 2, 3, 4 and 5.

Figure 2 shows the relationship between the average risk \bar{W} and the expected return at the ending period. Figures 3 and 4 show the expected return and the average risk at the ending period, respectively, for each value of the required expected return α . From Figure 3, we see that the expected return at the ending period in the proposed model is almost the same as in the conventional model. However, Figure 4 indicates that the proposed model can reduce the risk compared with the conventional model. From this, we conclude that the proposed model is useful in finding a portfolio with lower risk. Figure 5 shows the shortfall risk for each value of the required expected return α . It may be observed from the figure that the proposed model is also useful in securing the floors.

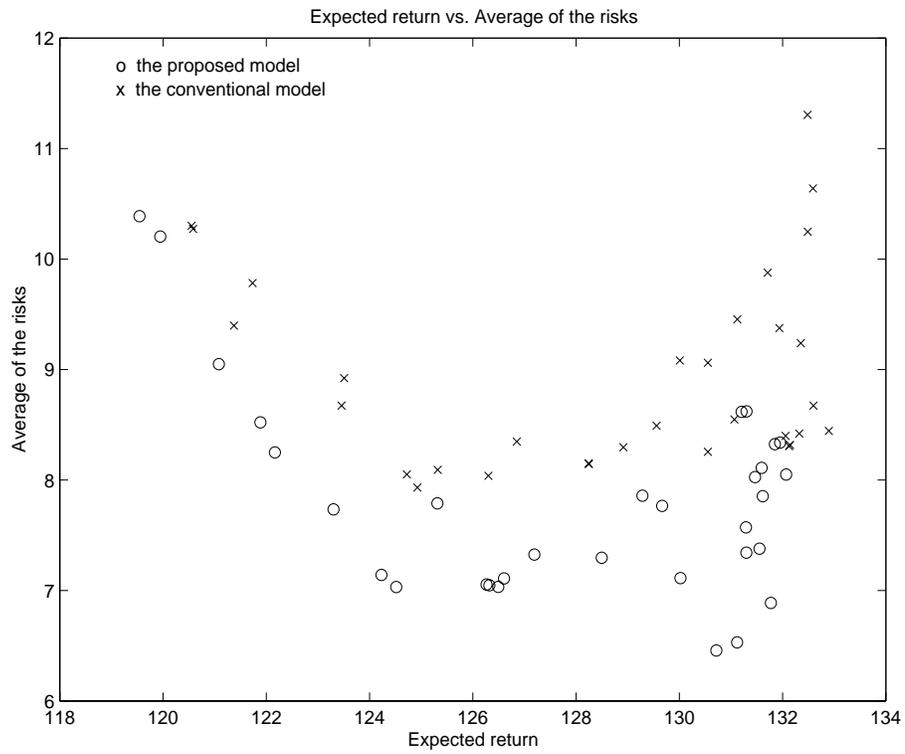


Figure 2: Expected return vs. the average of risks

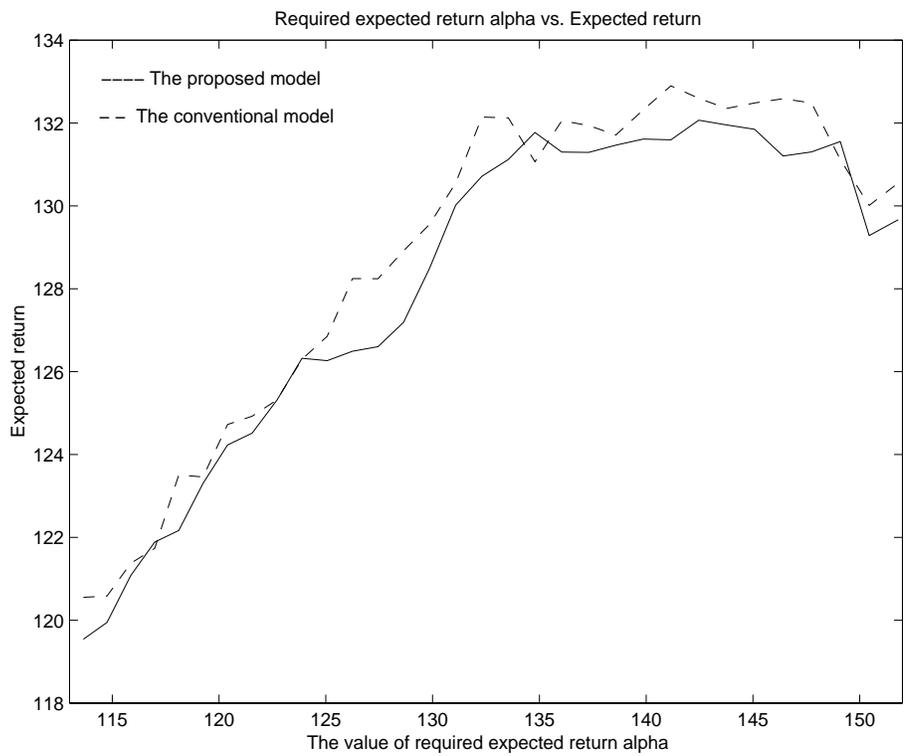


Figure 3: The required expected return α vs. expected return

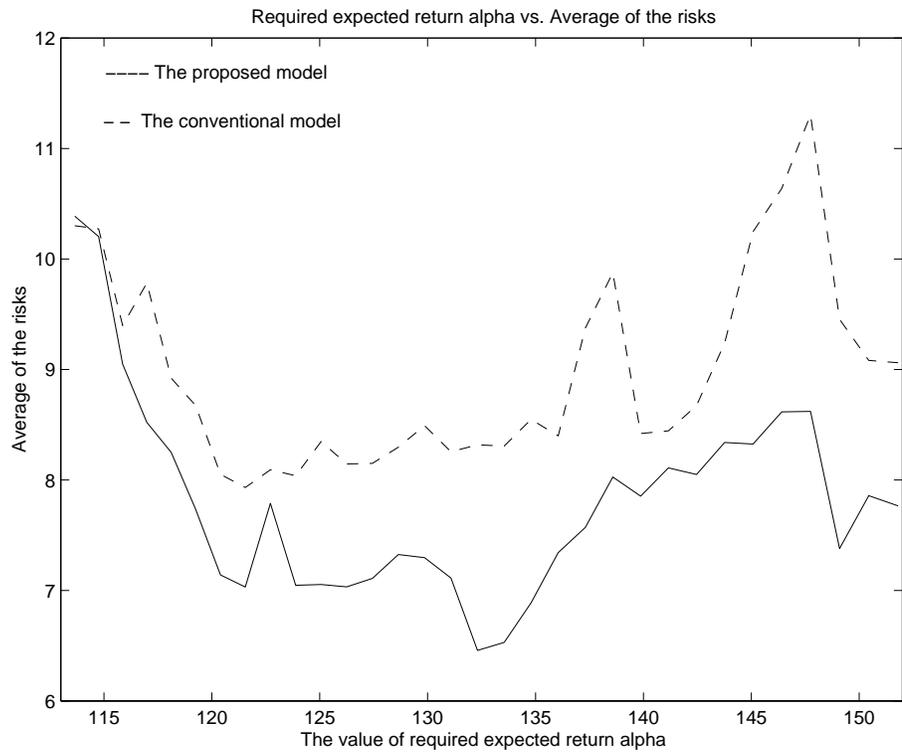


Figure 4: The required expected return α vs. the average of risks

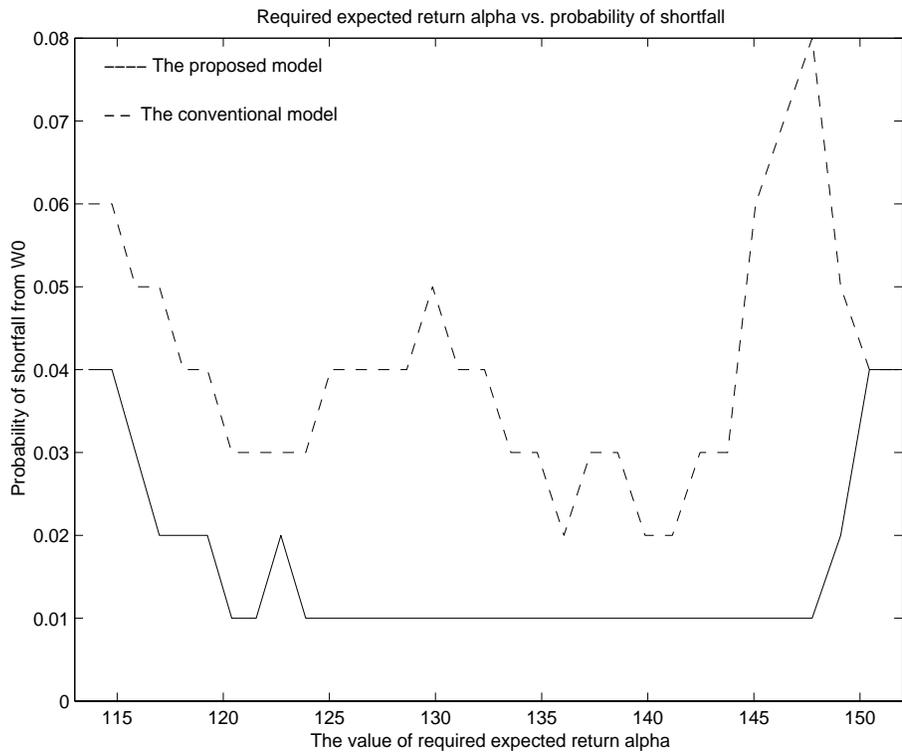


Figure 5: Shortfall risk

5 Conclusion

In this paper, we have proposed the model that extends the scenario tree model. The proposed model takes into account the worst return of the portfolio at each node by means of second-order cone constraints and is easy to solve by the interior point method. Moreover, to show the usefulness of this model, we did some simulations. The numerical results indicate that the proposed model is useful in reducing the shortfall risk.

