A Robust User Equilibrium in the Traffic Assignment Problem under Uncertainty

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Abstruct

Most traffic equilibrium models assume that the situation which may happen is unique and deterministic. However, the situation which actually happens is not deterministic but stochastic. Therefore it is natural to suppose that each network user chooses his or her route by taking into consideration the probability of each situation. We call an equilibrium that results from such users' route choice behavior a robust user equilibrium. In this paper, we propose three deterministic formulations of the robust user equilibrium model; the expected value (EV) method, the expected residual minimization (ERM) method, and the mathematical program with equilibrium constraints (MPEC) approach. We have conducted some numerical experiments for the three methods by using a simple network that has an uncertainty in the link cost functions and/or the traffic demands. We have confirmed that the solutions obtained by the ERM method and the MPEC approach are more risk averse than the solutions obtained by the EV method. This shows that the ERM method and the MPEC approach are useful in modeling the actual route choice behavior of the network users.

Contents

1 Introduction

The traffic assignment problem is to forecast the amount of flow in each link on a network when OD demands and the distribution principle are given for a realistic or virtual traffic network. It is very important to forecast traffic flows on the network, in building a new road or evaluating a traffic network. A lot of approaches to the traffic assignment problem have been studied until now. In those approaches, the equilibrium analysis is intensively studied by many researchers and its practical use is expected. Though a lot of researches that derive from the equilibrium analysis exist, Wardrop (1952) was the first who clarified the traffic assignment problem as an equilibrium problem. He advocated the following principle [18]:

> *"If the links of the network have unlimited capacities, the travel times on all the routes actually used between an OD pair are equal and less than those which would be experienced by a single user on any unused route."*

This principle is realized by natural route choice behavior of network users that they want to minimize their travel time selfishly. It is restated that *"no user can improve his or her travel time by unilaterally changing his or her routes."* It corresponds with Nash Equilibrium in the non–cooperative N–person game [2][15]. An equilibrium that results from Wardrop's princple is called a user equilibrium (UE).

In the equilibrium analysis, a lot of progressive researches which derive from the user equilibrium, like the stochastic user equilibrium and the dynamic user equilibrium and so on, have advanced so far [1][11][12][16]. These equilibrium models are categorized according to whether OD demands are fixed or variable. The dynamic user equilibrium models are also categorized according to whether departure times of users are fixed or variable [16].

Conventionally it is assumed that the situation which may happen is unique and deterministic, where the situation means an external factor unrelated to users' behavior. In other words, in every equilibrium model, the link cost function for each link, which represents a required time to pass through that link, is always assumed to be deterministic. However, the situation that actually happens is not deterministic but uncertain and the link cost function may change with situation. An example of situation is weather. The link cost function of the day when the visibility is good at fine weather and the road surface condition is good will probably differ from that of the day when the visibility is poor at bad weather and the road surface condition is bad. The OD demand may also change with situation.

In this paper, we propose a new equilibrium model where the situation is not deterministic but probabilistic. We take each situation into consideration according to the probability which may happen, and consider the equilibrium as a result of it. We show that this new model may yield an equilibrium which takes two or more situations into consideration simultaneously. This model will be useful when the actual situation is uncertain and the link cost function and/or the OD demand may change with situation. We call an equilibrium in the new model a "*robust user equilibrium*", since this model considers two or more possible situations, unlike conventional equilibrium models that only consider a deterministic situation.

The remainder of the paper is organized as follows: In Section 2, we define the mixed complementarity problem, an NCP function and a merit function that are required as preliminary knowledge to understand and deal with an equilibrium model. In Section 3, we introduce the user equilibrium model and briefly explain an extension to the model under uncertainty. In Section 4, we propose three formulations of robust user equilibria. In Section 5, we present numerical results for the three methods proposed in Section 4. In Section 6, we make some remarks to conclude the paper.

2 Preliminaries

The mixed complementarity problem (MCP) is represented as follows:

$$\begin{aligned} \text{Find} \quad & (x,u) \in \Re^{n+m} \\ \text{s.t.} \quad & g(x,u) = 0, \\ & x \geq 0,\ h(x,u) \geq 0,\ x^\top h(x,u) = 0, \end{aligned} \tag{1}$$

where $g : \Re^{n+m} \to \Re^m$ and $h : \Re^{n+m} \to \Re^n$ are given mappings. This is the problem of finding a solution which satisfies both the equality condition ($g(x,u) = 0$) and the complementarity conditions ($0 \leq x \perp h(x,u) \geq 0$) at the same time. In general, an equilibrium problem can be formulated as an MCP (1).

Next, we describe a standard method to deal with MCP (1). First, we introduce an NCP function. A function $\phi : \Re^2 \to \Re$ is called an NCP function if it has the property

$$\phi(a,b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0.$$

Two popular NCP functions are the "min" function and the Fischer–Burmeister (FB) function [8].

$$\begin{aligned} \text{[min function]} \quad & \phi(a,b) = \min(a,b), \\ \text{[FB function]} \quad & \phi(a,b) = \sqrt{a+b} - a - b. \end{aligned}$$

Using an NCP function, MCP (1) can be reformulated as a system of nonlinear equations as follows:

$$\Phi(x,u) = \begin{pmatrix} \phi(x_1, h_1(x,u)) \\ \vdots \\ \phi(x_n, h_n(x,u)) \\ g(x,u) \end{pmatrix} = 0. \tag{2}$$

Moreover, the nonlinear equations (2) can be reformulated as an unconstrained minimization problem as follows:

$$\text{minimize} \quad \Psi(x,u), \tag{3}$$

where

$$\Psi(x,u) = \|\Phi(x,u)\|^2.$$

Here, the value of $\Psi(x,u)$ is always larger than or equal to 0. In particular, $\Psi(x,u) = 0$ if and only if (x,u) is a solution of MCP (1).

The real–valued function which takes the value 0 at a solution of an equilibrium problem and takes a positive value otherwise is called a merit function of the equilibrium problem. So $\Psi(x,u)$ is a merit function of the equilibrium problem (1).

In general, it is not guaranteed that a solution of a minimization problem found by a gradient method like a quasi Newton method is a global minimum solution. It may be only a stationary point like a local minimum solution or a saddle point. However, if we use a merit function as an objective function of a minimization problem, we can confirm whether the obtained solution is a global minimum solution or not, that is, an equilibrium solution or not, by checking whether the value of the objective function is 0 or not.

3 Extension from deterministic model to stochastic model

In this section, we introduce the formulation of the user equilibrium (UE) model first. And then, we briefly explain an extension to the stochasticn model.

3.1 User equilibrium model

Here, we show the formulation of the UE model which is the basis of the robust user equilibrium model that we propose in this paper.

