Optimal Transaction Strategy Incorporating Liquidity Risk

Guidance

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Abstract

Among market players, there has been a necessity of considering not only the current prices of assets but also their liquidity when they trade various kinds of assets in the market. Liquidity means a barometer which expresses whether investors can sell their holding assets with proper prices or not. When investors intend to sell their holding assets which have low liquidity, they may not find trading partners or they are forced to trade them with much lower prices than they expected. We call the possibility that investors may face such an inconvenient situation for them liquidity risk. In this paper, we assume the siutation where an investor sells his/her holding assets which have low liquidity, and propose an optimal transaction strategy incorporating liquidity risk. Particularly, we take into account the market impact which shows an influence of a price drop accompanied by the investor's own tranasaction. When the investor has significantly large assets, selling all of them at once will result in a substantial price drop due to the market impact. To alleviate the influence of a price drop, we assume that the investor sells his/her holding assets at several times by dividing them. Hence, the proposed model is formulated as a multiperiod decision-making problem. This paper adopts a scenario tree model, and then we formulate the model as a quadratically constrained convex programming problem. This problem may be transformed into a second-order cone programming (SOCP) problem, which can be solved efficiently by using an interior point method. Finally we conduct numerical experiments and report the results.

Contents

1 Introduction

Recently, market players have noticed liquidity of financial assets. Liquidity means a barometer which expresses whether investors can sell their holding assets with proper prices or not. When investors intend to sell their holding assets which have low liquidity, they may not find trading partners or they are forced to trade them with much lower prices than they expected. So far, investors have assumed that financial assets have enough liquidity. However, seeing the examples of Asian currency crisis in 1997 and Russian currency crisis in 1998, investors have realized the case where liquidity of assets could be low. Hence, it is hard to say that the assumption that financial assets have a plenty of liquidity always holds, and investors have to trade their assets considering the possibility that liquidity of their assets can be low. In this paper, we call this possibility the liquidity risk and define it as the risk which has the possibility that investors cannot sell (liquidize) their holding assets with proper prices. The purpose of this paper is to introduce an optimal transaction strategy for investors when they sell their holding assets which have low liquidity. Nowdays, there exists the necessity of developing a model which quantifies the liquidity risk and adopting it to trade assets among corporate investors.

When investors sell their holding assets in the market, some conventional risk measurements are used under the following assumptions: 1) An influence of a price drop accompanied by an investor's own tranasaction (we call the influence the market impact) is not considered. 2) Investors can sell all of their assets in short time period. However, it is doubtful that these assumptions always hold in the usual state of the market. For example, in the case where investors intend to sell a large portion of stocks at once, the influence on the market cannot be neglected. So they may not deal with transactions without taking into account the market impact. To remove the above assumptions, we consider the influence of a price drop called the market impact that is accompanied by an investor's own tranasaction, and express liquidity risk by doing so. While various approaches [1, 5, 6] have been proposed to formulate the market impact, there is no consensus of specific formulation. This paper extends the previous research [4], which expresses the market impact as a function which has volumes of assets as variables.

Next, we suppose that the investor sells his/her holding assets at several times by dividing them to reduce the influence of the market impact. In short, the investor is subject to make more than one decision [10], so we formulate a problem as a multiperiod model, especially a scenario tree model. The advantage of this model is that we can describe uncertainties of the future discretely at each node of a scenario tree model. Particularly in this paper, we express the price variations by each node of the scenario tree (see Figure 1), where the investor decides the sales volumes at

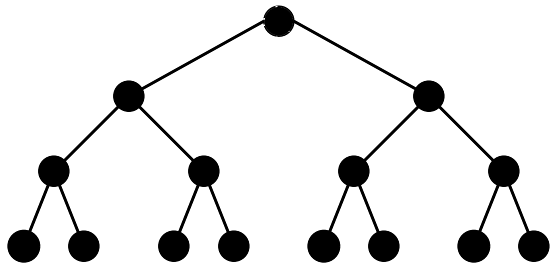

Figure 1: Scenario Tree

each node. However, the scenario tree model has the disadvantage that the size of the problem increases explosively [7] as we describe many situations in detail.

The model developed in this paper is formulated as a second-order cone programming (SOCP) problem, which is generally written as follows:

$$\begin{array}{ll} \text{minimize} & f^\top x \\ \text{subject to} & \|A_i x + b_i\| \le c_i^\top x + d_i, \quad (i = 1, \ldots, I) \\ & Bx = r, \end{array} \tag{1.1}$$

where $x \in \mathbf{R}^n$ is the decision variables, $f \in \mathbf{R}^n, A_i \in \mathbf{R}^{(n_i-1)\times n}, b_i \in \mathbf{R}^{n_i-1}, c_i \in \mathbf{R}^n, d_i \in \mathbf{R}, B \in \mathbf{R}^{m\times n}$ and $r \in \mathbf{R}^m$ are problem parameters. This problem is solved efficiently by using an interior point method.

This paper consists of five sections. In section 2, we formulate the model incorporating liquidity risk. In section 3, we transform the model proposed in section 2 into an SOCP problem. In section 4, we conduct some numerical experiments and report the results. Section 5 presents the conclusion.

2 The proposed model

2.1 Transaction strategy

In this section, we consider the following situation. The investor has a certain kind of assets and intends to sell the volumes N. If N is large to some extent and the investor sells all of N at once, a significant price drop will happen by the influence of the market impact, and then he/she will suffer from a big loss. Hence, we assume that the investor divides his/her holding assets to alleviate a price drop. Specifically, the investor aims at completing selling his/her assets until a certain period. Let the initial time be $t = 0$, and the investor will sell his/her assets at each time $t = 1, \ldots, T-1$, and will end up with selling them at the sales completion time $t = T$. We call the interval between time $t-1$ and time t term t.

