Optimal Design of PAC-Companion Structure for Mortgage Backed Securities Using Cash Reserve

Guidance

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Abstract

Owing to the agentification of the Government Housing Loan Corporation, Japanese private financial institutions are now developing long-term fixed-rate loans. In order to reduce the risk of yielding mismatches between asset and liability, financial institutions will sell out their loans by securitization instead of holding loans themselves. Mortgage backed security (MBS) is a product of mortgage securitization, which is issued backed by the repayment cash flow from a loan pool consisting of many mortgage loans. In mortgage loans, borrowers are basically allowed to make prepayment. Repayment cash flow from a loan pool changes due to the prepayment. So there is a prepayment risk in MBS, which means that investors cannot invest excessive liquidity in a planned yield or duration when prepayment occurs faster than forecast. Collateralized Mortgage Obligation (CMO) is a form of MBS, in which repayment cash flow from a loan pool is reorganized and bonds with various risks are issued. In this paper, we propose a method of designing CMO with PAC-Companion structure. We divide repayment cash flow, which is unstable due to uncertainty of prepayment, into two parts; a part in which principal repayment schedule must be satisfied (planned amortization class, PAC) and an unstable high-prepayment risk part (companion). We allow the repaid cash to be reserved in order to repay PAC bondholders in the following periods, which will make it possible to issue more PAC bond. We formulate the problem of determining an optimal PAC-companion structure as a mathematical programming problem and use a simulation-based approach to approximate the problem. We show that our model can be reformulated as an equivalent linear programming problem. Furthermore, we propose a modified model which yields a higher performance than the basic model. Finally we conduct numerical experiments and report the results.

Contents

1 Introduction

Financial conditions of the Government Housing Loan Corporation (GHLC) have been getting worth due to the economic downturn and the decline of interest rates in recent years in Japan. In order to cope with this situation, the "Reorganization and Rationalization Plan for Special Public Corporations" has been approved in a Cabinet meeting in 2001. On this plan, GHLC will be abolished within five years, and conventional mortgage loan business will be cut down gradually. On the other hand, GHLC will start a service to support mortgage securitization, and it is planned to establish an independent administrative institution that provides securitization service specially after abolishment of GHLC [6]. Long-term fixed-rate has been considered difficult to hedge risk loans and has not been treated by private financial institutions in Japan. However, owing to the agentification of GHLC, Japanese private financial institutions are now developing long-term fixed-rate loans. In order to reduce the risk of yielding mismatches between asset and liability, financial institutions will sell out their loans by securitization instead of holding loans themselves.

The risk of mortgage loans includes credit risk, liquidity risk, interest rate risk, prepayment risk and so on [3, 7]. By securitization, it becomes possible that many investors and financial institutions share these risks, instead of a financial institution taking the whole risks. The product of mortgage securitization is called Mortgage Backed Securities (MBS). MBS was first issued in 1970 by GMNA (Government National Mortgage Association, Ginnie Mae) which is a government institution in U.S. [5]. MBS is a security backed by the repayment cash flow from a loan pool consisting of many mortgage loans. The amount of MBS issued in U.S. reaches 500-800 billion dollars at present, and MBS market has become the second largest next to the government bond market [4]. In Japan, MBS has been issued on a large scale since 1999, and the issuance has been increasing steadily [6]. It is expected that MBS will become more common in future due to the agentification of GHLC.

In mortgage loans, borrowers are basically allowed to make prepayment, which implies that borrowers have a call option. Repayment cash flow from a loan pool changes due to the prepayment. So there is a prepayment risk in MBS, which means that investors cannot invest excessive liquidity in a planned yield or duration when prepayment occurs faster than forecast.

Pass-through is a common form of MBS [3]. With the pass-through structure, all principal and interest payments (less a servicing fee) from the pool of mortgages are passed directly to investors each month. Although pass-through is simple and easy to understand, investors have to take prepayment risk directly. Alternative form of MBS is to reorganize repayment cash flow and issue bonds with various risks, which is called CMO (Collateralized Mortgage Obligation). By issuing bonds with various risks, the MBS issuer can meet needs of various investors, and it results in a reduction of issuance cost.

In this paper, we propose a method of designing CMO with PAC-Companion structure optimally. We divide repayment cash flow, which is unstable due to uncertainty of prepayment, into two parts; a part in which principal repayment schedule must be satisfied (planned amortization class, PAC) and an unstable high-prepayment risk part (companion). The PAC bond is expected to be stable because the companion bond plays a role of a buffer by receiving excessive flow when prepayment is fast and taking on the shortage of liquidity when prepayment is slow. PAC bond is desirable for MBS issuer because its issue price is high. We assume that the repaid cash can be reserved to the next period in order to repay PAC bondholders in the following periods. This will make it possible to issue more PAC bond. We set the upper bound of cash reserve at each time, and payments are made according to the following procedure at time t.

Out of the sum of the amount of repayment cash flow at time t and cash reserve at time $t-1$:

1 First, make scheduled payment for the PAC bond;

2 Then, reserve the maximum cash that does not exceeds the upper bound of cash reserve;

3 Finally, if any, the amount that exceeds the upper bound of cash reserve is paid to the companion bond.

By the above procedure, the whole cash flow is determined by the amount of payment for the PAC bond and the upper bound of cash reserve. Note that the amount of payment for the PAC bond is not completely fixed, so the actual amount of payment for the PAC bond may become less than the planned amount.

In section 2, we present a mathematical model of cash flow in CMO and formulate the problem of determining an optimal PAC-companion structure as a mathematical programming problem. In section 3, we describe a simulation-based approach to the problem. In section 4, we show that the problem can be reformulated as a linear programming problem. A modified model is suggested in section 5. In section 6, we report the result of a numerical experiment, and we conclude in seciton 7.

2 Modeling CMO with PAC-Companion Structure

2.1 Parameters and Variables

First, we define parameters and variables.

T : maturity date;

C_t $(t = 1, \dots, T)$: repayment cash flow at time t (random variables);

a_t $(t = 1, \dots, T)$: planned amount of payment for the PAC bond at time t;

v_t $(t = 1, \dots, T-1)$: upper bound of cash reserve at time t;

We use the vector notation $\boldsymbol{a}_t = (a_1, \dots, a_t)$, $\boldsymbol{v}_t = (v_1, \dots, v_t)$, $\boldsymbol{a} = \boldsymbol{a}_T$, $\boldsymbol{v} = \boldsymbol{v}_T$, $\boldsymbol{C}_t = (C_1, \dots, C_t)$, $\boldsymbol{R} = (R_1, \dots, R_T)$

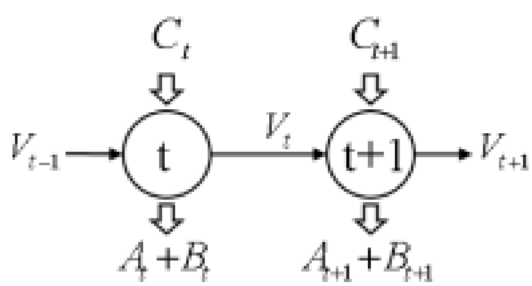

Figure 1: Cash Flow of CMO

Let A_t and B_t be the actual payments for the PAC bond and the companion bond respectively, and V_t be the cash reserve at time t. At each time t, payment is made for the PAC bond firstly out of the sum of repayment cash flow C_t and cash reserve V_{t-1} from the previous period. After that, we reserve cash for time $t+1$, and the amount that exceeds the upper bound of reserve is paid to the companion bond. From this, we get the following equations with $v_0 = V_0 = V_T = 0$:

$$\begin{aligned}
A_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t) &= \min\{a_t,\ C_t + V_{t-1}(\boldsymbol{a}_{t-1}, \boldsymbol{v}_{t-1}, \boldsymbol{C}_{t-1})\} \quad (t = 1, \ldots, T) && (1)\\
V_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{C}_t) &= \min\{v_t,\ C_t + V_{t-1}(\boldsymbol{a}_{t-1}, \boldsymbol{v}_{t-1}, \boldsymbol{C}_{t-1}) - A_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t)\} \quad (t = 1, \ldots, T-1) && (2)\\
B_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{C}_t) &= C_t + V_{t-1}(\boldsymbol{a}_{t-1}, \boldsymbol{v}_{t-1}, \boldsymbol{C}_{t-1}) - A_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t) - V_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{C}_t) && (3)\\
&\qquad (t = 1, \ldots, T)
\end{aligned}$$

For simplicity, we hereafter denote $A_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t), V_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{C}_t), B_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{C}_t)$ as A_t, V_t, B_t, respectively.

2.2 The Loss Function of the PAC Bond

The loss of the PAC bond occurs when the actual payment A_t is less than the scheduled payment a_t. We define the loss function of the PAC bond L_t as

$$\begin{aligned}
L_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t) &:= a_t - A_t && (4)\\
&= a_t - \min\{a_t,\ C_t + V_{t-1}\}\\
&= \max\{0, a_t - C_t - V_{t-1}\}. && (5)
\end{aligned}$$

From the equation (4), we get

$$A_t = a_t - L_t \tag{6}$$

and by substituting this into the equations (2) and (3), we get

$$\begin{aligned}
V_t &= \min\{v_t,\ C_t + V_{t-1} + L_t - a_t\}, && (7)\\
B_t &= C_t + V_{t-1} + L_t - a_t - V_t. && (8)
\end{aligned}$$

2.3 Present Value of the Bond

Let r_0 be the interest rate for the housing loans, and r' $(< r_0)$ be the interest rate for the PAC bond to issue. The function $W(\boldsymbol{a})$ gives the present value of the PAC bond with scheduled payment $\boldsymbol{a} = (a_1, \ldots, a_T)$, which is defined as

$$W(\boldsymbol{a}) := \sum_{t=1}^{T} \frac{a_t}{(1+r')^t}\ . \tag{9}$$

2.4 Basic Model

The aims of a CMO designer may be listed as follows:

- Increase the amount of the PAC bond to issue (Increase a_t).
- Keep the principal payment schedule of the PAC bond (Decrease loss L_t).
- Decrease the amount of reserve V_t (Decrease the upper bound of reserve v_t).
- Improve the quality of the companion bond (Make the variance of B_t small, and so on).

The reason why the issuer wants to decrease the amount of reserve may be described as follows: since the period of reserve is short and uncertain, it is difficult to invest the reserved cash. This means the reserved cash yields no profit and it leads to the issuer's loss.

We consider a model that takes into account all the above mentioned aims except the fourth one about the companion bond. Specifically, we propose the following optimization problem:

$$\begin{array}{ll} \max\limits_{\boldsymbol{a},\boldsymbol{v}} & W(\boldsymbol{a}) - \sum\limits_{t=1}^{T-1} \rho_t \cdot v_t \\ \text{s.t.} & \mathrm{E}\left[\sum\limits_{t=1}^{T} \gamma_t \cdot L_t\left(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t\right) \right] \leq \mathrm{U}_L, \\ & \boldsymbol{a} \geq 0, \quad \boldsymbol{v} \geq 0, \end{array} \tag{10}$$

where γ_t and ρ_t are weighting coefficients, and $\mathrm{E}[X]$ denotes the expectation of X. U_L is an upper boundary condition for the expectation of loss of the PAC bond.