3.1.1 Preliminaries

We assume that the traffic network is composed of a number of nodes and a number of links connecting nodes, that is, we consider the connected graph $G = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ denotes the set of nodes and $\mathcal{L}$ denotes the set of links. Each network user has each origin node and destination node. We call a pair of origin node and destination node an OD pair. Let $\mathcal{S}$ denote the set of OD pairs and m denote the number of OD pairs. Let K_s be the set of all available routes connecting the OD pair s. Let n denote the number of all routes (not including cyclic ones) in the traffic network.

Let x_i denote the flow of route i and $f_i(x)$ denote the travel time which is experienced by a user using route i to go from his origin to his destination. $x \in \Re^n$ is the vector such that $x = (x_1, \cdots, x_n)^\top$ and $f(x) \in \Re^n$ is the vector such that $f(x) = (f_1(x), \cdots, f_n(x))^\top$, where $\top$ means transpose of a vector. We summarize the notations in Table 1.

3.1.2 Formulation of user equilibrium model

Let $i \in K_s$. Then Wardrop's principle is represented as follows:

$$x_i > 0 \quad \Longrightarrow \quad f_i(x) \le f_j(x) \quad \forall j \in K_s.$$

This means if $x_i > 0$, route i must be the minimum cost route of all available routes connecting the OD pair s. It also means flow must not exist ($x_i = 0$) if route i is not such a minimum cost route. The above condition can be rewritten as a complementarity condition as follows:

$$x_i \ge 0, \; f_i(x) - f^s_{\min}(x) \ge 0, \; x_i(f_i(x) - f^s_{\min}(x)) = 0, \qquad \forall i \in K_s, \; \forall s \in \mathcal{S},$$

$G = (\mathcal{N}, \mathcal{L})$	$\cdots$	traffic network
$\mathcal{N}$	$\cdots$	set of nodes
$\mathcal{L}$	$\cdots$	set of links
$\mathcal{S}$	$\cdots$	set of OD pairs
K_s	$\cdots$	set of available routes connecting the OD pair $s \in \mathcal{S}$
n	$\cdots$	the number of all routes in the network
m	$\cdots$	the number of all OD pairs in the network
x_i	$\cdots$	flow on route i
$f_i(x)$	$\cdots$	route travel cost experienced by a user of route i
$x \in \Re^n$	$\cdots$	vector of route flows ($x = (x_1, \cdots, x_n)^\top$)
$f(x) \in \Re^n$	$\cdots$	vector of route travel costs ($f(x) = (f_1(x), \cdots, f_n(x))^\top$)
d_s	$\cdots$	OD demand associated with $s \in \mathcal{S}$
d	$\cdots$	vector of OD demands ($d = (d_1, \cdots, d_m)^\top$)

Table 1: Notations

where

$$f^s_{\min}(x) = \min_{i \in K_s} f_i(x), \qquad \forall s \in \mathcal{S}.$$

Moreover, we define $f_{\min}(x) \in \Re^m$ and $D \in \Re^{m \times n}$ as follows:

$$f_{\min}(x) = (f^1_{\min}(x), \cdots, f^m_{\min}(x))^\top \in \Re^m$$
$$D(s,j) = \begin{cases} 1 & \text{if } x_j \in K_s \\ 0 & \text{othewise.} \end{cases}$$

Then, Wardrop's principle can be represented as a complementarity condition in the vector form as follows:

$$x \geq 0,\ f(x) - D^\top f_{\min}(x) \geq 0,\ x^\top (f(x) - D^\top f_{\min}(x)) = 0. \tag{4}$$

Next, we consider the condition which must be satisfied by the OD demands and network flows. Let d_s denote the OD demand associated with $s \in \mathcal{S}$. Then we must have

$$\sum_{i \in K_s} x_i - d_s = 0, \qquad \forall s \in \mathcal{S},$$

which is called the flow conservation equation. Let $d = (d_1, \cdots, d_m)^\top$. Then the above equation can be restated as follows:

$$Dx - d = 0. \tag{5}$$

A flow x satisfies both Wardrop's principle (4) and the flow conservation equation (5), if and only if x is a user equilibrium flow pattern. So the user equilibrium problem is formulated as an MCP as follows:

$$\begin{array}{ll} \text{Find} & x \in \Re^n \\ \text{s.t.} & x \geq 0,\ f(x) - D^\top f_{\min}(x) \geq 0, \\ & x^\top (f(x) - D^\top f_{\min}(x)) = 0, \\ & Dx - d = 0. \end{array} \tag{6}$$

When we solve this problem actually, we can replace $f_{\min}(x)$ by a variable vector u. That is, we can consider the following problem instead of (6):

$$\begin{array}{ll} \text{Find} & (x, u) \in \Re^{n+m} \\ \text{s.t.} & x \geq 0,\ f(x) - D^\top u \geq 0, \\ & x^\top (f(x) - D^\top u) = 0, \\ & Dx - d = 0. \end{array} \tag{7}$$

If $d > 0$, an equilibrium solution (x^*, u^*) of (7) satisfies

$$u^* = f_{\min}(x^*),$$

so (6) and (7) are equivalent [14].

3.2 Extension to stochastic model

The user equilibrium model assumes that the link cost functions and OD demands are given deterministically. However, the actual situation may happen probabilistically rather than deterministically in many cases, for instance, depending on weather. The link cost function when the visibility is good at fine weather and the road surface condition is good will probably differ from that at bad weather. Moreover, there may be some users who cancel going out under the situation of bad weather.

We propose a robust user equilibrium model where the situation is probabilistic and the link cost functions and/or the OD demands change with situation. Under such a setting, users will consider not only a certain event but also two or more events according to the occurrence probability of them. In this section, we briefly explain the meaning of the robust user equilibrium. We introduce the formulation of the robust user equilibrium model in the next section.

For instance, we consider the case where there are two events, A, B, that can happen. Some existing equilibrium models only consider the case where either A or B happens, that is, they only consider the situation where either A happens with probability 1 or B happens with probability 1. In other words, they consider deterministic link cost functions or OD demands in the situation where either A or B happens. Actually, however, it is not deterministic but probabilistic which event happens. Next, we consider the situation where the occurrence probability of A is 0.7 and that of B is 0.3. Although users may expect to some extent that A will happen, it cannot be trusted completely. In such a case, it is natural that they also consider the situation where B happens even a little in selecting their routes, although they consider the situation where A happens much more. This is considered to be risk–averse route choice behavior as compared with the case where only one of the events is taken into consideration. In this sense, the equilibrium produced by such users' risk–averse behavior has the property of robustness.