We express the price variations by using scenarios because the investor cannot know how they will change in the future. Let us denote the set of scenarios as Ω, and the sales price and the sales volumes of assets at time t in scenario $\omega \in \Omega$ as π_t^ω and n_t^ω, respectively. The investor can gain profit $\pi_t^\omega n_t^\omega$ at time t in scenario ω. We introduce a discount rate $\rho \in (0, 1)$, so that we discount profit gained at each time t and view it as a current value. Accordingly, at $t = T$ in scenario ω, the investor can gain the following total sales profit:

$$R^\omega = \sum_{t=1}^{T} \rho^{t-1} \pi_t^\omega n_t^\omega, \quad (\omega \in \Omega), \tag{2.1}$$

where

$$\sum_{t=1}^{T} n_t^\omega = N, \quad (\omega \in \Omega). \tag{2.2}$$

Next, we will show the fractuation of the asset price in the future. Let ϵ_t^ω denotes the price variations during term t in scenario ω, and then the asset price at time t can be represented as $\pi_{t-1}^\omega + \epsilon_t^\omega$. The previous research [4] defines the market impact as the function $f(n_t^\omega)$. We will follow this idea, and express the sales price π_t^ω at time t in scenario ω which is influenced by the market impact as follows:

$$\pi_t^\omega = \pi_{t-1}^\omega + \epsilon_t^\omega - f(n_t^\omega), \quad (t = 1, \ldots, T;\ \omega \in \Omega). \tag{2.3}$$

As for n_t^ω, the state of the market may force the investor to restrict the sales volumes of assets because there may be few trading partners who can trade with him/her if he/she intends to sell huge volumes of assets. Then, the following constraints are needed:

$$\bar{n} \geq n_t^\omega \geq 0, \quad (t = 1, \ldots, T;\ \omega \in \Omega), \tag{2.4}$$

where $\bar{n}$ denotes an upper bound on sales volumes.

Next, we explain the nonanticipativity conditions. These constraints mean that the investor cannot make a decision with his/her knowing the state of scenario in the future. For example, let us assume that scenario ω shares the same node until a certain time t with scenario ζ ($\zeta \neq \omega$), and it does not after time $t + 1$. In such a case, with respect to scenarios ω and ζ, the investor's decisions until time t in scenario ω have to be equivarent to those in scenario ζ. For all $\omega, \zeta \in \Omega$ and any $t \in \{1, \ldots, T\}$, the nonanticipativity conditions are written as follows [3]:

$$n_t^\omega = n_t^\zeta \quad \text{if} \quad \epsilon_\tau^\omega = \epsilon_\tau^\zeta \quad \text{for} \quad \tau = 1, \ldots, t.$$

To describe these conditions in detail, let us denote the last time at which scenarios ω and ζ share the same node as

$$t^{max}(\omega, \zeta) = \max\{t : \epsilon_\tau^\omega = \epsilon_\tau^\zeta, \quad \tau = 1, \ldots, t\}. \tag{2.5}$$

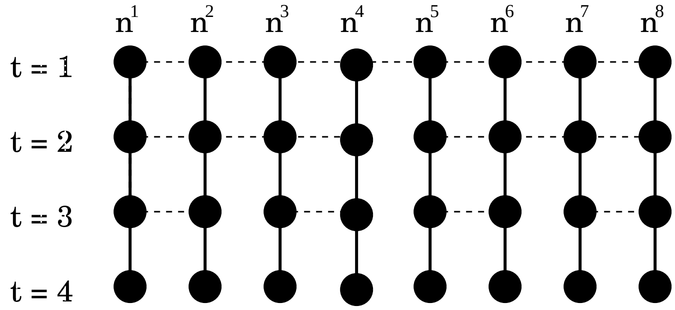

Figure 2: Sequences of decisions and nonanticipativity

Time	Scenario							
	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	1
2	2	3	4	1	6	7	8	5
3	2	1	4	3	6	5	8	7
4	1	2	3	4	5	6	7	8

Table 1: Relatives of Scenarios

We order scenarios in Ω by assigning to them numbers $j = 1, \ldots, S$ in such a way that for every scenario j, scenario $j+1$ has the largest last common time with j among all scenarios $i > j$:

$$t^{max}(j, j+1) = \max\{t^{max}(j, i) : i > j\}. \tag{2.6}$$

Scenarios in Figure 1 are ordered as shown in Figure 2. In short, at $t = 1$, the equalities $n_1^1 = n_1^2 = \cdots = n_1^8$ have to hold. At time t, one of scenarios which shares the same node with scenario j is given by

$$v(j,t) = \begin{cases} j+1 & \text{if } t^{max}(j, j+1) \geq t, \\ \min\{i : t^{max}(j,i) \geq t\} & \text{otherwise.} \end{cases} \tag{2.7}$$

For the example of Figures 1 and 2, the values of $v(j,t)$ are shown in Table 1. Note that it is easy to observe that $v(j,t) \neq j$, if the bundles of scenario j at time t contains more than one member, and $v(j,T) = j$ otherwise.

Finally, we introduce a lower bound of the expected total sales profit denoted by W_E. Letting p^j and S be the probability that scenario j is generated and the number of scenarios, respectively, we assume that the constraint which requires that the expected total sales profit exceeds W_E is given by

$$\sum_{j=1}^{S} p^j R^j \geq W_E. \tag{2.8}$$

As to the objective function, we aim at minimizing shortages of the total sales profit against the target total sales profit expressed by W_G. Therefore, we employ the risk measure

$$\sum_{j=1}^{S} p^j \max\{0, W_G - R^j\}. \tag{2.9}$$

This risk measure is called the lower partial moment of dimension 1.

2.2 Formulation

Below is the list of symbols used in the formulation of the model. Note that input variables for the model are parameters, and output variables for the model are decision variables.