This problem is to maximize the utility function of the MBS issuer that takes into account the amount of the PAC bond and the cash reserve under the constraint that the expectation of loss of the PAC bond is less than a preset value U_L. By solving problem (10), the distribution of the payment for the companion bond B_t may also be determined from the equation (3).

3 Approximation by Simulation Paths

3.1 Calculating Expectation

The expectation which appears in problem (10) can be written as

$$\mathrm{E}\left[\sum_{t=1}^{T} \gamma_t \cdot L_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t) \right] = \int_{c \in \Re^T} \sum_{t=1}^{T} \gamma_t \cdot L_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{c}_t) p(\boldsymbol{c}) d\boldsymbol{c}, \tag{11}$$

where $p(\boldsymbol{c})$ denotes the probability density functions of the repayment cash flow $\boldsymbol{C} = (C_1, \ldots, C_T)$. However, because the distribution of $\boldsymbol{C}_t$ is not known in practice, it is difficult to calculate the integral in (11). Here, we calculate the integral approximately by using sample paths of $\boldsymbol{C}_t$ generated by Monte Carlo method with "an interest rate fluctuation model" and "a mortgage prepayment model", which will be described later in detail.

We introduce some parameters.

I : the number of generated sample paths;

$\boldsymbol{r}^{(i)} = (r_1^{(i)}, \ldots, r_T^{(i)})$ $(i = 1, \ldots, I)$: sample paths of interest rates;

$\boldsymbol{c}^{(i)} = (c_1^{(i)}, \ldots, c_T^{(i)})$ $(i = 1, \ldots, I)$: sample paths of repayment cash flow;

With these sample paths, the calculation of the expectation (11) can be approximated as

$$\mathrm{E}\left[\sum_{t=1}^{T} \gamma_t \cdot L_t(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}, \boldsymbol{C}_t) \right] \approx \frac{1}{I} \sum_{i=1}^{I} \sum_{t=1}^{T} \gamma_t \cdot L_t(\boldsymbol{a}_t, \boldsymbol{v}_t, \boldsymbol{c}_t^{(i)}), \tag{12}$$

where $\boldsymbol{c}_t^{(i)} = (c_1^{(i)}, \ldots, c_t^{(i)})$. In the following subsections, we describe "an interest rate fluctuation model" and "a mortgage prepayment model", which will be used to generate sample paths $\boldsymbol{r}^{(i)}$ and $\boldsymbol{c}^{(i)}$, respectively.

3.2 Interest Rate Fluctuation Model

Interest rates tend to fluctuate around a certain average level m. As an interest rate model that takes this character into consideration, the following Vasicek model has been proposed in [9]:

$$r_{t+\triangle t} = r_t + \epsilon\,(m - r_t)\,\triangle t + \sigma \triangle B_t,$$

where r_t is the interest rate at time t, $\triangle t$ is a very short time interval, ϵ is a regression coefficient, σ is a diffusion coefficient, and $\triangle B_t$ is a Brownian motion with mean 0 and variance $\triangle t$. This model is easy to treat, and the behavior of interest rates is described well. However, this model has the serious drawback that interest rates may become negative. Once r_t becomes small, the possibility of r_t being negative in the next period becomes large under the influence of a diffusion coefficient. The following Cox-Ingersoll-Ross (CIR) model has been proposed in order to overcome this drawback [2]:

$$r_{t+\triangle t} = r_t + \epsilon\,(m - r_t)\,\triangle t + \sigma\sqrt{r_t}\triangle B_t. \tag{13}$$

In this model, even if r_t becomes small, it can prevent $r_{t+\triangle t}$ from being negative since volatility will also become small. In consideration of the present Japanese market where interest rates are very low, we use the CIR model to generate sample paths of interest rates. Specifically, one sample path is obtained by (13) recursively with the initial value r_0 and generating $\triangle B_t$ randomly. Repeating this procedure I times, we get I sample paths $\boldsymbol{r}^{(i)}$. In the numerical experiments reported in section 6, we generate a sequence $\{r_t\}$ recursively by (13) with $\triangle t = 0.01$, and determine a sample path $\boldsymbol{r}^{(i)}$ by picking up r_t from the sequence generated by (13) every one hundred recursions.

3.3 Sample Paths of Repayment Cash Flow

In a mortgage loan, it is necessary to take the prepayment risk into consideration since advanced repayment is basically accepted. In this section, we describe how to generate sample paths of repayment cash flow that include prepayments. We assume that there is a guarantee allowance to borrowers' default and a loan is performed by the fixed rate.

3.3.1 Mortgage Loan

We consider the mortgage loan of which maturity is T years, original principal balance is x_0, and interest rate is r_0. From

$$x_0 = \sum_{t=1}^{T} \frac{y}{(1+r_0)^t},$$

the amount of planned payment per year is calculated by

$$y = x_0 \times \frac{r_0(1+r_0)^T}{(1+r_0)^T - 1}.$$

Then, the outstanding principal balance at time t without prepayment, which we denote $\overline{x}_t$ ($t = 1, \ldots, T$), is calculated by

$$\overline{x}_t = \sum_{k=t+1}^{T} \frac{y}{(1+r_0)^{k-t}} = x_0 \times \frac{(1+r_0)^T - (1+r_0)^t}{(1+r_0)^T - 1}. \tag{14}$$

3.3.2 Mortgage Prepayment Model

There are economic and no-economic factors in prepayment. A typical economic factor is the refinancing of a loan due to the interest rate fall. No-economic factors may include loan age, region, age, seasonal one and so on. Moreover, there is a phenomenon called "burnout effect", which means that once the prepayment progresses, the quality of the left-behind loan pool changes and the sensitivity over interest rates is lost gradually. There are also some factors peculiar to Japan such as bonus-combined payment, step-up interest rates, a tax reduction period for mortgage holders, etc.

Schwartz and Torous model [8] has been used to model prepayment of mortgage loan. In this model, the prepayment rate π_t is provided in relation to a gap between the lending interest rate and the refinancing interest rate, the burnout effect, and seasonal influences. The prepayment rate π_t represents the probability that a loan obligor performing prepayment at time $t+1$ under the conditions of having not performed prepayment till time t. In Schwartz and Torous model, the prepayment rate π_t is calculated as follows:

$$\begin{aligned} \pi_t &= \hat{\pi}_t \cdot \exp\left\{\beta_1 k_t + \beta_2 l_t + \beta_3 m_t\right\}, \\ \hat{\pi}_t &= \kappa \frac{\omega\nu(\omega t)^{\nu-1}}{1+(\omega t)^{\nu}}, \\ k_t &= r_0 - r_{t-s}, \qquad s \geq 0, \\ l_t &= (k_t)^3, \\ m_t &= \ln\left(\frac{x_t}{\overline{x}_t}\right), \end{aligned} \tag{15}$$

where $\beta_1, \beta_2, \beta_3$ are weighting coefficients, κ, ω, ν are parameters, r_t is a refinancing interest rate at time t (generated by a interest rate fluctuation model), s is a time interval needed for refinancing (set to be 0 in this paper), x_t is the outstanding principal balance with prepayment, k_t is the difference between the lending interest rate and the refinicing interest rate, and m_t expresses the burnout effect. It is not clear whether this model can be applied to Japanese market as this model has been obtained by observing many MBS markets in U.S. However, since MBS market in Japan is still in the early stage and there are no other approved models, we employ this model.

3.3.3 Sample Paths of Repayment Cash Flow

Let $\overline{x}_t$ $(t = 1, \ldots, T)$ be given by equation (14) and sample paths of refinancing rates $\boldsymbol{r}^{(i)}$ $(t = 1, \ldots, T)$ be generated by the CIR model in advance. Prepayment occurs with probability π_{t-1} out of the outstanding principal balance x_{t-1}, and moreover, the outstanding principal balance decreases at the rate of planned repayment rate $\overline{x}_t/\overline{x}_{t-1}$. The difference between x_{t-1} and x_t becomes a repayment cash flow, and loan holders should pay the interest $r_0 \cdot x_{t-1}$ for the outstanding principal balance. Thus, sample paths of repayment cash flow $\boldsymbol{c}^{(i)}$ $(i = 1, \ldots, I)$ are generated as follows:

step1. Set $t := 1,\ \overline{x}_0 := x_0,\ r_0^{(i)} := r_0$.

step2. Calculate π_{t-1} from (15) with $r_{t-1}^{(i)},\ \overline{x}_{t-1},\ x_{t-1}$.

step3. Calculate the outstanding principal balance at time t

$$x_t := \frac{\overline{x}_t}{\overline{x}_{t-1}} \cdot (1 - \pi_{t-1}) \cdot x_{t-1}.$$

step4. Calculate repayment cash flow at time t

$$\begin{aligned} c_t^{(i)} &:= (x_{t-1} - x_t) + r_0 \cdot x_{t-1} \\ &= (1 + r_0)\, x_{t-1} - x_t. \end{aligned}$$

step5. If $t = T$, stop. Otherwise, set $t := t + 1$ and go to step2.

Applying this procedure I times for $i = 1, \ldots, I$, we obtain sample paths of repayment cash flow $\boldsymbol{c}^{(i)}$ $(i = 1, \ldots, I)$.