In the robust user equilibrium model, the route travel cost $f(x)$ in the UE model is replaced with $f(x, \omega)$ depending on the event $\omega \in \Omega$, and the OD demand d is replaced with $d(\omega)$. The problem (7) with $f(x)$ and d being replaced by $f(x, \omega)$ and $d(\omega)$, respectively, is not deterministic, so that we must use a certain deterministic formulation of the robust user equilibrium. In the following section, we introduce three kinds of formulations for the robust user equilibrium model.

4 Formulation of the robust user equilibrium model

Let Ω denote the sample space and the occurrence probability p is assigned to each $\omega \in \Omega$. When the route cost function is represented as $f(x,\omega)$, we may consider three kinds of formulations for the robust user equilibrium model; expected value (EV) method [9], expected residual minimization (ERM) method, and the formulation as a mathematical program with equilibrium constraints (MPEC). For EV method and ERM method, we also consider the case where the OD demand d is also replaced with $d(\omega)$.

4.1 Expected value method

Expected value method considers the following deterministic problem:

$$\begin{array}{ll} \text{Find} & (x,u) \in \Re^{n+m} \\ \text{s.t.} & x \geq 0,\ E_\omega[f(x,\omega)] - D^\top u \geq 0, \\ & x^\top (E_\omega[f(x,\omega)] - D^\top u) = 0, \\ & Dx - E_\omega[d(\omega)] = 0. \end{array} \tag{8}$$

From the user equilibrium model (7),

$$u_i^* = \min_{j \in K_i} E_\omega[f_j(x^*,\omega)]$$

if and only if (x^*, u^*) is the solution of (8). That is, the equilibrium solution (x^*, u^*) of (8) is equivalent to the user equilibrium solution in the case where the route cost function is $E_\omega[f(x,\omega)]$ and the OD demand is $E_\omega[d(\omega)]$.

To use the expected value method, we need to evaluate the expected value functions, $E_\omega[f(x,\omega)]$ and $E_\omega[d(\omega)]$. When the sample space is finite, that is, $\Omega = \{\omega_1, \cdots, \omega_L\}$, we can explicitly calculate them as follows:

$$E_\omega[f(x,\omega)] = \sum_{i=1}^{L} p_i f(x,\omega_i),$$

$$E_\omega[d(\omega)] = \sum_{i=1}^{L} p_i d(\omega_i).$$

However, in the case of continuous distribution, we must choose a sufficiently many samples and use normalized probabilities p_i $(p_1 + \cdots + p_L = 1)$ in order to calculate the expected values approximately. Once $E_\omega[f(x,\omega)]$ and $E_\omega[d(\omega)]$ are calculated, problem (8) can be considered to be a deterministic MCP. There are a number of researches on solution methods, existence and uniqueness of solutions and so on [4] [7] for deterministic MCPs.

Using an NCP function ϕ introduced in § 2, we define the function $\Phi_{\text{ev}} : \Re^{n+m} \to \Re^{n+m}$ as follows:

$$\Phi_{\text{ev}}(x,u) = \begin{pmatrix} \phi(x_1, E_\omega[f_1(x,\omega)] - (D^\top u)_1) \\ \vdots \\ \phi(x_n, E_\omega[f_n(x,\omega)] - (D^\top u)_n) \\ Dx - E_\omega[d(\omega)] \end{pmatrix}. \tag{9}$$

Then, we can derive the following unconstrained minimization problem which is equivalent to the equilibrium problem (8):

$$\text{minimize} \quad \Psi_{\text{ev}}(x,u), \tag{10}$$

where

$$\Psi_{\text{ev}}(x,u) = \|\Phi_{\text{ev}}(x,u)\|^2.$$

4.2 Expected residual minimization method

The ERM method minimizes the expected residual for the equilibrium. The residual for the equilibrium of each event ω is represented by using an NCP function introduced in § 2 as follows:

$$\Phi_{\text{er}}(x,u,\omega) = \begin{pmatrix} \phi(\alpha x_1, f_1(x,\omega) - (D^\top u)_1) \\ \vdots \\ \phi(\alpha x_n, f_n(x,\omega) - (D^\top u)_n) \end{pmatrix}. \tag{11}$$

Note that $\Phi_{\text{er}}(x^*, u^*, \omega_i) = 0$ means (x^*, u^*) is an equilibrium solution for the event $\omega_i \in \Omega$. Thus, $\|\Phi_{\text{er}}(x,u,\omega_i)\|$ can be regarded as the "distance" from the equilibrium of the event ω_i. In (11), α is a positive constant for adjusting the scale of the value of x_i and $f_i(x,\omega) - (D^\top u)_i$, because one shows the value of "flow" and the other shows that of "cost". It is necessary to set up a suitable value depending on the problem and the size of $f(x,\omega)$.

We assume that the sample space is finite, that is, $\Omega = \{\omega_1, \cdots, \omega_L\}$. Then, the ERM problem for the robust user equilibrium is given as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{L} p_i \Psi_{\text{er}}(x,u,\omega_i) \\ \text{s.t.} \quad & Dx - d = 0, \end{aligned} \tag{12}$$

where

$$\Psi_{\text{er}}(x,u,\omega) = \|\Phi_{\text{er}}(x,u,\omega)\|^2.$$

Here, since the equation $Dx - d = 0$ expresses the flow conservation equation which must be satisfied, we treat it as the constraint.

Next, if the OD demand also depends on $\omega \in \Omega$, that is, the OD demand is given as $d(\omega)$, we cannot deal with the flow conservation equation as the constraint. So we consider the following problem:

$$\text{minimize} \quad \sum_{i=1}^{L} p_i \left((1-\beta)\Psi_{\text{er}}(x,u,\omega_i) + \beta\|Dx - d(\omega)\|^2 \right). \tag{13}$$

where $\beta \in (0,1)$ is a weight parameter.

4.3 MPEC approach

An optimization problem involving equilibrium conditions like complementarity conditions or variational inequalities in the constraints is called a *mathematical program with equilibrium constraints* (MPEC). The idea of the formulation here is similar to § 4.2, but the problem to solve appears quite different from (12), because the residual for the equilibrium is defined in a different manner.