(A) Parameters

N	:	The total volumes of assets the investor intends to sell
T	:	The sales completion time
W_E	:	the lower bound on the expected total sales profit
W_G	:	The target total sales profit
p^j	:	The probability that scenario j is generated $(j = 1, \dots, S)$
π_0	:	The sales price at the initial time 0
ϵ_t^j	:	The price variations during term t in scenario j $(t = 1, \dots, T;\ j = 1, \dots, S)$
ρ	:	The discount rate
$\bar{n}$	:	The upper bound on the sales volumes

(B) Decision variables

R^j	:	The total sales profit in scenario j $(j = 1, \dots, S)$
π_t^j	:	The sales price at time t in scenario j $(t = 1, \dots, T;\ j = 1, \dots, S)$
n_t^j	:	The sales volumes at time t in scenario j $(t = 1, \dots, T;\ j = 1, \dots, S)$

Note that only n_t^j will appear in the final formulation of the problem, because R^j and π_t^j can be eliminated from equations (2.1) and (2.3). With the objective function (2.9) and constraints (2.1), (2.2), (2.3), (2.4) and (2.8), the problem discussed in the previous section is formulated as follows:

$$\begin{aligned}
&\text{minimize} && \sum_{j=1}^{S} p^j \max\{0, W_G - R^j\} \\
&\text{subject to} && \sum_{j=1}^{S} p^j R^j \geq W_E \\
& && R^j = \sum_{t=1}^{T} \rho^{t-1} \pi_t^j n_t^j, \quad (j = 1, \ldots, S) \\
& && \pi_t^j = \pi_{t-1}^j + \epsilon_t^j - f(n_t^j), \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\
& && \pi_0^j = \pi_0, \quad (j = 1, \ldots, S) \\
& && \sum_{t=1}^{T} n_t^j = N, \quad (j = 1, \ldots, S) \\
& && \bar{n} \geq n_t^j \geq 0, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\
& && n_t^j = n_t^{v(j,t)}, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S),
\end{aligned} \tag{2.10}$$

where the last constraints show the nonanticipativity conditions involving $v(j,t)$ defined by (2.5), (2.6) and (2.7). As shown in (2.3), π_t^j can be written using n_t^j. Furthermore, we assume that the market impact is linear in the sales volumes [4]. Under this assumption, the market impact is expressed as folows:

$$f(n_t^j) = a n_t^j, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S), \tag{2.11}$$

where $a > 0$ is a constant called the market impact constant. In the above model (2.10), we introduce variables q^j such that

$$q^j \geq \max\{0, W_G - R^j\}, \quad (j = 1, \ldots, S). \tag{2.12}$$

Then, the problem (2.10) can be rewritten as follows:

$$\begin{aligned}
&\text{minimize} && \sum_{j=1}^{S} p^j q^j \\
&\text{subject to} && q^j \geq W_G - R^j, \quad (j = 1, \ldots, S) \\
& && q^j \geq 0, \quad (j = 1, \ldots, S) \\
& && \sum_{j=1}^{S} p^j R^j \geq W_E \\
& && R^j = \sum_{t=1}^{T} \{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j) \rho^{t-1}\} n_t^j - a \sum_{k=1}^{T} \sum_{l=1}^{T} n_k^j \delta_{kl} n_l^j, \quad (j = 1, \ldots, S) \\
& && \sum_{t=1}^{T} n_t^j = N, \quad (j = 1, \ldots, S) \\
& && \bar{n} \geq n_t^j \geq 0, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\
& && n_t^j = n_t^{v(j,t)}, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S),
\end{aligned} \tag{2.13}$$

where

$$\delta_{kl} = \begin{cases} \rho^{l-1} & k \leq l \\ 0 & k > l. \end{cases} \tag{2.14}$$

Since a matrix $A = (\delta_{kl})$ that appears in the problem (2.13) is an upper triangular matrix, it is more convenient to rewrite the quadratic term in the problem (2.13) using the symmetric matrix $V = (\sigma_{kl})$ given by $V = (A + A^T)/2$. In addition, eliminating R^j, we transform (2.13) into the following problem that involves only variables n_t^j and q^j:

【Problem】

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{S} p^j q^j \\ \text{subject to} & q^j \geq W_G - \displaystyle\sum_{t=1}^{T}\{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j)\rho^{t-1}\} n_t^j + a \sum_{k=1}^{T}\sum_{l=1}^{T} n_k^j \sigma_{kl} n_l^j, \quad (j = 1, \ldots, S) \\ & q^j \geq 0, \quad (j = 1, \ldots, S) \\ & \displaystyle\sum_{j=1}^{S} p^j \{\sum_{t=1}^{T}\{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j)\rho^{t-1}\} n_t^j - a \sum_{k=1}^{T}\sum_{l=1}^{T} n_k^j \sigma_{kl} n_l^j\} \geq W_E \\ & \displaystyle\sum_{t=1}^{T} n_t^j = N, \quad (j = 1, \ldots, S) \\ & \bar{n} \geq n_t^j \geq 0, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\ & n_t^j = n_t^{v(j,t)}, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S), \end{array} \tag{2.15}$$

where

$$\sigma_{kl} = \begin{cases} \rho^{l-1} & k = l \\ \frac{1}{2}\rho^{l-1} & k < l \\ \frac{1}{2}\rho^{k-1} & k > l. \end{cases} \tag{2.16}$$

Here, we prove that when $\rho = 1$, the matrix V is positive definite. If all eigenvalues of V are positive, V is positive definite because V is a symmetric matrix. Letting λ and I be eigenvalues and an identity matrix, respectively, we introduce the characteristic equation

$$|V - \lambda I| = (\tfrac{1}{2} - \lambda)^{T-1}(\tfrac{T+1}{2} - \lambda).$$

From this, all eigenvalues of V are positive, hence V is a positive definite matrix.