3.4 Mathematical Model

By using the approximation (12) and introducing variables $\boldsymbol{L} = (L_1^{(1)}, \ldots, L_T^{(1)}, \ldots, L_1^{(I)}, \ldots, L_T^{(I)})$ and $\boldsymbol{V} = (V_0^{(1)}, \ldots, V_T^{(1)}, \ldots, V_0^{(I)}, \ldots, V_T^{(I)})$ (with $V_0^{(i)} = V_T^{(i)} = 0$), defined by equations (5) and (7), we consider the following problem in place of problem (10):

$$\begin{aligned} \max_{\boldsymbol{a}, \boldsymbol{v}, \boldsymbol{L}, \boldsymbol{V}} \quad & W(\boldsymbol{a}) - \sum_{t=1}^{T-1} \rho_t \cdot v_t \\ \text{s.t.} \quad & \frac{1}{I} \sum_{i=1}^{I} \sum_{t=1}^{T} \gamma_t \cdot L_t^{(i)} \leq \mathrm{U}_L, \\ & L_t^{(i)} = \max\left\{0,\ a_t - c_t^{(i)} - V_{t-1}^{(i)}\right\}, \\ & V_t^{(i)} = \min\left\{v_t, c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t\right\}, \\ & \boldsymbol{a} \geq 0, \quad \boldsymbol{v} \geq 0, \quad V_0^{(i)} = V_T^{(i)} = 0. \end{aligned} \tag{16}$$

Also from equation (6) and (8), sample paths of payment for the PAC bond $\boldsymbol{A}^{(i)} = (A_1^{(i)}, \ldots, A_T^{(i)})$ and the companion bond $\boldsymbol{B}^{(i)} = (B_1^{(i)}, \ldots, B_T^{(i)})$ can be calculated respectively by

$$A_t^{(i)} = a_t - L_t^{(i)}, \tag{17}$$

$$B_t^{(i)} = c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t - V_t^{(i)}. \tag{18}$$

4 Linear Programming Model

Problem (16) is equivalent to the following problem (P_1):

$$
\begin{aligned}
(\mathrm{P}_1) \qquad \max_{\boldsymbol{a},\boldsymbol{v},\boldsymbol{L},\boldsymbol{V}} \quad & W(\boldsymbol{a}) - \sum_{t=1}^{T-1} \rho_t \cdot v_t \\
\text{s.t.} \quad & \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot L_t^{(i)} \le \mathrm{U}_L, \\
& L_t^{(i)} \ge 0, \\
& L_t^{(i)} \ge a_t - c_t^{(i)} - V_{t-1}^{(i)}, \\
& L_t^{(i)} = 0 \quad \text{or} \quad L_t^{(i)} = a_t - c_t^{(i)} - V_{t-1}^{(i)}, \qquad (19) \\
& V_t^{(i)} \le v_t, \\
& V_t^{(i)} \le c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t, \\
& V_t^{(i)} = v_t \quad \text{or} \quad V_t^{(i)} = c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t, \qquad (20) \\
& a_t \ge 0, \quad v_t \ge 0, \quad V_0^{(i)} = V_T^{(i)} = 0.
\end{aligned}
$$

Since this problem has the complementarity conditions (19) and (20), it is very difficult to solve this problem as it is. In this section, we show that this problem can be reformulated as an equivalent linear programming problem, for which very efficient solvers are available.

By removing the complementarity constraints (19) and (20) from problem (P_1) and adding nonnegative constraints with respect to $V_t^{(i)}$, we obtain the following problem:

$$
\begin{aligned}
\max_{\boldsymbol{a},\boldsymbol{v},\boldsymbol{L},\boldsymbol{V}} \quad & W(\boldsymbol{a}) - \sum_{t=1}^{T-1} \rho_t \cdot v_t \\
\text{s.t.} \quad & \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot L_t^{(i)} \le \mathrm{U}_L, \\
& L_t^{(i)} \ge 0, \\
& L_t^{(i)} \ge a_t - c_t^{(i)} - V_{t-1}^{(i)}, \qquad (21) \\
& V_t^{(i)} \le v_t, \qquad (22) \\
& V_t^{(i)} \le c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t, \qquad (23) \\
& V_t^{(i)} \ge 0, \qquad (24) \\
& a_t \ge 0, \\
& v_t \ge 0, \qquad (25) \\
& V_0^{(i)} = V_T^{(i)} = 0.
\end{aligned}
$$

Constraints (21) can be eliminated from constraints (23) and (24). Moreover constraints (25) can also be eliminated from constraints (22) and (24). As a result, we obtain the following problem (P_2):

$$(\mathrm{P}_2) \qquad \max_{\boldsymbol{a},\boldsymbol{v},\boldsymbol{L},\boldsymbol{V}} \qquad W(\boldsymbol{a}) - \sum_{t=1}^{T-1} \rho_t \cdot v_t$$

$$\text{s.t.} \qquad \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot L_t^{(i)} \le \mathrm{U}_L, \tag{26}$$

$$V_t^{(i)} \le c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t, \tag{27}$$

$$0 \le V_t^{(i)} \le v_t,$$

$$L_t^{(i)} \ge 0, \quad a_t \ge 0, \quad V_0^{(i)} = V_T^{(i)} = 0.$$

In the rest of this section, we prove that problem (P_1) and problem (P_2) are equivalent under certain natural assumptions. Problem (P_2) is a linear programming problem, so it can be solved by the simplex method or interior point method efficiently [1].

Lemma 1. *At an optimal solution of problem (P_2), constraint (26) is active.*

Proof. We prove by contradiction. Let $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ be an optimal solution of problem (P_2). Assume

$$\frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot \overline{L}_t^{(i)} < \mathrm{U}_L \tag{28}$$

holds. Define δ_1, $\breve{L}_1^{(i)}$ and $\breve{a}_1$ as

$$\begin{aligned} \delta_1 &:= \mathrm{U}_L - \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot \overline{L}_t^{(i)} > 0, \\ \breve{L}_1^{(i)} &:= \overline{L}_1^{(i)} + \frac{\delta_1}{\gamma_1} \quad (i = 1, \ldots, I), \\ \breve{a}_1 &:= \overline{a}_1 + \frac{\delta_1}{\gamma_1}\,. \end{aligned}$$

Let $(\breve{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \breve{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ be obtained by replacing $\overline{L}_1^{(i)}$ and $\overline{a}_1$ in $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ by $\breve{L}_1^{(i)}$ and $\breve{a}_1$, respectively. From

$$\begin{aligned} \frac{1}{I}\sum_{i=1}^{I}\left(\gamma_1 \breve{L}_1^{(i)} + \sum_{t=2}^{T} \gamma_t \cdot \overline{L}_t^{(i)}\right) &= \frac{1}{I}\sum_{i=1}^{I}\left(\gamma_1 \overline{L}_1^{(i)} + \delta_1 + \sum_{t=2}^{T} \gamma_t \cdot \overline{L}_t^{(i)}\right) \\ &= \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot \overline{L}_t^{(i)} + \delta_1 = \mathrm{U}_L, \end{aligned}$$

$(\breve{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \breve{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ satisfies constraint (26). Moreover, from $\overline{V}_t^{(i)} \le c_t^{(i)} + \overline{V}_{t-1}^{(i)} + \overline{L}_t^{(i)} - \overline{a}_t$, we have

$$\overline{V}_t^{(i)} \le c_t^{(i)} + \overline{V}_{t-1}^{(i)} + (\overline{L}_t^{(i)} + \frac{\delta_1}{\gamma_1}) - (\overline{a}_t + \frac{\delta_1}{\gamma_1}),$$

and hence

$$\overline{V}_t^{(i)} \le c_t^{(i)} + \overline{V}_{t-1}^{(i)} + \breve{L}_t^{(i)} - \breve{a}_t.$$

Thus $(\breve{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \breve{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ satisfies constraint (27), which implies $(\breve{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \breve{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ is feasible for problem (P$_2$). However, we have

$$W(\breve{\boldsymbol{a}}) - W(\overline{\boldsymbol{a}}) = \frac{\delta_1}{\gamma_1(1+r')} > 0.$$

This implies that the objective function value of $(\breve{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \breve{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ is larger than that of $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$. This contradicts the assumption that $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ is optimal. Therefore constraint (26) must become active at an optimal solution of problem (P$_2$). □

The following theorem implies that problem (P$_1$) and problem (P$_2$) are equivalent under the natural assumption that $\gamma_t > \gamma_{t+1}$ $(t = 1, \ldots, T-1)$.

Theorem 1. *Let $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ be the optimal solution of problem (P$_2$). If $\gamma_t > \gamma_{t+1}$ $(t = 1, \ldots, T-1)$, then $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ becomes an optimal solution of problem (P$_1$), where $\tilde{\boldsymbol{V}} = (\tilde{V}_0^{(1)}, \ldots, \tilde{V}_T^{(1)}, \ldots, \tilde{V}_0^{(I)}, \ldots, \tilde{V}_T^{(I)})$ is given by*

$$\tilde{V}_1^{(i)} := \min\{\overline{v}_1,\ c_1^{(i)} + \overline{L}_1^{(i)} - \overline{a}_1\}, \tag{29}$$

$$\tilde{V}_t^{(i)} := \min\{\overline{v}_t,\ c_t^{(i)} + \tilde{V}_{t-1}^{(i)} + \overline{L}_t^{(i)} - \overline{a}_t\} \quad (t = 2, \ldots, T-1), \tag{30}$$

$$\tilde{V}_0^{(i)} := 0, \quad \tilde{V}_T^{(i)} := 0\ .$$

Proof. It is easy to see that $\overline{V}_0^{(i)} = 0$ and

$$\overline{V}_1^{(i)} \leq c_1^{(i)} + \overline{L}_1^{(i)} - \overline{a}_1,$$
$$0 \leq \overline{V}_1^{(i)} \leq \overline{v}_1$$

hold at the optimal solution $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$. From this, $\tilde{V}_1^{(i)} \geq \overline{V}_1^{(i)}$ holds, where $\tilde{V}_1^{(i)}$ is given by (29), and therefore $\tilde{V}_1^{(i)} \geq 0$ holds. Let $\tilde{V}_t^{(i)}$ for $t = 2, \ldots, T-1$ be determined by (30) recursively. Assume $\tilde{V}_{t-1}^{(i)} \geq \overline{V}_{t-1}^{(i)}$. Then we have

$$\begin{aligned}\tilde{V}_t^{(i)} &= \min\{\overline{v}_t,\ c_t^{(i)} + \tilde{V}_{t-1}^{(i)} + \overline{L}_t^{(i)} - \overline{a}_t\} \\ &\geq \min\{\overline{v}_t,\ c_t^{(i)} + \overline{V}_{t-1}^{(i)} + \overline{L}_t^{(i)} - \overline{a}_t\} \\ &\geq \overline{V}_t^{(i)} \geq 0,\end{aligned}$$

where the second and the last inequalities follow from the fact that $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ satisfies the constraints of problem (P$_2$). Thus $\tilde{V}_t^{(i)} \geq 0$ $(t = 2, \ldots, T-1)$ hold inductively from $\tilde{V}_1^{(i)} \geq \overline{V}_1^{(i)}$. Therefore $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ becomes feasible for problem (P$_2$). Moreover, it is optimal for problem (P$_2$), since the objective function value of $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is equal to that of the optimal solution $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$.