We introduce recourse variables $z^i \in \Re^n$ to satisfy complementarity conditions for each ω_i $(i = 1, \cdots, L)$ and we regard $\|z^i\|$ as the residual to the equilibrium for ω_i. We try to find a vector (x, u) that minimizes the expected residual. Thus, we obtain the following MPEC:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{L} p_i \|z^i\|^2 \\ \text{s.t.} \quad & x \geq 0, \; f(x, \omega_i) - D^\top u - z^i \geq 0, \\ & x^\top (f(x, \omega_i) - D^\top u - z^i) = 0, \; i = 1, \cdots, L, \\ & Dx - d = 0. \end{aligned} \tag{14}$$

Lin and Fukushima [13] proposed a solution method for a stochastic MPEC which has complementarity conditions as constraints. In this paper, we solve constrained minimization problems approximating (14), instead of using the method specialized to MPEC. In particular, we consider the following problem where the complementarity comditions are relaxed:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{L} p_i \|z^i\|^2 \\ \text{s.t.} \quad & x \geq 0, \; f(x, \omega_i) - D^\top u - z^i \geq 0, \\ & x^\top (f(x, \omega_i) - D^\top u - z^i) \leq \varepsilon_k, \; i = 1, \cdots, L, \\ & Dx - d = 0. \end{aligned} \tag{15}$$

Here, $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\lim_{k \to \infty} \varepsilon_k = 0$. In numerical experiments, we produce a sequence $\{(x^k, u^k, z^{1,k}, \cdots, z^{L,k})\}$ converging to a solution $(x^*, u^*, z^{1*}, \cdots, z^{L*})$ of (14), by solving the sequence of problems (15).

5 Numerical Experiments

In this section, we show some numerical results for the three methods presented in the previous section.

5.1 Preliminaries

We conducted some numerical experiments by using a simple network with one OD pair (1, 7) as shown in Figure 1. All available routes are as follows:

route 1 : link 1 $\longrightarrow$ link 4 $\longrightarrow$ link 9,

route 2 : link 1 $\longrightarrow$ link 3 $\longrightarrow$ link 6 $\longrightarrow$ link 9,

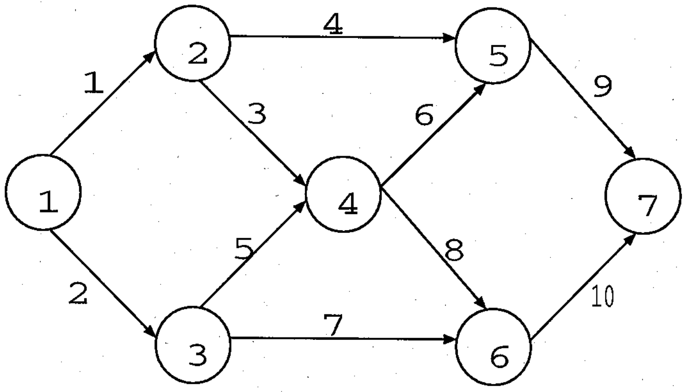

Figure 1: A network example

$$\begin{aligned}
\text{route } 3: &\quad \text{link } 1 \to \text{link } 3 \to \text{link } 8 \to \text{link } 10,\\
\text{route } 4: &\quad \text{link } 2 \to \text{link } 5 \to \text{link } 6 \to \text{link } 9,\\
\text{route } 5: &\quad \text{link } 2 \to \text{link } 5 \to \text{link } 8 \to \text{link } 10,\\
\text{route } 6: &\quad \text{link } 2 \to \text{link } 7 \to \text{link } 10.
\end{aligned}$$

Let x_i denote the route–flow on route i and y_j denote the link–flow on link j. We can express the relationship between the route–flow and the link–flow as follows:

$$y = Mx,$$

where $M \in \Re^{10\times 6}$ is the path–link incidence matrix such that

$$M_{ij} = \begin{cases} 1 & \text{(if route } j \text{ contains link } i) \\ 0 & \text{(otherwise)}. \end{cases}$$

Let $c_i(y)$ denote the link cost of link i and $c(y)$ denote the vector $(c_1(y), \cdots, c_{10}(y))^\top$. Then we can express the route cost $f(x) \in \Re^6$ as follows:

$$f(x) = M^\top c(y).$$

Next, we introduce two types of link cost functions with which we conduct some numerical experiments. One is a linear function and the other is a nonlinear function.

5.1.1 Linear link cost function

We use the following linear link cost function:

$$c(y) = Ay + b, \quad A \in \Re^{10\times 10},\ b \in \Re^{10}, \tag{16}$$

where A_{ij} represents the magnitude of the effect of flows on link j to the link cost of link i, and b_i represents the free travel cost of link i.

5.1.2 Nonlinear link cost function

We used the BPR function [17] as a nonlinear link cost functon. The BPR function is expressed as follows:

$$c_i(y) = t_i \left\{ 1 + \mu \left(\frac{y_i}{V_i} \right)^\gamma \right\}, \tag{17}$$

where, t_i is the free travel cost of link i and V_i is the traffic capacity of link i. The link cost becomes $1+\mu$ times the free travel cost when the link flow y_i reachs V_i. The parameter γ represents the magnitude of the effect of the congestion in each link to the corresponding link cost. The effect of γ is dominant in the region $y_i \geq V_i$. Figure 2 shows the shape

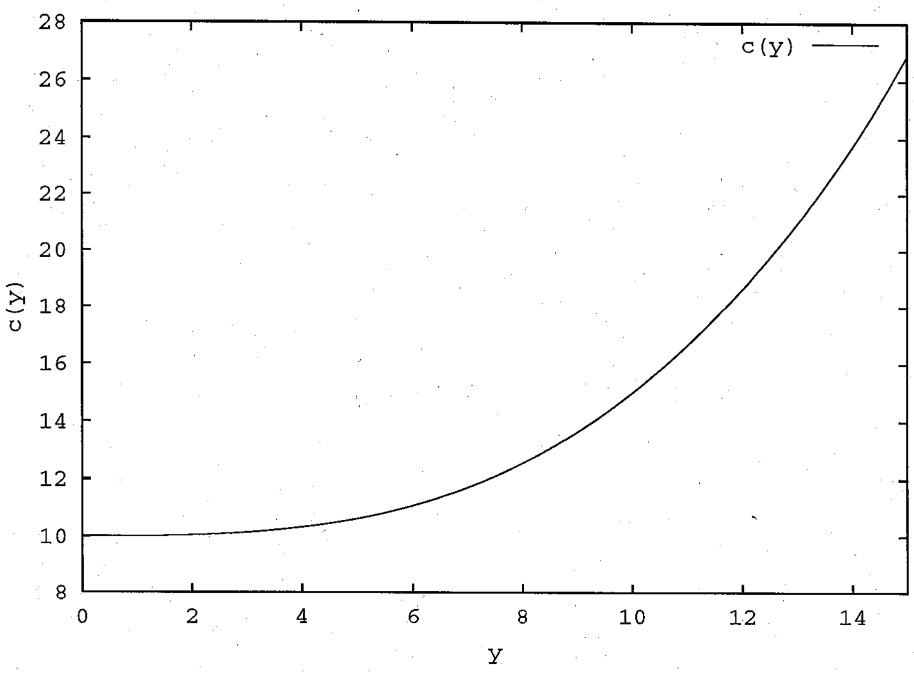

Figure 2: BPR function : $c(y) = 10\{1 + 0.5(y/10)^3\}$

of the BPR function with $\alpha = 0.5$, $\beta = 3$, $t = 10$, $V = 10$. Here the link cost does not diverge to the infinity when the link flow reachs the traffic capacity, because the BPR function is a modified version of the Davidson function [6], which diverges to the infinity as $y_i \to V_i$, so that the link cost should be defined in the region $y_i \geq V_i$.