3 Transformation of the problem

3.1 SOCP problem

In this section, we reformulate the problem given in the previous chapter as an SOCP problem. The typical SOCP problem [2] is written as

$$\begin{array}{ll} \text{minimize} & f^\top x \\ \text{subject to} & \|A_i x + b_i\| \le c_i^\top x + d_i, \quad (i = 1, \dots, I) \\ & Bx = r, \end{array} \tag{3.1}$$

where $\|\cdot\|$ denotes the standard Euclidean norm, $\|z\| = \sqrt{z^\top z}$, $x \in \mathbf{R}^n$ is the vector of optimization variables, $f \in \mathbf{R}^n$, $A_i \in \mathbf{R}^{(n_i-1)\times n}$, $b_i \in \mathbf{R}^{n_i-1}$, $c_i \in \mathbf{R}^n$, $d_i \in \mathbf{R}$, $B \in \mathbf{R}^{m\times n}$ and $r \in \mathbf{R}^m$ are problem parameters. The standard or unit second-order (convex) cone of dimension k is defined as

$$\mathcal{C}_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \,\middle|\, u \in \mathbf{R}^{k-1},\ t \in \mathbf{R},\ \|u\| \le t \right\}.$$

For $k = 1$, the second-order cone reduces to the set of nonnegative reals

$$\mathcal{C}_1 = \{\ t \mid t \in \mathbf{R},\ 0 \le t\}.$$

Since we have

$$\|A_i x + b_i\| \le c_i^\top x + d_i \iff \begin{bmatrix} A_i \\ c_i^\top \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in \mathcal{C}_{n_i},$$

the constraint

$$\|A_i x + b_i\| \le c_i^\top x + d_i \tag{3.2}$$

in the problem (3.1) is expressed using a second-order cone. The set of points satisfying a second-order cone constraint is the inverse image of the second-order cone under the affine mapping

$$x \mapsto \begin{bmatrix} A_i \\ c_i^\top \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix}$$

and hence is convex. Therefore the SOCP problem (3.1) is a convex programming problem.

3.2 Transformation

The problem (2.15) is a quadratically constrained mathmatical programming problem, and the coefficient matrix in the constraints is a positive semidefinite symmetric matrix. Accordingly, we can simply write the problem (2.15) as

$$\begin{array}{ll} \text{minimize} & f^\top x \\ \text{subject to} & x^\top P_i x + p_i^\top x + r_i \leq 0, \quad (i = 1, \dots, I) \\ & A_j x + b_j \leq 0, \quad (j = 1, \dots, J), \end{array} \tag{3.3}$$

where $x \in \mathbf{R}^n$ is the vector of decision variables, and $P_i \in \mathbf{R}^{n\times n}$ are positive semidefinite symmetric matrices, and $A_j \in \mathbf{R}^{m\times n}, f \in \mathbf{R}^n, p_i \in \mathbf{R}^n, b_i \in \mathbf{R}^m$ and $r_i \in \mathbf{R}$ are problem parameters.

If P_i is a positive semidefinite matrix, then we can decompose it as $P_i = C_i C_i^\top$ with some matrix C_i. Thus the quadratic constraints are transformed as follows:

$$\begin{array}{ll} & x^\top P_i x + p_i^\top x + r_i \leq 0 \\ \Longleftrightarrow & \|C_i^\top x\|^2 + p_i^\top x + r_i \leq 0 \\ \Longleftrightarrow & (1 + p_i^\top x + r_i)^2 + \|2C_i^\top x\|^2 \leq (1 - p_i^\top x - r_i)^2 \\ \Longleftrightarrow & \sqrt{(1 + p_i^\top x + r_i)^2 + \|2C_i^\top x\|^2} \leq 1 - p_i^\top x - r_i. \end{array} \tag{3.4}$$

As discussed in the previous section, the last inequality is a second-order cone constraint, so the problem (3.3) can be reduced to an SOCP problem. In other words, the problem (2.15) is formulated as an SOCP problem.

4 Numerical experiments

In this section, we report some numerical experience with the proposed model described in section 2. We first describe the experimental environment, and then report the experimental results. We code programs with MATLAB Version 5 by using SeDuMi [11], which is an interior point solver for SOCP problems and semidefinite programming (SDP) problems.

4.1 Experimental environment

We set the values of parameters as follows:

The total volumes of assets	$N = 100$
The sales completion time	$T = 4$
The target total sales profit	$W_G = 8700$
The probability that scenario j is generated	$p^j = 1/S \quad (j = 1, \dots, S)$
The sales price at the initial time	$\pi_0 = 100$
The discount rate	$\rho = 0.98$

Note that we set the number of branchings at each node to be 3, so we have the number of scenarios $S = 27$. At the initial time $t = 0$, the current aggregate value the investor has is 10000. The price variations ϵ_t^j during term t in scenario j are fabricated appropriately.

4.2 Numerical results

4.2.1 Experiment A : Comparison by the expected total sales profit

We solve the problem (2.15) by changing the lower bound on the expected total sales profit W_E to see the optimal sales volumes of assets at each time. We solve eight problems that are called Case 1 to Case 8. We set the upper bound on sales volumes $\bar{n}$ and the market impact constant a to be $+\infty$ and 0.100, respectively. To examine extreme cases, we formulate Case 1 and Case 8 as the risk minimization problem and the total sales profit maximization problem, respectively.

【Case 1】

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{S} p^j q^j \\ \text{subject to} & \displaystyle q^j \geq W_G - \sum_{t=1}^{T}\{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j)\rho^{t-1}\}n_t^j + a\sum_{k=1}^{T}\sum_{l=1}^{T} n_k^j \sigma_{kl} n_l^j, \quad (j = 1, \ldots, S) \\ & q^j \geq 0, \quad (j = 1, \ldots, S) \\ & \displaystyle\sum_{t=1}^{T} n_t^j = N, \quad (j = 1, \ldots, S) \\ & \bar{n} \geq n_t^j \geq 0, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\ & n_t^j = n_t^{v(j,t)}, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S). \end{array}$$

【Case 8】

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{S} p^j \{\sum_{t=1}^{T}\{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j)\rho^{t-1}\}n_t^j - a\sum_{k=1}^{T}\sum_{l=1}^{T} n_k^j \sigma_{kl} n_l^j\} \\ \text{subject to} & \displaystyle\sum_{t=1}^{T} n_t^j = N, \quad (j = 1, \ldots, S) \\ & \bar{n} \geq n_t^j \geq 0, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S) \\ & n_t^j = n_t^{v(j,t)}, \quad (t = 1, \ldots, T;\ j = 1, \ldots, S). \end{array}$$

We first solved Case 1 and Case 8, and found that the expected total sales profits of Case 1 and Case 8 are given by 8714.4 and 8938.8, respectively. So, we set the lower bound on the expected total sales profit of Case 2 to Case 7 to be 8746 to 8906. In Table 2, we show the expected total sales profit and the minimum value of the objective function, which we call the risk value obtained by solving the problem with the given lower bound of the expected total sales profit. Let us add that in Case 8, the risk value is obtained by

$$\sum_{j=1}^{S} p^j \max\{0, W_G - \sum_{t=1}^{T}\{(\pi_0 + \sum_{i=1}^{t} \epsilon_i^j)\rho^{t-1}\}n_t^j + a\sum_{k=1}^{T}\sum_{l=1}^{T} n_k^j \sigma_{kl} n_l^j\}.$$

Table 10 shows the optimal sales volumes of Case 1 to Case 8 except Cases 4 and 6. Next, we conduct the same experiments to each problem with a certain value of the market impact constant, and Tables 3 to 6 show the expected total sales profit and the corresponding risk value. Furthermore, to see the relation between the lower bound on the expected total sales profit and the risk value, Figure 3 illustrates efficient frontiers for various market impact constants.