Consider the following constraints in problem (P$_2$):

$$L_t^{(i)} \geq 0, \quad L_t^{(i)} \geq a_t + V_t^{(i)} - c_t^{(i)} - V_{t-1}^{(i)}. \tag{31}$$

We show that at least one of the inequalities in (31) becomes active at $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$. Assume to the contrary that there exist $i_1 \in \{1, \ldots, I\}$ and $t_1 \in \{1, \ldots, T\}$ such that

$$\overline{L}_{t_1}^{(i_1)} > 0 \quad \text{and} \quad \overline{L}_{t_1}^{(i_1)} > \overline{a}_{t_1} + \tilde{V}_{t_1}^{(i_1)} - c_{t_1}^{(i_1)} - \tilde{V}_{t_1-1}^{(i_1)}.$$

Then define δ_2, $\check{L}_{t_1}^{(i)}$ and $\check{a}_{t_1}$as

$$\begin{aligned}
\delta_2 &:= \min\{\overline{L}_{t_1}^{(i_1)},\ \overline{L}_{t_1}^{(i_1)} - \overline{a}_{t_1} - \tilde{V}_{t_1}^{(i_1)} + c_{t_1}^{(i_1)} + \tilde{V}_{t_1-1}^{(i_1)}\} > 0, \qquad (32)\\
\check{L}_{t_1}^{(i_1)} &:= \overline{L}_{t_1}^{(i_1)} - \delta_2 + \frac{\delta_2}{I} = \max\left\{\frac{\delta_2}{I}, \left(\overline{a}_{t_1} + \frac{\delta_2}{I}\right) + \tilde{V}_{t_1}^{(i_1)} - c_{t_1}^{(i_1)} - \tilde{V}_{t_1-1}^{(i_1)}\right\} > 0,\\
\check{L}_{t_1}^{(i)} &:= \overline{L}_{t_1}^{(i)} + \frac{\delta_2}{I} > 0 \qquad (i = 1, \ldots, I,\ i \neq i_1),\\
\check{a}_{t_1} &:= \overline{a}_{t_1} + \frac{\delta_2}{I} > 0.
\end{aligned}$$

Let $(\check{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \check{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ be obtained by replacing $\overline{L}_{t_1}^{(i)}$ and $\overline{a}_{t_1}$ in $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \overline{\boldsymbol{V}})$ by $\check{L}_1^{(i)}$ and $\check{a}_1$, respectively. Then we have

$$\check{L}_{t_1}^{(i_1)} = \max\left\{\frac{\delta_2}{I}, \check{a}_t + \tilde{V}_{t_1}^{(i_1)} - c_{t_1}^{(i_1)} - \tilde{V}_{t_1-1}^{(i_1)}\right\} \geq \check{a}_{t_1} + \tilde{V}_{t_1}^{(i_1)} - c_{t_1}^{(i_1)} - \tilde{V}_{t_1-1}^{(i_1)}.$$

Moreover, from

$$\overline{L}_{t_1}^{(i)} \geq \overline{a}_{t_1} + \tilde{V}_{t_1}^{(i)} - c_{t_1}^{(i)} - \tilde{V}_{t_1-1}^{(i)} \quad (i = 1, \ldots, I, i \neq i_1),$$

we obtain

$$\left(\overline{L}_{t_1}^{(i)} + \frac{\delta_2}{I}\right) \geq \left(\overline{a}_{t_1} + \frac{\delta_2}{I}\right) + \tilde{V}_{t_1}^{(i)} - c_{t_1}^{(i)} - \tilde{V}_{t_1-1}^{(i)} \quad (i = 1, \ldots, I, i \neq i_1),$$

and hence

$$\check{L}_{t_1}^{(i)} \geq \check{a}_{t_1} + \tilde{V}_{t_1}^{(i)} - c_{t_1}^{(i)} - \tilde{V}_{t_1-1}^{(i)} \quad (i = 1, \ldots, I, i \neq i_1).$$

Thus $(\check{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \check{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ satisfies constraint (27). Moreover, the definition of $\check{L}_{t_1}^{(i)}$ implies

$$\sum_{i=1}^{I} \check{L}_{t_1}^{(i)} = \sum_{i=1}^{I} \overline{L}_{t_1}^{(i)},$$

which implies that $(\check{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \check{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ satisfies constraint (26). Therefore $(\check{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \check{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ becomes feasible for problem (P_2). However, we have

$$W(\check{\boldsymbol{a}}) - W(\overline{\boldsymbol{a}}) = \frac{\delta_2}{I(1+r')^{t_1}} > 0.$$

This implies that the objective function value of $(\check{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \check{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is larger than that of $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$. This contradicts the assumption that $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is optimal. Therefore at least one of the inequalities in (31) becomes active at $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$. Consequently, the following equalities hold for all $i \in \{1, \ldots, I\}$ and $t \in \{1, \ldots, T\}$:

$$\overline{L}_t^{(i)} = \max\{0,\ \overline{a}_t + \tilde{V}_t^{(i)} - c_t^{(i)} - \tilde{V}_{t-1}^{(i)}\}. \qquad (33)$$

Now, we define index sets J_t and K_t as

$$\begin{aligned}
J_t &:= \left\{j \in \{1, \ldots, I\} \mid \overline{L}_t^{(j)} = 0,\ \overline{L}_t^{(j)} \geq \overline{a}_t + \tilde{V}_t^{(j)} - c_t^{(j)} - \tilde{V}_{t-1}^{(j)}\right\},\\
K_t &:= \left\{k \in \{1, \ldots, I\} \mid \overline{L}_t^{(k)} \geq 0,\ \overline{L}_t^{(k)} = \overline{a}_t + \tilde{V}_t^{(k)} - c_t^{(k)} - \tilde{V}_{t-1}^{(k)}\right\}.
\end{aligned}$$

Then we have $J_t \cup K_t = \{1, \ldots, I\}$ from (33).

We denote the complement of S as S^c. We show $\tilde{V}_t^{(i)} = 0$ for all $i \in J_t^c \cap K_t$ and $t \in \{1, \ldots, T-1\}$ by contradiction. Assume that there exist $i_1 \in J_t^c \cap K_t$ and $t_1 \in \{1, \ldots, T-1\}$ such that $\tilde{V}_{t_1}^{(i_1)} > 0$. Since $i_2 \in J_{t_2}^c \cap K_{t_2}$, we have

$$\overline{L}_{t_2}^{(i_2)} = \overline{a}_{t_2} + \tilde{V}_{t_2}^{(i_2)} - c_{t_2}^{(i_2)} - \tilde{V}_{t_2-1}^{(i_2)} > 0. \tag{34}$$

Let us define δ_3, $\hat{V}_{t_2}^{(i_2)}, \hat{L}_{t_2}^{(i_2)}$ and $\hat{L}_{t_2+1}^{(i_2)}$ as

$$\begin{aligned}
\delta_3 &:= \min\left\{\tilde{V}_{t_2}^{(i_2)},\ \overline{a}_{t_2}^{(i_2)} + \tilde{V}_{t_2}^{(i_2)} - c_{t_2}^{(i_2)} - \tilde{V}_{t_2-1}^{(i_2)}\right\} > 0, && (35)\\
\hat{V}_{t_2}^{(i_2)} &:= \tilde{V}_{t_2}^{(i_2)} - \delta_3\\
&= \max\left\{0,\ c_{t_2}^{(i_2)} + \tilde{V}_{t_2-1}^{(i_2)} - \overline{a}_{t_2}^{(i_2)}\right\} \geq 0,\\
\hat{L}_{t_2}^{(i_2)} &:= \overline{L}_{t_2}^{(i_2)} - \delta_3\\
&= \max\left\{\overline{L}_{t_2}^{(i_2)} - \tilde{V}_{t_2}^{(i_2)},\ \overline{L}_{t_2}^{(i_2)} - \overline{a}_{t_2}^{(i_2)} - \tilde{V}_{t_2}^{(i_2)} + c_{t_2}^{(i_2)} + \tilde{V}_{t_2-1}^{(i_2)}\right\}\\
&= \max\left\{\overline{L}_{t_2}^{(i_2)} - \tilde{V}_{t_2}^{(i_2)},\ 0\right\} \qquad \text{(by (34))}\\
&\geq 0,\\
\hat{L}_{t_2+1}^{(i_2)} &:= \overline{L}_{t_2+1}^{(i_2)} + \delta_3 > 0 \qquad \left(\text{by } \overline{L}_{t_2+1}^{(i_2)} \geq 0 \text{ and } (35)\right).
\end{aligned}$$

Let $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{V}})$ be obtained by replacing $\tilde{V}_{t_2}^{(i_2)}$, $\overline{L}_{t_2}^{(i_2)}$ and $\overline{L}_{t_2+1}^{(i_2)}$ in $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ by $\hat{V}_{t_2}^{(i_2)}$, $\hat{L}_{t_2}^{(i_2)}$ and $\hat{L}_{t_2+1}^{(i_2)}$, respectively. It then follows from $\tilde{V}_{t_2}^{(i_2)} \leq v_{t_2}$ that $\hat{V}_{t_2}^{(i_2)} < v_{t_2}$ and from $\tilde{V}_{t_2}^{(i_2)} \leq c_{t_2}^{(i_2)} + \tilde{V}_{t_2-1}^{(i_2)} + \overline{L}_{t_2}^{(i_2)} - \overline{a}_{t_2}$ that

$$\left(\tilde{V}_{t_2}^{(i_2)} - \delta_3\right) \leq c_{t_2}^{(i_2)} + \tilde{V}_{t_2-1}^{(i_2)} + \left(\overline{L}_{t_2}^{(i_2)} - \delta_3\right) - \overline{a}_{t_2},$$

namely

$$\hat{V}_{t_2}^{(i_2)} \leq c_{t_2}^{(i_2)} + \tilde{V}_{t_2-1}^{(i_2)} + \hat{L}_{t_2}^{(i_2)} - \overline{a}_{t_2}.$$

Moreover, from $\tilde{V}_{t_2+1}^{(i_2)} \leq c_{t_2+1}^{(i_2)} + \tilde{V}_{t_2}^{(i_2)} + \overline{L}_{t_2+1}^{(i_2)} - \overline{a}_{t_2+1}$, we obtain

$$\begin{aligned}
\tilde{V}_{t_2+1}^{(i_2)} &\leq c_{t_2+1}^{(i_2)} + \left(\tilde{V}_{t_2}^{(i_2)} - \delta_3\right) + \left(\overline{L}_{t_2+1}^{(i_2)} + \delta_3\right) - \overline{a}_{t_2+1}\\
&= c_{t_2+1}^{(i_2)} + \hat{V}_{t_2}^{(i_2)} + \hat{L}_{t_2+1}^{(i_2)} - \overline{a}_{t_2+1}.
\end{aligned}$$

Thus $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{V}})$ is feasible for problem (P$_2$). Furthermore, $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{V}})$ is optimal since the objective function value of $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{V}})$ is equal to that of the optimal solution $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$. Since $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ and $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{V}})$ are both optimal, we obtain the following equalities from Lemma 1:

$$\frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\gamma_t \cdot \overline{L}_t^{(i)} = \mathrm{U}_L, \tag{36}$$

$$\frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\gamma_t \cdot \hat{L}_t^{(i)} = \mathrm{U}_L, \tag{37}$$

where $\hat{L}_t^{(i)} := \overline{L}_t^{(i)} \ \ ((i,t) \neq (i_2, t_2), (i_2, t_2+1))$. By (36) and (37), we obtain

$$\frac{1}{I}\left(\gamma_{t_2} \cdot \overline{L}_{t_2}^{(i_2)} + \gamma_{t_2+1} \cdot \overline{L}_{t_2+1}^{(i_2)}\right) - \frac{1}{I}\left(\gamma_{t_2} \cdot \hat{L}_{t_2}^{(i_2)} + \gamma_{t_2+1} \cdot \hat{L}_{t_2+1}^{(i_2)}\right) = 0,$$

that is,

$$(\gamma_{t_2} - \gamma_{t_2+1})\delta_3 = 0. \tag{38}$$

This contradicts the assumption $\gamma_{t_2} > \gamma_{t_2+1}$ and (35). Therefore $\tilde{V}_t^{(i)} = 0$ holds for all $i \in J_t^c \cap K_t$ and $t \in \{1, \ldots, T-1\}$. This implies that $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ satisfies constraint (19). Moreover, from (29) and (30), $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ satisfies constraint (20). Consequently, $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is feasible for problem (P_1).