5.2 Experiment 1: Uncertainty in the link cost

First, we state some materials common to the experiment with a linear cost function and that with a nonlinear cost function.

We consider the case where ω is uniformly distributed in the interval $[1/2-\delta, 1/2+\delta]$. Then, the expectation of ω is $1/2$ and the variance of ω is $2\delta^3/3$.

We choose L samples $\omega_i = 1/2 - \delta + (i-1) \cdot \Delta$ $(i = 1, \cdots, L)$ from the interval $[1/2-\delta, 1/2+\delta]$ to approximate the continuous distribution, where $\Delta = 2\delta/(L-1)$.

We set $p_i = 1/L$, the probability of event ω_i. Note that $\sum_{i=1}^{L} p_i = 1$. We set the demand of OD pair (1, 7) at 200. We consider the following six cases: $\delta = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$.

5.2.1 The case of a linear link cost

We consider the case where the coefficient A, b in (16) are given as follows:

$$A(\omega) = \begin{bmatrix} 2.5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & 1.5 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 1.5 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 1.5 & 0 & 3.5 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 3.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2+10\omega & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.5 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.5 \end{bmatrix},$$

$$b(\omega) = [50,\ 30,\ 40,\ 40,\ 40,\ 50,\ 20+80\omega,\ 60,\ 40,\ 70]^\top.$$

Here, only the link cost of link 7 depends on the random variable ω.

In the expected value (EV) method, we set $L = 21$.

In the expected residual minimization (ERM) method, we set $L = 21$ and scaling parameter $\alpha = 100$.

In the MPEC approach, we set $L = 21$.

Results

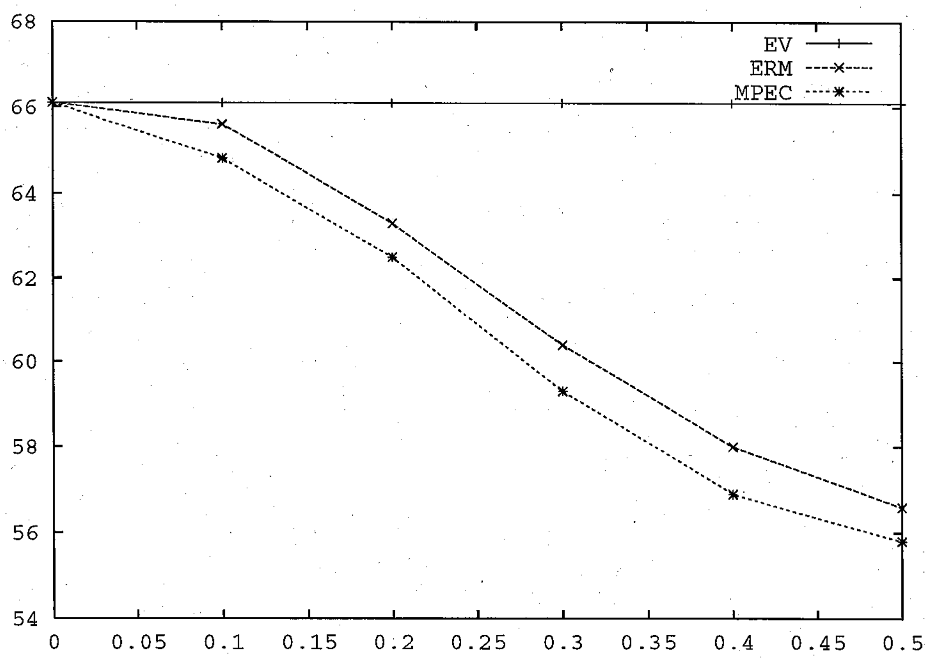

Figure 3: Flow on link 7 : horizontal axis: δ, vertical axis: y_7

Figure 3 shows the amount of flow on link 7 obtained by the EV method, the ERM method, and the MPEC approach.

In the EV method, for all δ, the same user equilibrium solution as in the deterministic case where $P(\omega = 0.5) = 1$ was obtained, because the expectation of the route cost does not change with the value of δ. Since the link cost function is linear with respect to ω, the expectation of the cost does not change with the value of δ, so that the solution is not influenced by the magnitude of the variance of ω.

On the other hand, in the ERM method and the MPEC approach, the larger the value of δ, that is, the larger the variance of ω, the less the flow on link 7 which depends on ω.

5.2.2 The case of a nonlinear link cost

We consider the case where the link cost function is represented as the BPR function (17). We set the parameters as follows:

$$t = (t_1, \cdots, t_{10}) = (30, 20, 30, 60 + 10\omega, 40, 40, 60, 30, 30, 30)$$
$$V = (V_1, \cdots, V_{10}) = (40, 40, 40, 20 + 40\omega, 40, 40, 40, 40, 40, 40)$$
$$\alpha = 0.5,\ \beta = 3.$$

Here, only the cost of link 4 depends on the random variable ω.

In the expected value (EV) method, we set $L = 21$.

In the expected residual minimization (ERM) method, we set $L = 21$ and scaling parameter $\alpha = 100$.

In the MPEC approach, we set $L = 21$.