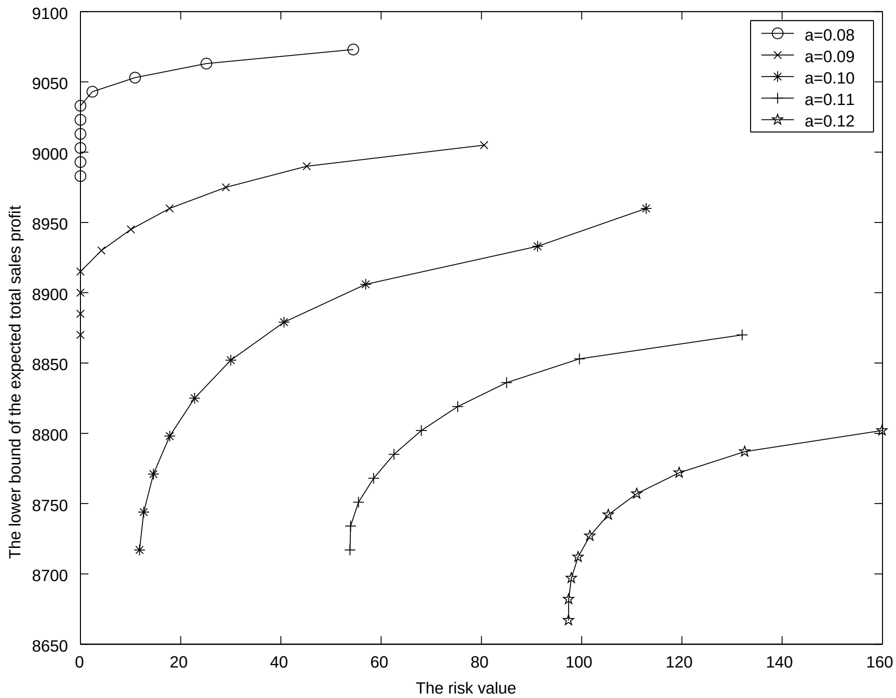

Figure 3: Efficient frontiers for various market impact constants

4.2.2 Experiment B : Comparison by the upper bound on the sales volumes

Next, we assume that there exists a constraint on the sales volumes at each time because of the market environment, and show the optimal sales volumes of assets by changing the parameter $\bar{n}$ that represents the upper bound on the sales volumes. Table 11 shows the optimal sales volumes obtained by solving the problems with $\bar{n} = +\infty, 70$ and 60. As to other paramters, we set the market impact constant a to be 0.100, and the lower bound on the expected total sales volumes W_E to be 8842 or 8906.

In addition, to see the risk value and the lower bound on the expected total sales volumes, we solve Case 1 to Case 8 like in Experiment A, and report the results in Tables 7 to 9. Figure 4 shows efficient frontiers obtained from the results of Experiment B.

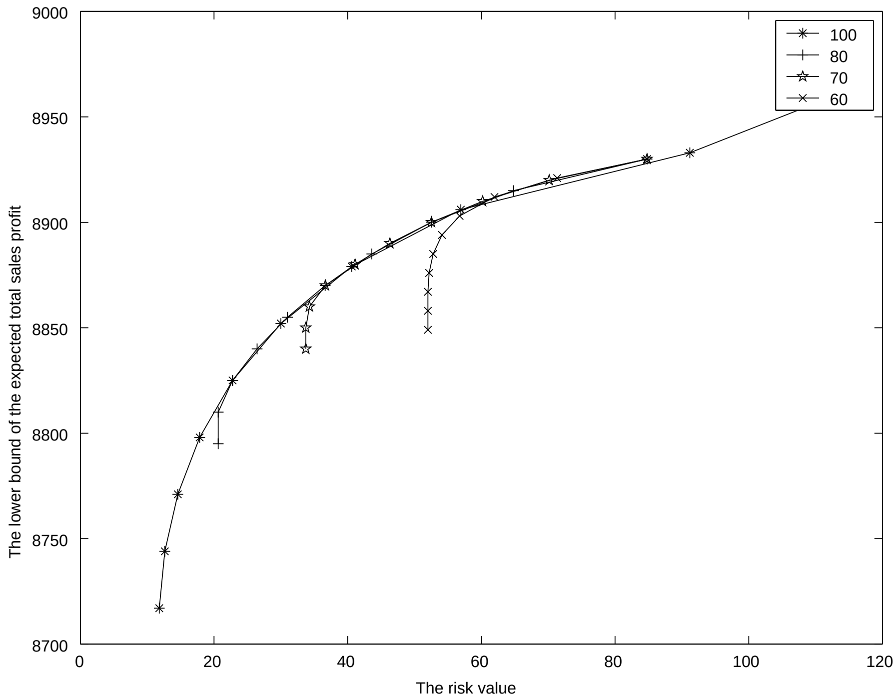

Figure 4: Efficient frontiers for various upper bounds on the sales volumes

4.2.3 Experiment C : Comparison by the market impact constant

Finally, we conduct numerical experiments to see the optimal sales volumes for the different values of the market impact constant. We report the results in Table 12. As to parameters, we set the lower bound on the expected total sales profit W_E and the upper bound on sales volumes $\bar{n}$ to be 8800 and $+\infty$, respectively. Figure 5 illustrates how the risk value varies with the market impact constant.