Let the feasible region of problem (P_1) and problem (P_2) be X_1 and X_2, respectively. Since $X_1 \subseteq X_2$ and the objective functions of problem (P_1) and problem (P_2) are the same, we obtain

$$(\text{optimal value of } (P_1)) \ \leq \ (\text{optimal value of } (P_2)) \,. \tag{39}$$

Since $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is feasible for both problem (P_1) and problem (P_2) and is optimal for problem (P_2), it follows from (39) that $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{L}}, \tilde{\boldsymbol{V}})$ is optimal for problem (P_1). □

5 Modification of the Model

5.1 Features of Mortgage Repayment Cash Flow

The mortgage repayment cash flow with prepayment has a strong time-correlation: When prepayment occurs frequently, a large amount of repayment flow arises in the first half of the maturity period, and repayment flow in the last half becomes small. Figure 2 illustrates two typical sample paths: the dotted curve shows the case where prepayment occurs frequently, while the solid curve shows the case where prepayment occurs less frequently. The amount of

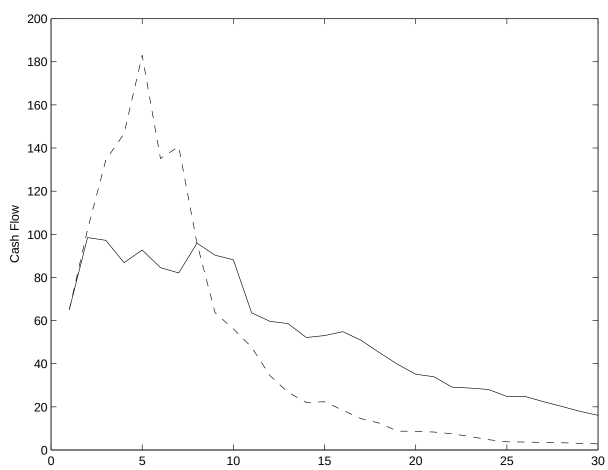

Figure 2: Shift of repayment cash flow due to prepayment

issuable PAC bond can be increased by reserving surplus yielded as a result of prepayment and paying this surplus to PAC bond holders in the subsequent periods. In the model proposed in previous sections, it is assumed that the upper bound of cash reserve v_t is fixed regardless of the pattern of repayment cash flow. However, in view of the nature of repayment cash flow as mentioned above, it may be meritorious to change the upper bound of cash reserve in accordance with the pattern of repayment cash flow.

5.2 Modified Model

Let Z_t be the total amount of prepayment up to time t, which is given by

$$Z_t := \sum_{k=1}^{t-1} \pi_k \cdot x_k,$$

where π_t is the prepayment rate defined by (15) and x_t is the outstanding principal balance at time t. Let the upper bound of cash reserve v_t be given as a function of the total prepayment amount Z_t, i.e.,

$$v_t(\tau_t, \xi_t, Z_t) := \tau_t + \xi_t Z_t \qquad (t = 1, \ldots, T-1),$$

where τ_t and ξ_t are parameters. Note that the variables in the model are changed from v_t to τ_t and ξ_t by this modification. In the following, we use the vector notation $\boldsymbol{\tau}_t = (\tau_1, \ldots, \tau_t)$, $\boldsymbol{\xi}_t = (\xi_1, \ldots, \xi_t)$, $\boldsymbol{\tau} = \boldsymbol{\tau}_{T-1}$, $\boldsymbol{\xi} = \boldsymbol{\xi}_{T-1}$, $\boldsymbol{Z}_t = (Z_1, \ldots, Z_t)$, $\boldsymbol{v}_t(\boldsymbol{\tau}_t, \boldsymbol{\xi}_t, \boldsymbol{Z}_t) = (v_1(\boldsymbol{\tau}_1, \boldsymbol{\xi}_1, \boldsymbol{Z}_1), \ldots, v_t(\boldsymbol{\tau}_t, \boldsymbol{\xi}_t, \boldsymbol{Z}_t))$.

Now we propose the following model:

$$\begin{aligned} \max_{\boldsymbol{a}, \boldsymbol{\tau}, \boldsymbol{\xi}} \quad & W(\boldsymbol{a}) - \sum_{t=1}^{T-1} \rho_t \cdot \mathrm{E}\left[v_t(\tau_t, \xi_t, Z_t)\right] \\ \text{s.t.} \quad & \mathrm{E}\left[\sum_{t=1}^{T} \gamma_t \cdot L_t\big(\boldsymbol{a}_t, \boldsymbol{v}_{t-1}(\boldsymbol{\tau}_{t-1}, \boldsymbol{\xi}_{t-1}, \boldsymbol{Z}_{t-1}), \boldsymbol{C}_t\big)\right] \le \mathrm{U}_L, \\ & a_t \ge 0, \quad v_t(\tau_t, \xi_t, Z_t) \ge 0. \end{aligned} \tag{40}$$

Let us call this model "the modified model". By applying the approximation method as described in section 3, we obtain the following problem:

$$\begin{aligned} \max_{\boldsymbol{a}, \boldsymbol{\tau}, \boldsymbol{\xi}, \boldsymbol{L}, \boldsymbol{V}} \quad & W(\boldsymbol{a}) - \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T-1} \rho_t \cdot (\tau_t + \xi_t z_t^{(i)}) \\ \text{s.t.} \quad & \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \gamma_t \cdot L_t^{(i)} \le \mathrm{U}_L, \\ & L_t^{(i)} = \max\left\{0,\ a_t - c_t^{(i)} - V_{t-1}^{(i)}\right\}, \\ & V_t^{(i)} = \min\left\{\tau_t + \xi_t z_t^{(i)}, c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t\right\}, \\ & a_t \ge 0, \quad \tau_t + \xi_t z_t^{(i)} \ge 0, \quad V_0^{(i)} = V_T^{(i)} = 0, \end{aligned} \tag{41}$$

where $z_t^{(i)}$ is the total prepayment up to time t on path i. Moreover, we can show in a manner similar to section 4 that a solution of problem (41) is obtained by solving the following problem:

$$\begin{array}{ll}\max\limits_{\boldsymbol{a},\boldsymbol{\tau},\boldsymbol{\xi},\boldsymbol{L},\boldsymbol{V}} & W(\boldsymbol{a}) - \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T-1}\rho_t\cdot(\tau_t+\xi_t z_t^{(i)}) \\ \text{s.t.} & \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\gamma_t\cdot L_t^{(i)} \le \mathrm{U}_L, \\ & V_t^{(i)} \le c_t^{(i)} + V_{t-1}^{(i)} + L_t^{(i)} - a_t, \\ & 0 \le V_t^{(i)} \le \tau_t + \xi_t z_t^{(i)}, \\ & L_t^{(i)} \ge 0, \quad a_t \ge 0, \quad V_0^{(i)} = V_T^{(i)} = 0.\end{array}$$

6 Numerical Experiments

In this section, we report the results of some numerical experiments with our proposed models. We first describe the experimental environment, and then show the computational results.

6.1 Numerical Environment

We assume that:

- the original principal balance of the loan pool is $x_0 = 1000$, the interest rate on loans is $r_0 = 0.05$, and maturity period is $T = 30$;
- the refinancing rate is generated by CIR model with the initial rate 0.05;
- the interest rate on the PAC bond is $r' = 0.04$;

Parameters in the CIR model (13) and the prepayment model (15) are set as shown in Table 1*.

Table 1: Parameter Settings

CIR model		Prepayment model	
ϵ	0.2	κ	1.5
m	0.05	ω	0.083
σ	0.02	ν	1.74
		β_1	34.2
		β_2	0
		β_3	0.3

We used $I = 1000$ sample paths for approximation. Figure 3 shows the actual sample paths of repayment cash flow we used.

We implement our models with modeling language SIMPLE and solve them by NUOPT Version 6 on the platform of RedHat 8 with Intel Pentium 4 CPU 2.80GHz, 1GB memory†. The computation time spent to solve problems with 1000 sample paths is about 45 seconds.

*Parameters in the prepayment model are referred from Kai [7]

†SIMPLE and NUOPT are Mathematical Programming Software developed by Mathematical Systems, Inc.

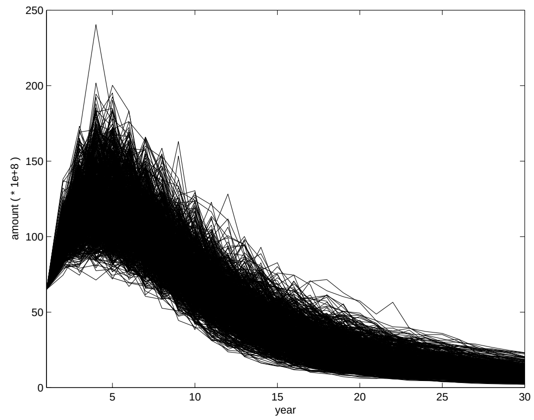

Figure 3: Sample paths of repayment cash flow ($I = 1000$)

6.2 Computational Results

6.2.1 Model 1

First, we report experimental results for the basic model (10) in which the upper bound of reserve v_t does not depend on the previous repayment cash flow. We call this model "Model 1" hereinafter.

We set parameters γ_t and ρ_t as

$$\gamma_t := \frac{1}{(1+r')^t} \quad , \qquad \rho_t := \frac{\rho_0}{(1+r')^t},$$

where ρ_0 is a parameter. Then, the objective function of Model 1 can be written as

$$W(\boldsymbol{a}) - \rho_0 \cdot W(\boldsymbol{v}),$$

where $W(\boldsymbol{v})$ represents the present value of cash reserve bounds $\boldsymbol{v}$ and is defined as

$$W(\boldsymbol{v}) \quad := \quad \sum_{t=1}^{T-1} \frac{v_t}{(1+r')^t}.$$

Relative weights for the PAC bond and cash reserve can be changed by adjusting ρ_0.

Table 2 shows $W(\boldsymbol{a})$ and $W(\boldsymbol{v})$ at the optimal solution when $\mathrm{U}_L = 0.1$ (fixed) and ρ_0 is changed from 0 to 1. Table 3 shows the corresponding results when $\mathrm{U}_L = 0.01$. From Table 2 and Table 3, we observe that both $W(\boldsymbol{v})$ and $W(\boldsymbol{a})$ at the optimal solution increase as ρ_0 become small. This implies that the amount of the PAC bond may be increased by allowing more cash reserve. Figure 4 illustrates the behavior of $W(\boldsymbol{a})$ versus $W(\boldsymbol{v})$. This graph shows that the marginal rate of increase of $W(\boldsymbol{a})$ relative to $W(\boldsymbol{v})$ becomes large as $W(\boldsymbol{v})$ decreases. Moreover, $W(\boldsymbol{a})$ becomes small as U_L becomes small. This implies that the more certainly the issuer wants to make payment for the PAC bond holders, the less the PAC bond can be issued.