Results

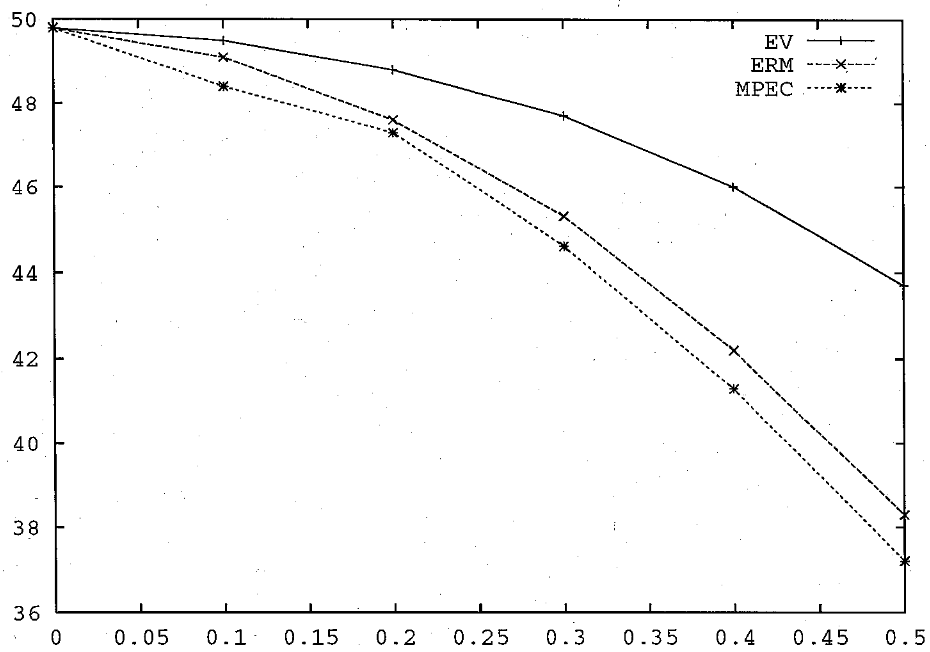

Figure 4: Flow on link 4 : horizontal axis: δ, vertical axis: y_4

Figure 4 shows the amount of flow on link 4 obtained by the EV method, the ERM method, and the MPEC approach.

Since the link cost function is nonlinear in ω, the expectation of the link cost changes with the value of δ. Thus, the solutions obtained by the EV method change with the value of δ. The larger the variance of ω, the larger the expectation of the cost for link 4. Thus, we can observe that the amount of flow on link 4, whose cost depends on ω, is gradually decreasing as δ increases.

In the two methods other than the EV method, we can also observe that the amount of flow on link 4 is gradually decreasing as δ increases, that is, the variance of ω becomes large.

Although in three methods we can observe the same result that the amount of flow on link 4 decreases gradually as δ increases, the solutions obtained by the ERM method and the MPEC approach are more risk averse than solutions obtained by the EV method. The difference between the solutions obtained by the EV method and the other methods becomes noticeable as the variance of ω increases.

5.3 Experiment 2: Uncertainty in the link cost and the demand

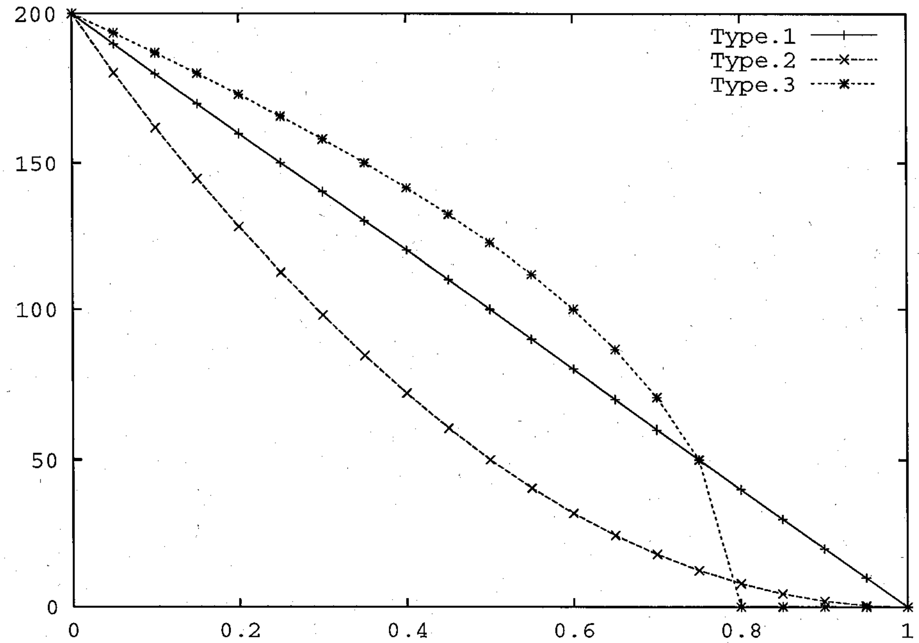

Figure 5: Type of the demand: horizontal axis: ω, vertical axis: $d(\omega)$

We consider the case where not only the link cost but also the demand has uncertainty, that is, depends on ω. We conduct some experiments for the EV method and the ERM method, using the network with the BPR link cost function as given in § 5.2.2 and the following three types of demand (Figure 5).

$$\text{Type 1.} \quad d(\omega) = -200\omega + 200,$$

$$\text{Type 2.}\quad d(\omega) = 200(\omega - 1)^2,$$

$$\text{Type 3.}\quad d(\omega) = \begin{cases} \sqrt{200^2(0.8-\omega)/0.8} & (0 \le \omega \le 0.8) \\ 0 & (0.8 < \omega) \end{cases}.$$

Type 1

We consider the case where the demand is decreasing linearly with respect to ω.

In the EV method, the value of $E[d(\omega)]$ always equals 100 irrespective of the value of δ. Thus, the same solution was obtained for all δ. This is because the expectation of the demand does not change with the value of δ when the demand is a linear function of ω, as we stated in the previous subsection.

On the other hand, in the ERM method, the demand obtained from the equilibrium solution for δ changes with the value of β, which is a weight parameter for the flow conservation condition in (13) (see Figure 6). The obtained demand is different by method

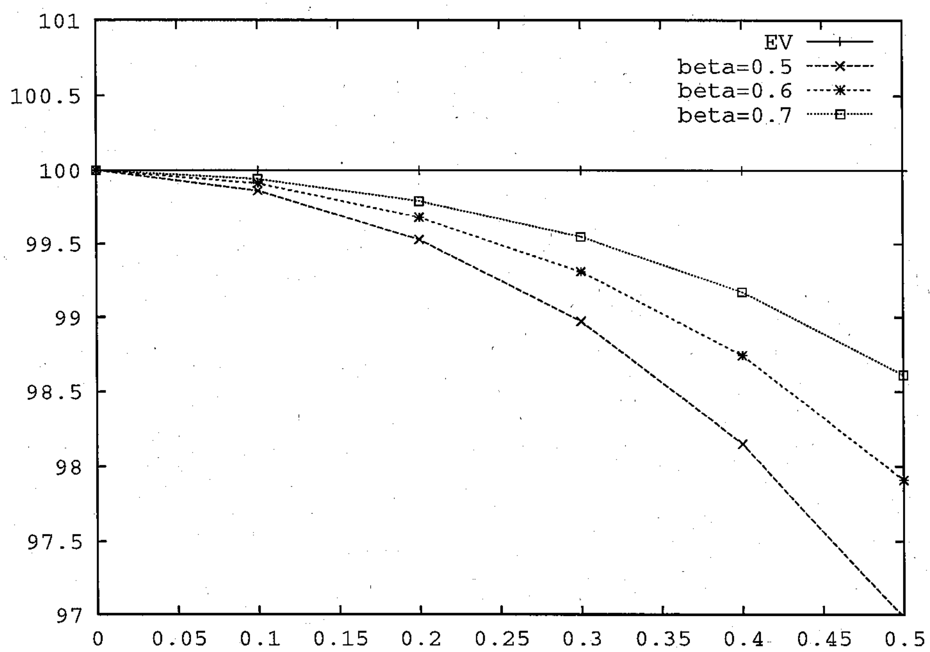

Figure 6: Type 1. Demand versus δ: horizontal axis: δ, vertical axis: demand

and the value of δ, and it decreases as δ increases and β decreases. To compare solutions obtained by the EV method and those obtained by the ERM method with different values of parameter β, we calculate the utilization rate of link 4, which is given by y_4/d. The results for the EV method and the ERM method ($\beta = 0.5,\ 0.6,\ 0.7$) are shown in Figure 7. The utilization rate of link 4 in the ERM method is less than that in the EV method. This indicates that the ERM method produces more risk averse solutions than the EV method.