4.3 Discussions

We first consider the results of Experiment A. From Table 10, we can see that in Case 1 the investor sells a large part of assets at $t = 1$. In Case 2 to Case 8, the sales volume at $t = 1$ decreases as the lower bound on the expected total sales profit gradually increases, although the sales volume after $t = 2$ increases. Especially, there is a clear difference in the optimal sales volumes in Case 8 compared to other cases. This result suggests that in the case where the lower bound on the expected total sales profit is relatively small, by selling assets as early as possible, the investor may avoid the diversification of the total sales profit among scenarios. This is because the objective function contains the target sales profit W_G, which means that the

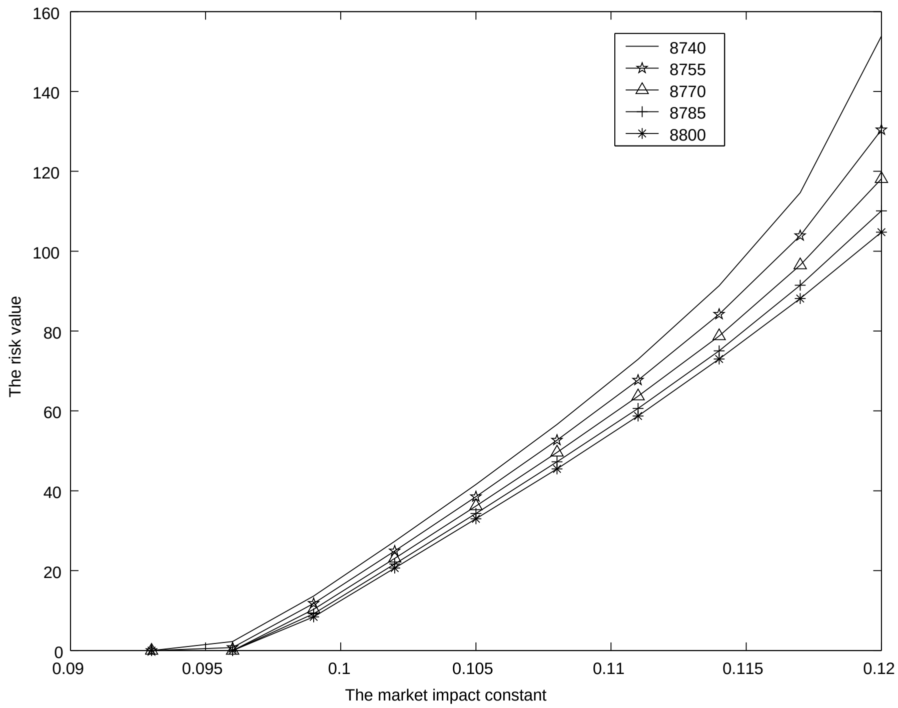

Figure 5: The risk value vs the market impact constant

investor's satisfaction does not change no matter how far or a little the total sales profit of each scenario exceeds W_G. Therefore, in such a case we suppose that the investor intends to sell a large part of his/her holding assets at $t = 1$ to decrease the diversification of the total sales profit among scenarios. On the other hand, we can see that in the case where the lower bound on the expected total sales profit is relatively large just like in Case 7 or Case 8, the investor sells his/her assets in the later stage, compared with the case where the lower bound on the expected total sales profit is small. This is why the investor sell his/her assets to some extent at each time in both scenarios where an asset price is rising and dropping, and therefore we can infer that the diversification of the total sales profit among scenarios will increase. However, by selecting the strategy of selling his/her assets in the later stage, the investor can satisfy the constraint on the lower bound on the expected total sales profit. Table 2 shows the risk values for the lower bound on the expected total sales profit. It is observed from this table that when the lower bound on the expected total sales profit is small, the diversification of the total sales profit can be small and the risk value can be decreased, although when the lower bound on the expected total sales profit is large, the diversification of the total sales profit can be large and the risk value can be increased. Accordingly, we realize that the relation of trade-off between risk and return, which corresponds to the lower bound on the expected total sales profit in this paper, holds. In addition, we confirm from Tables 3 to 6 that the same is true for various market impact constants other than 0.100. Figure 3 illustrates efficient frontiers showing the relation of trade-off mentioned above. We can easily observe that the risk value for the lower bound on the expected total sales profit increases as the market impact constant increases, because an asset price will drop substantially as the influence of the market impact becomes more significant.

Next, we consider the results of Experiment B. We can see from Table 11 that in the case of $W_E = 8842$ the investor sells the exact volumes of the upper bound value at $t = 1$ when $\bar{n} = 70$ or $\bar{n} = 60$, and meanwhile in the case of $W_E = 8906$ he/she does the same thing at $t = 1$ when $\bar{n} = 60$. In any case, the timing to sell assets is shifted to later stages compared with the case of $\bar{n} = +\infty$. Actually, Figure 4 indicates that the risk value increases, as the constraints on the sales volumes becomes more restrictive, and hence we realize that the risk value can increase when there exist constraints on the sales volumes. However, when the lower bound on the expected total sales profit is large, as shown in Experiment A, the timing to sell assets is delayed, which shows that the constraints on the sales volumes are not effective, and hence the risk value is almost the same regardless of the upper bound value on the sales volumes. Furthermore, from Tables 7 to 9, we can confirm in Experiment B the relation of the trade-off between risk and return just as in Experiment A.

Finally, we consider the results of Experiment C, in which we examined the optimal sales volumes for various values of the market impact constant. Table 12 shows that the investor tends to sell his/her assets in the later stages as the market impact constant increases. We can explain the reason as follows. In the case where the market impact is supposed to be siginificant, if the investor sells a large part of assets at first, then an asset price will drop substantially and afterward it remain at a low level. Therefore, the investor cannot expect the desiable total sales profit. To avoid such a situation, the investor needs to alleviate a price drop by diversifying the timing to sell his/her assets and satisfy the constraint on the lower bound on the expected total sales profit. Hence, we may conclude that the investor should not sell his/her assets at once, but sell them at each time by dividing them, when the influence of his/her asset's market impact is expected to be significant. We observe from Figure 5 that the risk value increases in proportion to the market impact constant, because the significant influence of the market impact results in a substantial price drop.