Figure 8 in Appendix illustrates in detail the results for the case $(\mathrm{U}_L, \rho_0) = (0.1, 0.1)$. Figure 8 shows the planned amount of payment for the PAC bond a_t, the upper bound of cash reserve v_t, the mean of actual reserve $\mathbf{E}[\tilde{\boldsymbol{V}}_t]$ and the loss of the PAC bond $L_t^{(i)}$ at the

Table 2: Results for Model 1 ($U_L = 0.1$)

ρ_0	$W(\boldsymbol{a})$	$W(\boldsymbol{v})$
1.00	665.329012	0.000016
0.90	670.356264	5.228164
0.80	670.356334	5.228175
0.70	675.934293	12.965120
0.60	675.934359	12.965192
0.50	687.478028	33.167633
0.40	700.987603	61.421008
0.30	716.179808	102.538409
0.20	746.297396	228.257527
0.10	834.935921	862.815392
0.00	964.916426	> 10000

Table 3: Results for Model 1 ($U_L = 0.01$)

ρ_0	$W(\boldsymbol{a})$	$W(\boldsymbol{v})$
1.00	623.453907	0.000497
0.90	656.148754	34.002963
0.80	656.148776	34.002962
0.70	656.148806	34.002980
0.60	656.148863	34.002979
0.50	656.148866	34.002999
0.40	691.417999	108.860996
0.30	696.513779	125.377859
0.20	713.763086	200.229550
0.10	817.424007	960.052624
0.00	951.113801	> 10000

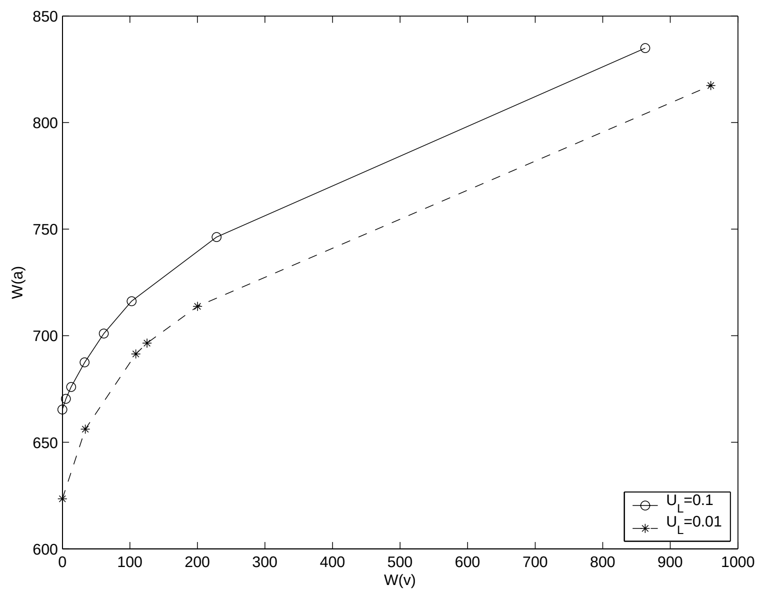

Figure 4: Behavior of $W(\boldsymbol{v})$ and $W(\boldsymbol{a})$ (Model 1).

optimal solution, where $\mathbf{E}[\tilde{\boldsymbol{V}}_t]$ is calculated as

$$\mathbf{E}[\tilde{\boldsymbol{V}}_t] := \frac{1}{I}\sum_{i=1}^{I}\tilde{V}_t^{(i)},$$

where $\tilde{V}_t^{(i)}$ is given by (29) and (30). The shaded area in Figure 8 represents sample paths shown in Figure 3. Figure 9 illustrates the payment for the companion bond $B_t^{(i)}$ at the optimal solution for $(\mathrm{U}_L, \rho_0) = (0.1, 0.1)$, where $B_t^{(i)}$ is calculated by (18). Figure 10 illustrates the loss of the PAC bond $L_t^{(i)}$ in the case where the PAC bond a_t in Figure 8 is issued without cash reserve. Figure 11 illustrates v_t and $\mathbf{E}[\tilde{\boldsymbol{V}}_t]$ in Figure 8 and $L_t^{(i)}$ in Figure 10 together. Table 6 shows the probability of having $L_t^{(i)} > 0$ at each time in Figure 8. The corresponding results for $(\mathrm{U}_L, \rho_0) = (0.1, 0.2)$ are shown in Figures 12-15 and Table 7, and the results for $(\mathrm{U}_L, \rho_0) = (0.01, 0.1)$ are shown in Figures 16-19 and Table 8. From Figure 8 and Figure 12, we may observe that the upper bound of cash reserve v_t becomes small by setting ρ_0 small, and the mean of actual reserve $\mathbf{E}[\tilde{\boldsymbol{V}}_t]$ also becomes small. Tables 6-8 show that the issuer can make payment for the PAC bond holders more certainly by setting U_L smaller. The probability of having $L_t^{(i)} > 0$, that is the probability of the payment for the PAC bond not being performed as planned, is less than 0.5% when $(\mathrm{U}_L, \rho_0) = (0.1, 0.1)$, less than 0.7% when $(\mathrm{U}_L, \rho_0) = (0.1, 0.2)$ and less than 0.2% when $(\mathrm{U}_L, \rho_0) = (0.01, 0.1)$. Figures 11, 15 and 19 show that the upper bound of cash reserve is determined so as to cover the loss without reserve. Payment for the companion bond has a very large variance, as shown in Figures 9, 13 and 17.

Let $\mathbf{E}[W_0(\boldsymbol{A})]$, $\mathbf{E}[W_0(\boldsymbol{B})]$ and $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$ be the mean present value of actual payment for the PAC bond $A_t^{(i)}$ (calculated by (17) and (18)), actual payment for the companion bond $B_t^{(i)}$ and actual cash reserve $\tilde{V}_t^{(i)}$, respectively, where the discount rate is $1 + r_0$ (rate on mortgage loans). There quantities are given by

$$\begin{aligned}
\mathbf{E}[W_0(\boldsymbol{A})] &:= \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\frac{A_t^{(i)}}{(1+r_0)^t},\\
\mathbf{E}[W_0(\boldsymbol{B})] &:= \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\frac{B_t^{(i)}}{(1+r_0)^t},\\
\mathbf{E}[W_0(\tilde{\boldsymbol{V}})] &:= \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T-1}\frac{\tilde{V}_t^{(i)}}{(1+r_0)^t}.
\end{aligned}$$

$\mathbf{E}[W_0(\boldsymbol{A})]$ and $\mathbf{E}[W_0(\boldsymbol{B})]$ represent the amount of the PAC bond and that of the companion bond in the original principal balance, respectively. Table 4 shows $\mathbf{E}[W_0(\boldsymbol{A})]$, $\mathbf{E}[W_0(\boldsymbol{B})]$, $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$ and $\mathbf{E}[W_0(\boldsymbol{A})] + \mathbf{E}[W_0(\boldsymbol{B})]$ at the optimal solution when $\mathrm{U}_L = 0.1$ (fixed) and ρ_0 is changed from 0 to 1. Figure 5 illustrates the results given in Table 4.

Figure 5 shows that $\mathbf{E}[W_0(\boldsymbol{A})]$ increases but $\mathbf{E}[W_0(\boldsymbol{B})]$ decreases as $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$ increases. This implies that the amount of the PAC bond increases and that of the companion bond decreases by allowing more cash reserve. However, from Figure 5, we see that the sum of $\mathbf{E}[W_0(\boldsymbol{A})]$ and $\mathbf{E}[W_0(\boldsymbol{B})]$ decreases as $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$ increases. This can be explained as follows: From (3), we have

$$A_t + B_t = C_t + \tilde{V}_{t-1} - \tilde{V}_t \quad (t = 1, \ldots, T).$$

Table 4: Results for Model 1 ($U_L = 0.1$)

ρ_0	$\mathbf{E}[W_0(\boldsymbol{A})]$	$\mathbf{E}[W_0(\boldsymbol{B})]$	$\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$	$\mathbf{E}[W_0(\boldsymbol{A})] + \mathbf{E}[W_0(\boldsymbol{B})]$
1.00	625.263237	374.736763	0.000014	999.999999
0.90	629.954983	369.808362	4.969755	999.763345
0.80	629.944548	369.818797	4.969762	999.763345
0.70	634.959381	364.464178	12.105262	999.423559
0.60	634.959454	364.464103	12.105309	999.423557
0.50	645.631514	352.894570	30.952223	998.526085
0.40	658.204883	339.079144	57.035418	997.284028
0.30	672.283194	323.208737	94.669458	995.491931
0.20	699.907879	290.147892	208.828794	990.055772
0.10	779.567270	187.451879	692.597871	967.019149
0.00	889.568543	27.547358	1740.566092	917.115900

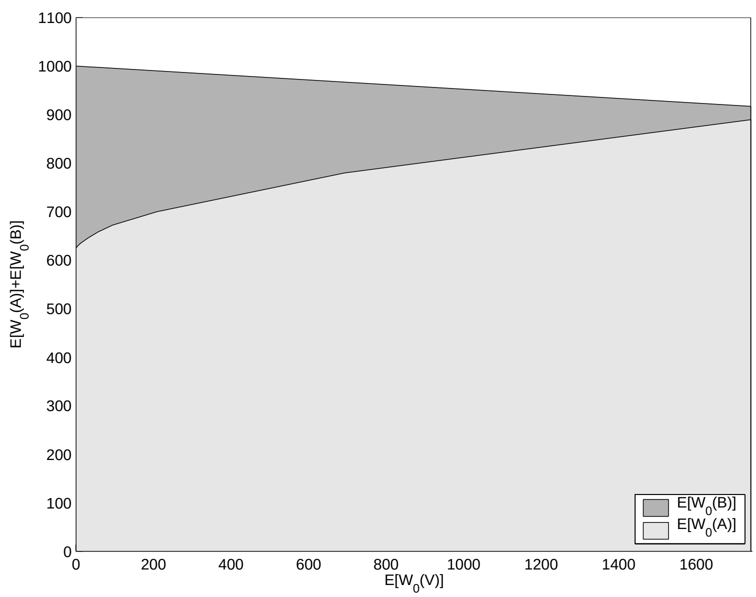

Figure 5: Behavior of $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$, $\mathbf{E}[W_0(\boldsymbol{A})]$ and $\mathbf{E}[W_0(\boldsymbol{B})]$ (Model 1)