Type 2 and Type 3

We consider the case where the demand is decreasing nonlinearly with respect to ω.

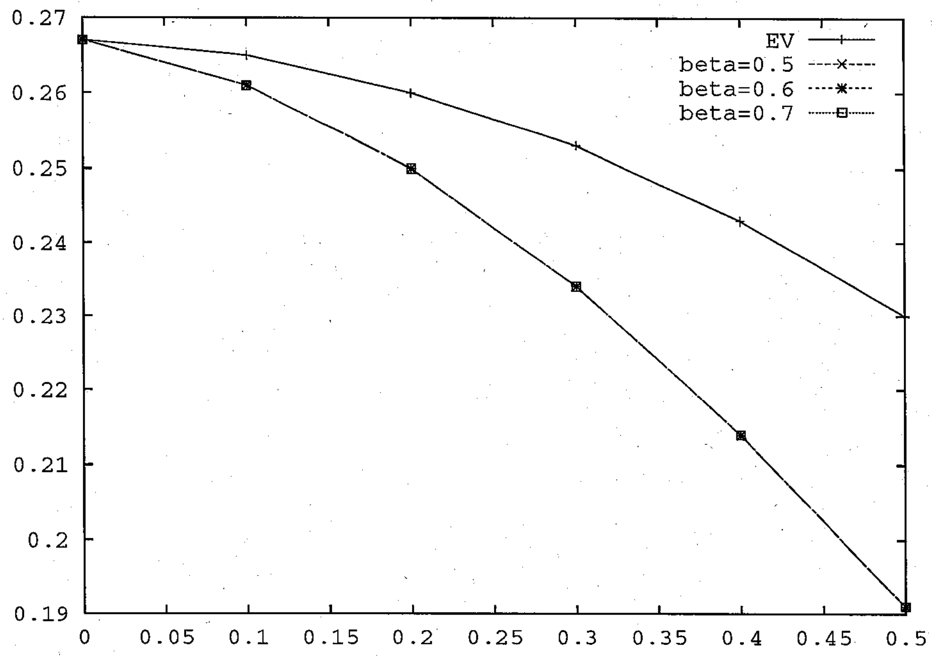

Figure 7: Type 1. Utilization rate of link 4: horizontal axis: δ, vertical axis: y_4/d

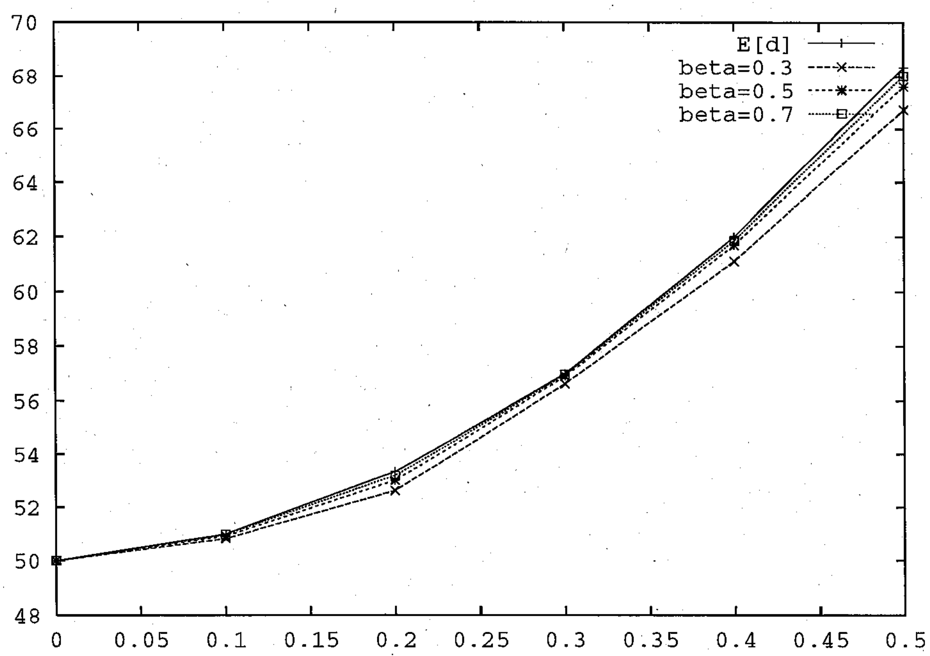

Figure 8: Type 2. Demand versus δ: horizontal axis: δ, vertical axis: demand

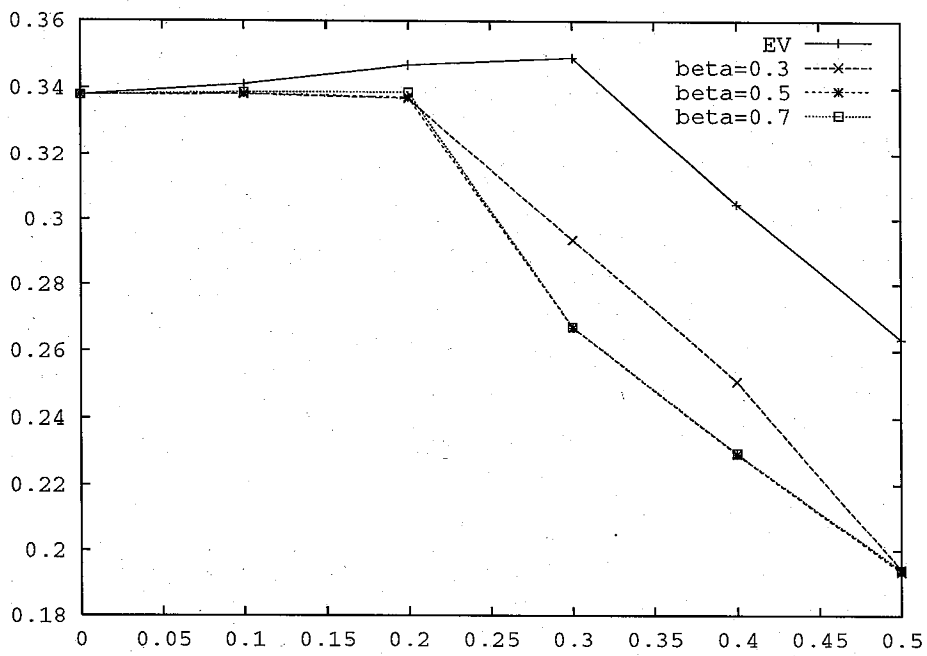

Figure 9: Type 2. Utilization rate of link 4: horizontal axis: δ, vertical axis: y_4/d

In the ERM method, for each δ, the demand of the solution asymptotically approaches $E[d(\omega)]$ as β increases (see Figure 8 and Figure 10).