Finally, we mention further research topics. In the simulations, we assumed that the asset follows the multivariate normal distribution, and we used the squared root of the variance-covariance matrix as H_v in the return set Γ_v . It may be worth considering how to construct a more practical scenario tree and H_v for business use. Another important task we have to do is to deal with models with practical (large) size.

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References

- [1] M. J. Brennan, E. S. Schwartz and R. Lagnado, Strategic asset allocation, *Journal of Economic Dynamics and Control* 21 (1997), 1377–1403.
- [2] J. Campbell and L. M. Viceira, Consumption and portfolio decisions when expected returns are time varying, *Quarterly Journal of Economics* 114 (1999), 433–495.
- [3] H. Konno, Financial engineering II, (in Japanese), Nikkagiren-syuppansya, tokyo, 1998.
- [4] H. Markowitz, *Portfolio selection: efficient diversification of Investments*, John Wiley & Sons, New York, 1959.
- [5] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous time case, *Review of Economics and Statistics* 51 (1969), 247–257
- [6] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3 (1971), 373–413.

- [7] J. Mossin, Optimal multiperiod portfolio policies, *Journal of Business* 41 (1968), 215–229.
- [8] J. M. Mulvey and A. Ruszczyński, A new scenario decomposition method for large-scale stochastic optimization, *Operations Research* 43 (1995), 477–490.
- [9] W. Ogryczak and A. Ruszczyński, From stochastic dominance to mean-risk models: Semideviations as risk measures, Interim Report 97–027, International Institute for Applied Systems Analysis, Laxenburg, Austria (1997).
- [10] T. Rockafellar and R. J. Wets, Scenarios and policy aggregation in optimization under uncertainty, *Mathematics of Operations Research* 16 (1987), 32–49.
- [11] P. A. Samuelson, Lifetime portfolio selection by dynamic stochastic programming, *Review of Economics and Statistics* 51 (1969), 238–246.
- [12] W. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance* 19 (1964), 425–442.
- [13] J. F. Sturm, Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones, *Optimization Methods and Software* 11 & 12 (1999), 625–653.
- [14] H. Takehara, optimization of portfolio, (in Japanese), Asakura-Shoten, Tokyo, 1997.
- [15] H. Takehara, An interior point algorithm for large scale portfolio optimization, *Annals of Operations Research* 45 (1993), 373–386.
- [16] T. Tsuchiya, A convergence analysis of the scaling-invariant primal-dual path-following algorithms for second-order cone programming, *Optimization Methods and Software* 11/12 (1999), 141–182.

A Second-Order Cone Programming (SOCP)

The Second-order Cone Programming (SOCP) problem is stated as follows.

[**SOCP**]

$$\begin{aligned} & \text{minimize} && f'x \\ & \text{subject to} && \|A_i x + b_i\| \leq c'_i x + d_i \quad i = 1, \dots, N \\ & && Fx = g, \end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm, i.e., $\|z\| = \sqrt{z'z}$ and $x \in R^n$, $f \in R^n$, $A_i \in R^{(n_i-1) \times n}$, $b_i \in R^{n_i-1}$, $c_i \in R^n$, $d_i \in R$, $F \in R^{m \times n}$, $g \in R^m$. The constraint

$$\|A_i x + b_i\| \leq c'_i x + d_i \quad i = 1, \dots, N$$

is called a second-order cone constraint. The second-order cone of dimension k is defined by

$$C_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \mid u \in R^{k-1}, t \in R, \|u\| \leq t \right\}.$$

For $k = 1$, the second-order cone is given by

$$C_1 = \{t \mid t \in R, 0 \leq t\}.$$

Therefore, the linear constraint

$$m_i x + q_i \geq 0 \quad i = 1, \dots, L$$

can be expressed as a second-order cone constraint with $n = 1$. So, the SOCP contains the linear programming (LP) problem as a special case.

A second-order cone is a convex set, and the set of points satisfying the second-order cone constraint $\|A_i x + b_i\| \leq c'_i x + d_i$ is the inverse image of the second-order cone under an affine mapping:

$$\|A_i x + b_i\| \leq c'_i x + d_i \iff \begin{bmatrix} A_i \\ c'_i \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in C_{n_i}.$$

Hence the set of points satisfying a second-order cone constraint is convex. The problem [SOCP] is a convex programming problem since the problem has a convex cost function and a convex constraint set.

The SOCP can be solved by using the primal-dual interior point method in $O(\sqrt{N})$ iterations, and the computational cost of every iteration is $O(n^3 + n^2 \sum_{i=1}^N n_i)$. Therefore, we can solve SOCP in $O(\sqrt{N} n^3 + \sqrt{N} n^2 \sum_{i=1}^N n_i)$ arithmetic operations [16].