5 Conclusion

This paper has proposed an optimal transaction strategy model incorporating the influence of a price drop called the market impact that is accompanied by the investor's own transaction in the situation where he/she sells assets which have low liquidity. We have assumed that the investor sells his/her holding assets at several times by dividing them to alleviate the influence of the market impact, and formulated the problem by adopting a scenario tree model. Then, we have transformed the problem into an SOCP problem, which is solved the lower bound on the expected total sales profitly by an interior point method, and obtained the optimal sales volumes, which yield the optimal transaction strategy for the investor. Moreover, we have examined optimal sales volumes and risk values for various cases where some parameters are changed.

The framework presented in this paper does not incorporate all the aspects of liquidity risk. Nevertheless, this model is effective as a method of evaluating financial risk when the influence of the market impact on specific financial products is significant.

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A Appendix

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8714.4	11.81
Case 2	$W_E = 8746$	8746.0	12.74
Case 3	$W_E = 8778$	8778.0	15.29
Case 4	$W_E = 8810$	8810.0	19.81
Case 5	$W_E = 8842$	8842.0	26.99
Case 6	$W_E = 8874$	8874.0	38.33
Case 7	$W_E = 8906$	8906.0	56.93
Case 8	Total sales profit maximization	8938.8	121.73

Table 2: The market impact constant a=0.100

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8981.6	0.00
Case 2	$W_E = 8995$	8996.6	0.00
Case 3	$W_E = 9008$	9008.6	0.00
Case 4	$W_E = 9021$	9021.2	0.00
Case 5	$W_E = 9034$	9034.1	0.00
Case 6	$W_E = 9047$	9047.0	5.45
Case 7	$W_E = 9060$	9060.0	20.04
Case 8	Total sales profit maximization	9074.3	71.27

Table 3: The market impact constant a=0.080

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8868.5	0.00
Case 2	$W_E = 8888$	8888.9	0.00
Case 3	$W_E = 8908$	8908.1	0.00
Case 4	$W_E = 8928$	8928.0	3.52
Case 5	$W_E = 8948$	8948.0	11.43
Case 6	$W_E = 8968$	8968.0	23.34
Case 7	$W_E = 8888$	8988.0	42.53
Case 8	Total sales profit maximization	9006.3	96.24

Table 4: The market impact constant a=0.090

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8713.7	53.78
Case 2	$W_E = 8735.5$	8735.5	53.95
Case 3	$W_E = 8758$	8758.0	56.61
Case 4	$W_E = 8780.5$	8780.5	61.36
Case 5	$W_E = 8803$	8803.0	68.39
Case 6	$W_E = 8825.5$	8825.5	78.64
Case 7	$W_E = 8848$	8848.0	94.54
Case 8	Total sales profit maximization	8871.7	148.13

Table 5: The market impact constant a=0.110

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8666.6	97.37
Case 2	$W_E = 8685$	8685.0	97.47
Case 3	$W_E = 8705$	8705.0	98.56
Case 4	$W_E = 8725$	8725.0	101.25
Case 5	$W_E = 8745$	8745.0	106.27
Case 6	$W_E = 8765$	8765.0	115.11
Case 7	$W_E = 8785$	8785.0	130.38
Case 8	Total sales profit maximization	8804.8	181.86

Table 6: The market impact constant a=0.120

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8792.8	20.62
Case 2	$W_E = 8813$	8813.0	20.62
Case 3	$W_E = 8834$	8834.0	24.89
Case 4	$W_E = 8855$	8855.0	30.98
Case 5	$W_E = 8876$	8876.0	39.23
Case 6	$W_E = 8897$	8897.0	50.55
Case 7	$W_E = 8918$	8918.0	67.91
Case 8	Total sales profit maximization	8938.8	121.73

Table 7: The upper bound on the sales volumes $\bar{n}$=80

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8831.2	33.73
Case 2	$W_E = 8848$	8848.0	33.73
Case 3	$W_E = 8863$	8863.0	34.79
Case 4	$W_E = 8878$	8878.0	40.15
Case 5	$W_E = 8893$	8893.0	48.07
Case 6	$W_E = 8908$	8908.0	58.52
Case 7	$W_E = 8923$	8923.0	73.86
Case 8	Total sales profit maximization	8938.8	121.73

Table 8: The upper bound on the sales volumes $\bar{n}$=70

Case	The lower bound on the expected total sales profit	The expected total sales profit	The risk value
Case 1	Risk minimization	8848.9	52.01
Case 2	$W_E = 8861$	8861.0	52.01
Case 3	$W_E = 8874$	8874.0	52.11
Case 4	$W_E = 8887$	8887.0	53.00
Case 5	$W_E = 8900$	8900.0	55.71
Case 6	$W_E = 8913$	8913.0	62.88
Case 7	$W_E = 8926$	8926.0	78.06
Case 8	Total sales profit maximization	8938.8	121.75

Table 9: The upper bound on the sales volumes $\bar{n}$=60

Case 1				Case 2				Case 3			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
93.57	4.27	0.00	2.16	89.94	7.58	0.00	2.48	85.95	11.04	0.00	3.01
			2.16				2.48				3.01
			2.16				2.48				3.01
		2.16	0.00			2.48	0.00			3.01	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		2.16	0.00			2.48	0.00			3.01	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	4.32	1.56	0.54		5.11	4.80	0.15		3.08	10.88	0.10
			0.54				0.15				0.10
			0.54				0.15				0.10
		1.32	0.78			4.95	0.00			10.97	0.00
			0.78				0.00				0.00
			0.78				0.00				0.00
		2.11	0.00			4.95	0.00			10.97	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	2.33	2.17	1.93		10.06	0.00	0.00		14.05	0.00	0.00
			1.93				0.00				0.00
			1.93				0.00				0.00
		2.10	2.00			0.00	0.00			0.00	0.00
			2.00				0.00				0.00
			2.00				0.00				0.00
		2.25	1.85			0.00	0.00			0.00	0.00
			1.85				0.00				0.00
			1.85				0.00				0.00