The utilization rate y_4/d of link 4 in the ERM method is less than that in the EV method for all δ. The larger the value of δ, the larger the difference between the EV method and the ERM method. That is, in the ERM method, the risk is reflected much more than the EV method, and this property becomes more evident as the variance increases (see Figure 9 and Figure 11).

6 Conclusion

In this paper, we have proposed a robust user equilibrium model that takes into account the risk in the actual non-deterministic situation.

When the cost and the demand are linear with respect to the random variable, solutions obtained by the EV method only depend on the expectation of the random variable. On the other hand, in the ERM method and the MPEC approach, even if the expectation of the random variable is the same, when the variance differs, different solutions were obtained. In the case where the cost and the demand are nonlinear with respect to the random variable, the expectation of the cost and the demand change with the value of the variance of the random variable. Thus, in the EV method, different solutions are obtained. However the ERM method or the MPEC approach produced solutions that reflect the risk more significantly than the EV method. But in the MPEC approach, as the number of samples L becomes large, the number of variables enormously increases and the problem becomes very difficult to solve.

We have also considered the case where not only the cost but also the demand has un-

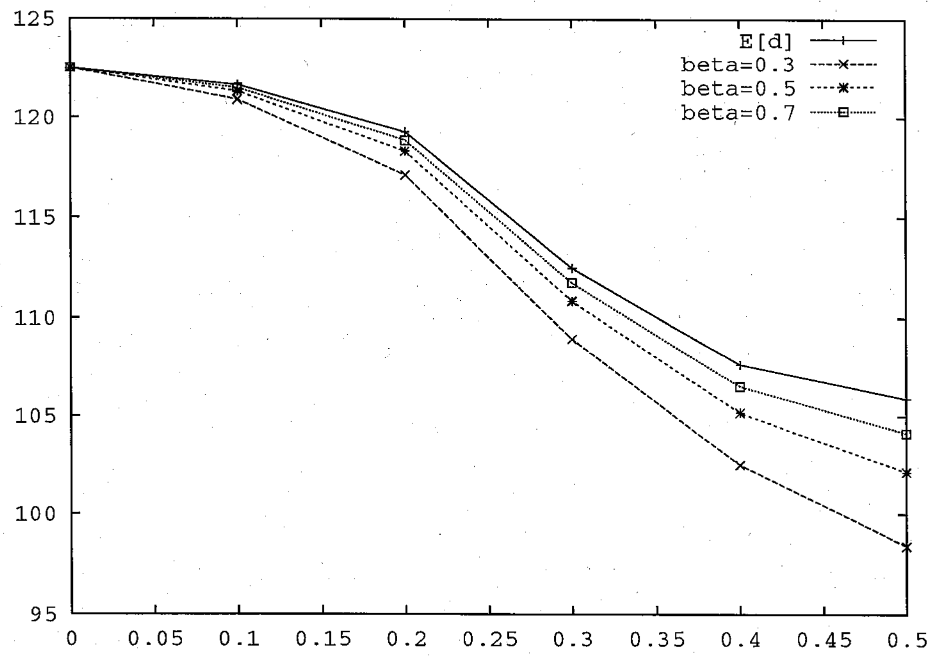

Figure 10: Type 3. Demand versus δ: horizontal axis: δ, vertical axis: demand

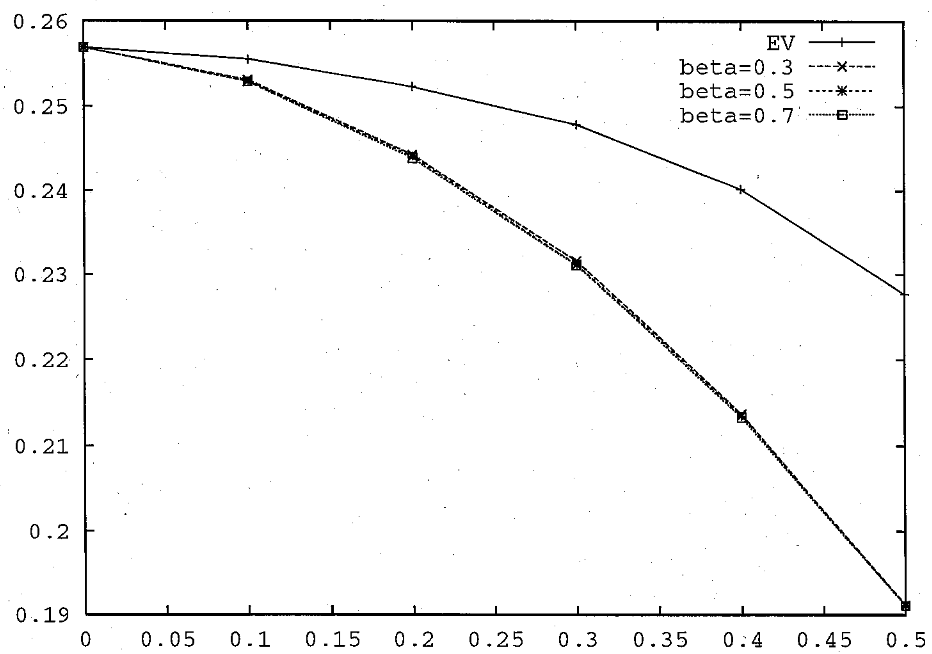

Figure 11: Type 3. Utilization rate of link 4: horizontal axis: δ, vertical axis: y_4/d

certainty. We conducted some numerical experiments with the EV method and the ERM method. We observed that the demand obtained by the ERM method approached the expectation of the demand $E_\omega[d(\omega)]$, as the weight parameter β for the flow conservation condition increases. In this case, the ERM method produced solutions which reflect the effect of the risk more clearly than the EV method.

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