From these equations, we obtain

$$\int_{c\in\Re^T}\sum_{t=1}^{T}\frac{A_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c}+\int_{c\in\Re^T}\sum_{t=1}^{T}\frac{B_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c}$$
$$=\int_{c\in\Re^T}\sum_{t=1}^{T}\frac{c_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c}-\frac{r_0}{1+r_0}\int_{c\in\Re^T}\sum_{t=1}^{T}\frac{\tilde{V}_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c}. \tag{42}$$

Now, we note that

$$\begin{aligned}\int_{c\in\Re^T}\sum_{t=1}^{T}\frac{A_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c} &\approx \mathbf{E}[W_0(\boldsymbol{A})],\\ \int_{c\in\Re^T}\sum_{t=1}^{T}\frac{B_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c} &\approx \mathbf{E}[W_0(\boldsymbol{B})],\\ \int_{c\in\Re^T}\sum_{t=1}^{T}\frac{c_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c} &= x_0,\\ \int_{c\in\Re^T}\sum_{t=1}^{T}\frac{\tilde{V}_t}{(1+r_0)^t}p(\boldsymbol{c})d\boldsymbol{c} &\approx \mathbf{E}[W_0(\tilde{\boldsymbol{V}})].\end{aligned}$$

Hence we obtain from (42)

$$\mathbf{E}[W_0(\boldsymbol{A})]+\mathbf{E}[W_0(\boldsymbol{B})]\approx x_0-\frac{r_0}{1+r_0}\mathbf{E}[W_0(\tilde{\boldsymbol{V}})].$$

Consequently, by substituting $x_0 = 1000$ and $r_0 = 0.05$, we obtain

$$\mathbf{E}[W_0(\boldsymbol{A})]+\mathbf{E}[W_0(\boldsymbol{B})]\approx 1000-0.0476\times\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]. \tag{43}$$

This indicates that the sum of the present value of the PAC bond and the companion bond, which corresponds to the original principal balance, decreases proportionally to the amount of cash reserve $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$. This results from the fact that the reserved cash yields no profit.

6.2.2 Model 2

Next, we report experimental results for the modified model (10) in which the upper bound of reserve v_t is defined as a function of the amount of prepayment. We call this model "Model 2" hereinafter. Sample paths and γ_t, ρ_t used in Model 2 are the same as those of Model 1.

Table 5 shows $\mathbf{E}[W_0(\boldsymbol{A})]$, $\mathbf{E}[W_0(\boldsymbol{B})]$, $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$ and $\mathbf{E}[W_0(\boldsymbol{A})]+\mathbf{E}[W_0(\boldsymbol{B})]$ at the optimal solution when $\mathrm{U}_L = 0.1$ (fixed) and ρ_0 is changed from 0 to 1. Figure 6 illustrates the results in Table 5.

Figure 6 shows that the amount of the PAC bond increases and that of the companion bond decreases by allowing more cash reserve. Moreover, $\mathbf{E}[W_0(\boldsymbol{A})]+\mathbf{E}[W_0(\boldsymbol{B})]$ decreases as cash reserve increases. This can be explained from the fact that equation (43) also holds in Model 2, and the coefficients on the right-hand of (43) are common with those of Model 1.

Figure 7 illustrates the results for Model 1 and Model 2 simultaneously. Figure 7 shows that $\mathbf{E}[W_0(\boldsymbol{A})]$ of Model 2 is larger than that of Model 1. This implies that it is possible to issue larger amount of the PAC bond in Model 2 than that of Model 1, indicating that Model 2 provides better performance than Model 1.

Table 5: Results for Model 2 ($U_L = 0.1$)

ρ_0	$\mathbf{E}[W_0(\boldsymbol{A})]$	$\mathbf{E}[W_0(\boldsymbol{B})]$	$\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$	$\mathbf{E}[W_0(\boldsymbol{A})] + \mathbf{E}[W_0(\boldsymbol{B})]$
1.00	634.071862	365.557142	7.790922	999.629004
0.90	634.491017	365.117217	8.227078	999.608234
0.80	638.837951	360.518112	13.522688	999.356062
0.70	642.498872	356.624970	18.399314	999.123842
0.60	656.457501	341.641524	39.920476	998.099025
0.50	668.560686	328.514590	61.419213	997.075276
0.40	677.081134	319.050343	81.238981	996.131477
0.30	707.557033	284.305264	170.891758	991.862297
0.20	776.951769	200.892829	465.263445	977.844598
0.10	865.809122	82.112535	1093.645202	947.921657
0.00	889.568546	27.547355	1740.566072	917.115901

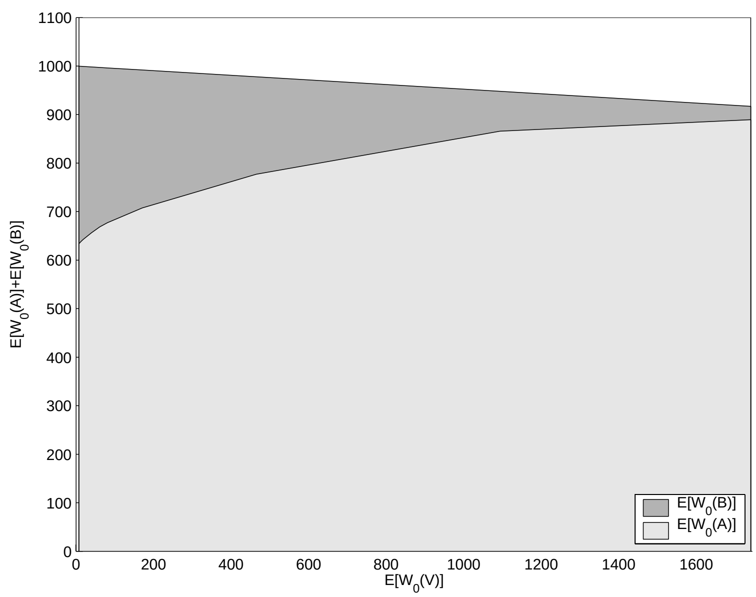

Figure 6: Behavior of $\mathbf{E}[W_0(\tilde{\boldsymbol{V}})]$, $\mathbf{E}[W_0(\boldsymbol{A})]$ and $\mathbf{E}[W_0(\boldsymbol{B})]$ (Model 2)

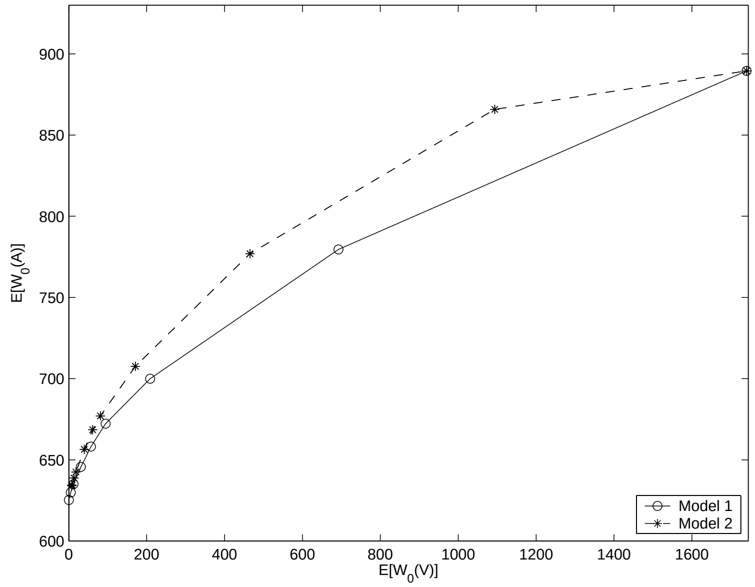

Figure 7: Comparison of Model 1 and Model 2

7 Conclusion

In this paper, we have proposed a new method of designing CMO with PAC-Companion structure for MBS, assuming that the repaid cash can be reserved to the next period in order to issue more PAC bond. We have approximated the probability density function of repayment cash flow by using sample paths generated by an interest rate fluctuation model and a mortgage prepayment model. We have shown that our model can be reformulated as an equivalent linear programming problem. Through some numerical experiments, we have confirmed that the repayment cash flow can be divided into a stable PAC portion and an unstable companion portion under the constraint that the loss of the PAC bond is less than a certain upper bound. We have also confirmed that the amount of the PAC bond can be increased by using cash reserve. Furthermore, we have proposed the modified model and observed that the modified model yields a higher performance than the basic one.

We have considered only the prepayment risk in our model. However, it is possible to take the default risk into consideration by generating sample path with default of loan holders incorporated [7]. It may yield higher performance by determining the amount of cash reserve at each time after issuance. In our model, the quality of the companion bond (such as variance) has not been considered. It may be better to consider the quality of the companion bond as well as the PAC bond and cash reserve, since the selling price of the companion bond depends on its quality. However, the problem then becomes very difficult to solve the problem if we consider the quality of the companion bond; for example, the problem is formulated as a large-scale nonlinear complementarity problem when we consider the variance of the companion bond. Further research is required to suggest a model in which the quality of the companion bond is considered as well as the PAC bond and cash reserve.