Case 5				Case 7				Case 8			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
75.92	16.41	0.00	7.66	61.43	21.47	0.00	17.11	42.40	25.86	0.00	31.74
			7.66				17.11				31.74
			7.66				17.11				31.74
		7.66	0.00			17.11	0.00			31.74	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		7.66	0.00			17.11	0.00			31.74	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	0.00	23.25	0.82		0.00	24.21	14.36		0.00	14.98	42.62
			0.82				14.36				42.62
			0.82				14.36				42.62
		24.07	0.00			38.58	0.00			50.81	6.79
			0.00				0.00				6.79
			0.00				0.00				6.79
		24.07	0.00			38.58	0.00			57.60	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	22.58	1.49	0.00		29.59	8.99	0.00		38.25	19.35	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		1.49	0.00			7.53	1.46			12.71	6.64
			0.00				1.46				6.64
			0.00				1.46				6.64
		1.49	0.00			8.99	0.00			19.35	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00

Table 10: The optimal sales volumes for the market impact $a = 0.100$

$W_E = 8842$ ($\bar{n} = +\infty$)				$W_E = 8842$ ($\bar{n} = 70$)				$W_E = 8842$ ($\bar{n} = 60$)			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
75.92	16.41	0.00	7.66	70.00	21.67	0.00	8.33	60.00	26.63	0.37	13.00
			7.66				8.33				13.00
			7.66				8.33				13.00
		7.66	0.00			8.33	0.00			13.37	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		7.66	0.00			8.33	0.00			13.37	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	0.00	23.25	0.82		13.17	16.07	0.76		25.32	14.29	0.39
			0.82				0.76				0.39
			0.82				0.76				0.39
		24.07	0.00			12.53	4.31			8.66	6.02
			0.00				4.31				6.02
			0.00				4.31				6.02
		24.07	0.00			15.23	1.60			12.92	1.76
			0.00				1.60				1.76
			0.00				1.60				1.76
	22.58	1.49	0.00		16.21	9.13	4.67		15.75	13.47	10.78
			0.00				4.67				10.78
			0.00				4.67				10.78
		1.49	0.00			7.40	6.40			12.48	11.77
			0.00				6.40				11.77
			0.00				6.40				11.77
		1.49	0.00			9.66	4.14			13.99	10.26
			0.00				4.14				10.26
			0.00				4.14				10.26

$W_E = 8906$ ($\bar{n} = +\infty$)				$W_E = 8906$ ($\bar{n} = 70$)				$W_E = 8906$ ($\bar{n} = 60$)			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
61.43	21.47	0.00	17.11	61.43	21.47	0.00	17.11	60.00	23.43	0.00	16.57
			17.11				17.11				16.57
			17.11				17.11				16.57
		17.11	0.00			17.11	0.00			16.57	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		17.11	0.00			17.11	0.00			16.57	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	0.00	24.21	14.36		0.00	24.21	14.36		1.27	29.79	8.94
			14.36				14.36				8.94
			14.36				14.36				8.94
		38.58	0.00			38.58	0.00			38.73	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		38.58	0.00			38.58	0.00			38.73	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	29.59	8.99	0.00		29.59	8.99	0.00		30.24	9.76	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		7.53	1.46			7.53	1.45			7.92	1.84
			1.46				1.45				1.84
			1.46				1.45				1.84
		8.99	0.00			8.99	0.00			9.76	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00

Table 11: The optimal sales volumes for the lower bound of the expected total sales profit and the upper bound on the sales volumes

$a = 0.095$				$a = 0.100$				$a = 0.105$			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
88.92	10.62	0.20	0.27	82.88	13.59	0.00	3.53	76.19	16.32	0.00	7.48
			0.27				3.53				7.48
			0.27				3.53				7.48
		0.43	0.04			3.53	0.00			7.48	0.00
			0.04				0.00				0.00
			0.04				0.00				0.00
		0.45	0.02			3.53	0.00			7.48	0.00
			0.02				0.00				0.00
			0.02				0.00				0.00
	1.48	4.58	5.02		1.64	15.25	0.24		1.21	20.00	2.59
			5.02				0.24				2.59
			5.02				0.24				2.59
		7.97	1.63			15.49	0.00			22.59	0.00
			1.63				0.00				0.00
			1.63				0.00				0.00
		8.91	0.69			15.49	0.00			22.59	0.00
			0.69				0.00				0.00
			0.69				0.00				0.00
	6.62	2.90	1.57		17.12	0.00	0.00		21.89	1.92	0.00
			1.57				0.00				0.00
			1.57				0.00				0.00
		2.39	2.08			0.00	0.00			1.92	0.00
			2.08				0.00				0.00
			2.08				0.00				0.00
		3.06	1.41			0.00	0.00			1.92	0.00
			1.41				0.00				0.00
			1.41				0.00				0.00

$a = 0.110$				$a = 0.115$				$a = 0.120$			
$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$	$t=1$	$t=2$	$t=3$	$t=4$
68.75	19.07	0.00	12.18	60.45	21.97	0.00	17.57	47.48	25.61	0.00	26.91
			12.18				17.57				26.91
			12.18				17.57				26.91
		12.18	0.00			17.57	0.00			26.91	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		12.18	0.00			17.57	0.00			26.91	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	0.94	25.15	5.17		0.14	25.87	13.54		0.00	21.43	31.09
			5.17				13.54				31.09
			5.17				13.54				31.09
		30.32	0.00			38.71	0.69			44.44	8.08
			0.00				0.69				8.08
			0.00				0.69				8.08
		30.32	0.00			39.41	0.00			52.52	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
	25.05	6.20	0.00		28.30	11.25	0.00		33.73	18.79	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00
		5.77	0.44			8.13	3.11			11.76	7.03
			0.44				3.11				7.03
			0.44				3.11				7.03
		6.20	0.00			11.25	0.00			18.79	0.00
			0.00				0.00				0.00
			0.00				0.00				0.00

Table 12: The optimal sales volumes for various market impact constant