Abstract

Future integrated-services packet networks will carry a wide range of applications which could differ significantly in their Quality-of-Service (QoS) requirements. Generalized processor sharing (GPS) is the most important and ideal fluid scheduling discipline, which is useful in guaranteeing QoS. Many practical packet scheduling algorithms build on it. However, most studies of GPS assume constant service rate. This assumption might be invalid, especially, in heterogeneous internetworking environments. In reality, the service rate at communication servers could be time-varying owing to multiple-access mechanisms in subnetworks, link-level flow/error control, and user mobility.

In this thesis, we analyze GPS networks with variable service rate in the deterministic and stochastic settings and extend the results of Parekh, Zhang, et al. to the variable service rate case. In the deterministic setting, we assume that sources are constrained by leaky buckets and the service processes are given. We first show bounds of backlog, delay, and output burstiness for each session in a single GPS server. We extend these results to networks with arbitrary topology, which belong to a broad class of GPS assignments, the so-called Consistent Relative Session Treatment (CRST) GPS assignment. We show the stability of CRST networks and then show bounds of backlog and delay for each session in CRST networks. Moreover, we relate the results for GPS to those for GPS, which closely approximates GPS. In the stochastic setting, on the other hand, we use exponentially bounded burstiness process and exponentially bounded fluctuation process to characterize source traffic and server fluctuation, respectively. We show statistical bounds on the distributions of backlog and delay for each session in CRST networks with arbitrary topology.

Contents

1 Introduction

The provision of Quality-of-Service (QoS) guarantees has become important in the design of high-speed networks. One important issue in the provision of QoS guarantees is what scheduling disciplines should be employed at network switches. Ideally these scheduling disciplines should, on the one hand, provide isolation between sessions, so that the misbehavior of one session will not affect other sessions and, on the other hand, exploit statistical multiplexing gain [8]. It is desirable that their (deterministic or statistical) bounds can be derived.

One of the most widely studied non-FCFS scheduling is the Generalized Processor Sharing (GPS) scheduling discipline (also known as Weighted Fair Queueing in [8]). In [1] and [2], Parakh and Gallager examined of GPS with leaky-bucket controlled incoming traffic [10]. If each session is leaky-bucket controlled and that the total arrival rate is smaller than the service rate, it was shown, in the case of a single server in isolation, that the backlog and delay of each session are bounded from above; and in the case of a network of GPS servers, that under a broad class of GPS assignments known as Consistent Relative Session Treatment (CRST) GPS assignments, the network is stable. These bounds are actually attained in the worst-case scenario. Simulation results in [9] show that deterministic upper bounds are usually very conservative, and that, if these bounds are used as admission control criteria, low utilization of network bandwidth will result. In [4], Zhi-Li Zhang presented bounds on the individual session backlog and delay distribution under GPS scheduling discipline in a stochastic setting. They model the source session traffic as an "exponentially bounded burstiness" (E. B. B.) process whose total "long-term upper rate" is smaller than the service rate.

These studies assume constant service rate. This assumption of steadiness, especially in a heterogeneous internetworking environment, might be invalid owing to subnetwork multiple-access mechanism, link-level flow/error control, and user mobility [7]. In [7], Lee presented the technique "exponentially fluctuated bounded" (E. B. F.) process and "fluctuation constrained" (F. C.) process to characterize and analyze work-conserving communication servers with varying service rate.

This work is motivated by, and is an extension of, [1], [2], and [4]. We will extend their result to a variable service rate case. We analyze variable service rate GPS servers and networks in both deterministic and stochastic settings by using the techniques similar to [1], [2], and [4]. In the deterministic setting, we show the existence of bounds of each session's delay and backlog in a single server and a CRST network under the following assumptions: the arrival process of all sessions in a server or network are leaky-bucket constrained, the average service rate is greater than the total arrival rate and the maximum service rate is upper-bounded. We also show that bounds can be computed and attainable in the worst-case scenario if the output rate is upper-bounded and fluctuation constrained process. Moreover, we relate results for GPS to those for PGPS, which closely approximates GPS. In the stochastic setting, we present statistical bounds on the distributions of backlog and delay of each session for a single server and a CRST network, under the assumptions that the arrival and output processes are E. B. B. and E. B. F. processes, respectively.

The rest of this thesis is organized as follows. In section 2, GPS and leaky bucket are defined and explained. In section 3 and 4, we proceed with an analysis of a single GPS server in isolation and a GPS network, respectively. In section 5, we analyze single and multi-node GPS systems in the stochastic setting. Finally, the conclusion is provided in section 6.