Acknowledgement

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A Appendix

$\underline{\mathrm{U}_L = 0.1,\ \rho_0 = 0.1.}$

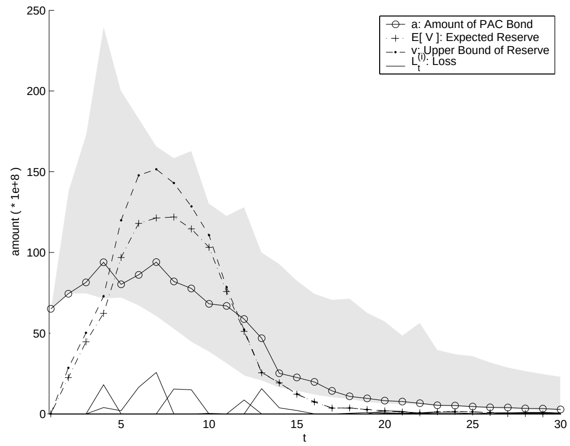

Figure 8: PAC bond a_t, upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$

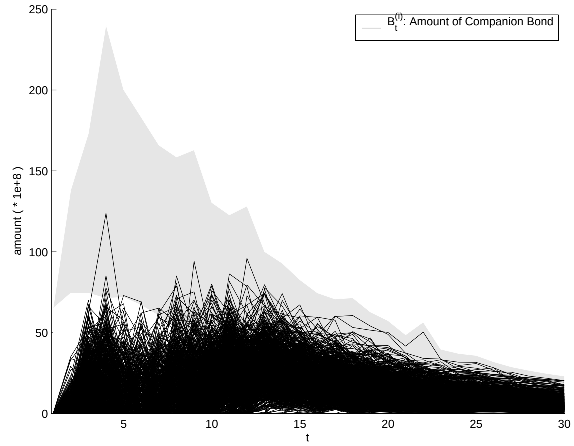

Figure 9: Companion bond $B_t^{(i)}$

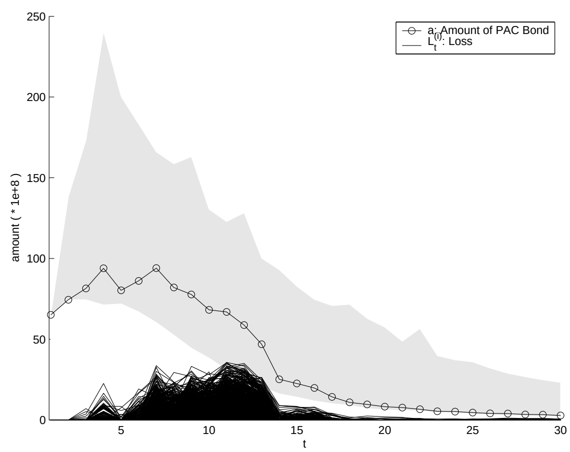

Figure 10: Loss $L_t^{(i)}$ without reserve

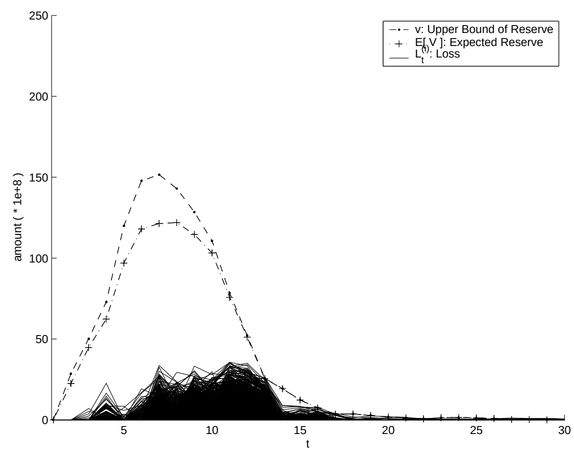

Figure 11: Upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$ without reserve

Table 6: Probability of having $L_t^{(i)} > 0$ in Figure 8

t	1	2	3	4	5	6	7	8	9	10
Probability (%)	0.0	0.0	0.1	0.2	0.2	0.1	0.2	0.2	0.1	0.1
	11	12	13	14	15	16	17	18	19	20
	0.3	0.1	0.2	0.2	0.1	0.1	0.0	0.1	0.2	0.3
	21	22	23	24	25	26	27	28	29	30
	0.2	0.2	0.1	0.2	0.2	0.1	0.4	0.5	0.4	0.5

$\mathrm{U}_L = 0.1,\ \rho_0 = 0.2.$

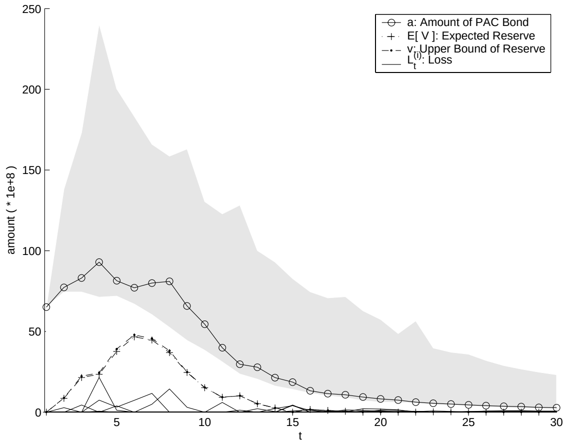

Figure 12: PAC bond a_t, upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$

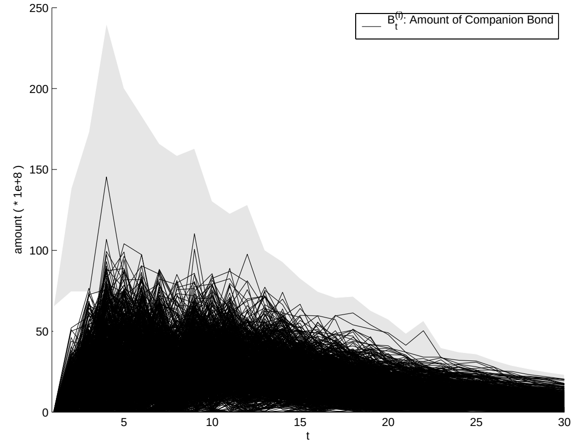

Figure 13: Companion bond $B_t^{(i)}$

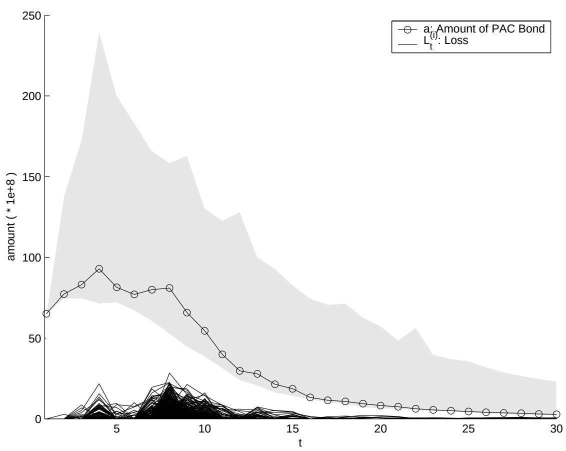

Figure 14: Loss $L_t^{(i)}$ without reserve

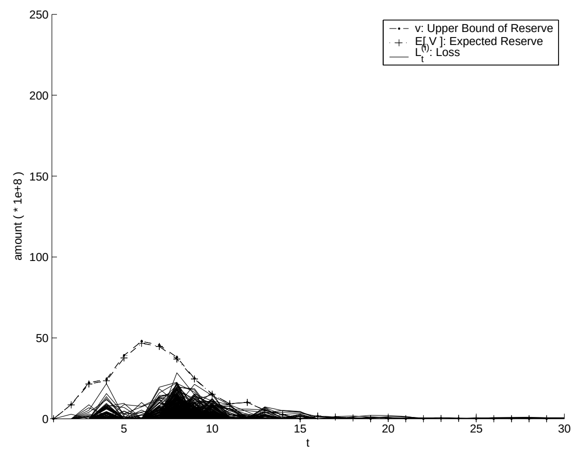

Figure 15: Upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$ without reserve

Table 7: Probability of having $L_t^{(i)} > 0$ in Figure 12

t	1	2	3	4	5	6	7	8	9	10
Probability (%)	0.0	0.2	0.1	0.3	0.3	0.1	0.2	0.2	0.2	0.2
	11	12	13	14	15	16	17	18	19	20
	0.2	0.1	0.3	0.2	0.7	0.3	0.4	0.5	0.5	0.6
	21	22	23	24	25	26	27	28	29	30
	0.6	0.5	0.3	0.4	0.6	0.5	0.5	0.5	0.6	0.5

$\underline{\mathrm{U}_L = 0.01,\ \rho_0 = 0.1.}$

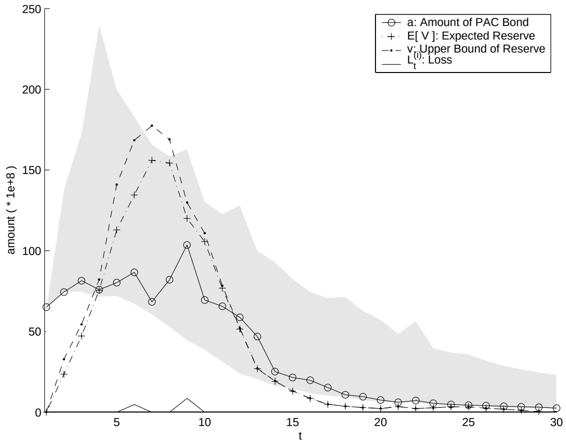

Figure 16: PAC bond a_t, upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$

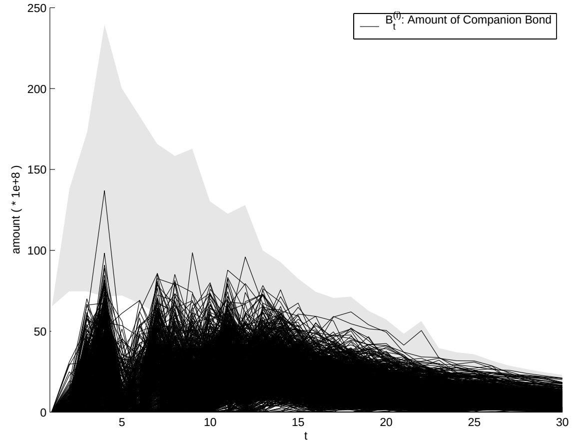

Figure 17: Companion bond $B_t^{(i)}$

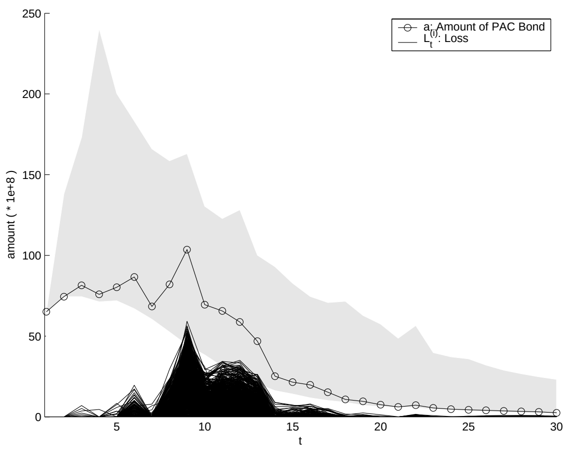

Figure 18: Loss $L_t^{(i)}$ without reserve

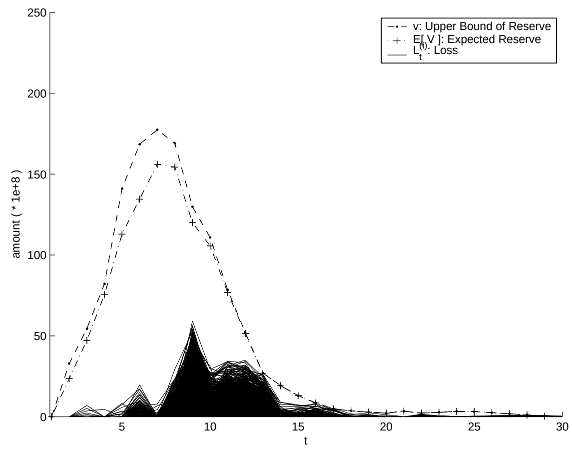

Figure 19: Upper bound v_t, expected reserve $\mathbf{E}[V_t]$, and loss $L_t^{(i)}$ without reserve

Table 8: Probability of having $L_t^{(i)} > 0$ in Figure 16

t	1	2	3	4	5	6	7	8	9	10
Probability (%)	0.0	0.0	0.1	0.1	0.1	0.1	0.0	0.0	0.1	0.0

t	11	12	13	14	15	16	17	18	19	20
Probability (%)	0.1	0.0	0.1	0.1	0.0	0.0	0.0	0.0	0.1	0.1

t	21	22	23	24	25	26	27	28	29	30
Probability (%)	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2