2 Preliminaries

2.1 A GPS server with variable service rate

We consider a GPS server that is work conserving and can operate at rate $r(t)$ at time t. By work conserving, we mean that the server must operate at full rate $r(t)$ if there are packets waiting in the system at t. A GPS server that serves N sessions is characterized by positive real numbers $\phi_1, \phi_2, \ldots, \phi_N$. Let $S_i(\tau, t)$ denote the amount of session i traffic served in an interval $(\tau, t]$. A session is backlogged at time t if a positive amount of that session's traffic is queued at time t. A GPS server is defined as one for which

$$\frac{S_i(\tau, t)}{\phi_i} \geq \frac{S_j(\tau, t)}{\phi_j}, \tag{1}$$

for any session i that continuously backlogged in the interval $(\tau, t]$. If both session i and j are continuously backlogged in $(\tau, t]$,

$$\frac{S_i(\tau, t)}{\phi_i} = \frac{S_j(\tau, t)}{\phi_j}. \tag{2}$$

Note that this is the same definition as in the constant rate case [1].

To simplify notations, we define $R(\tau, t) = \int_\tau^t r(x)dx$. $R(\tau, t)$ means the amount which the server can transmit during an interval $[\tau, t]$. In (2), summing over all session j,

$$S_i(\tau, t) \sum_{j=1}^{N} \phi_j \geq R(\tau, t)\phi_i,$$

and session i is served at greater than or equal to

$$g_i(t) = \frac{\phi_i}{\sum_{j=1}^{N} \phi_j} r(t),$$

at time t.

This is one of the important differences between the constant and variable cases. When $r(t) = r$, session i is guaranteed a rate of

$$g_i = \frac{\phi_i}{\sum_{j=1}^{N} \phi_j} r,$$

regardless of t or other sessions. This is very attractive. For example, if the session i is locally stable, i.e., ρ_i is less than g_i, the session can be guaranteed a throughput and delay bound and its bound can be obtained easily (σ_i/g_i). Moreover, it is well known that the delay and backlog upper-bounds of Rate Proportional Processor Sharing networks, where all sessions locally stable, can be obtained easily regardless the topology of the network. This is owing to the above point. Thus we should use another approach to analyze RPPS networks.

2.2 Leaky bucket

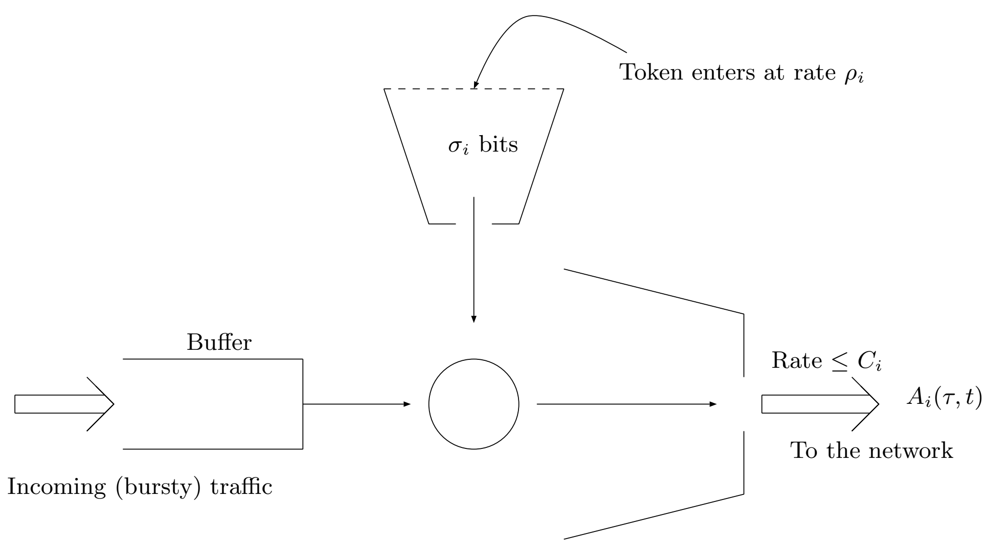

Figure 1: Leaky bucket.

We summarize some results on the leaky bucket scheme in [1]. Fig. 2.2, Fig. 2.2, and Fig. 2.2 are the same as Figs. 3, 4, and 5 in [1], respectively. Fig. 2.2 depicts the leaky bucket scheme that we will use

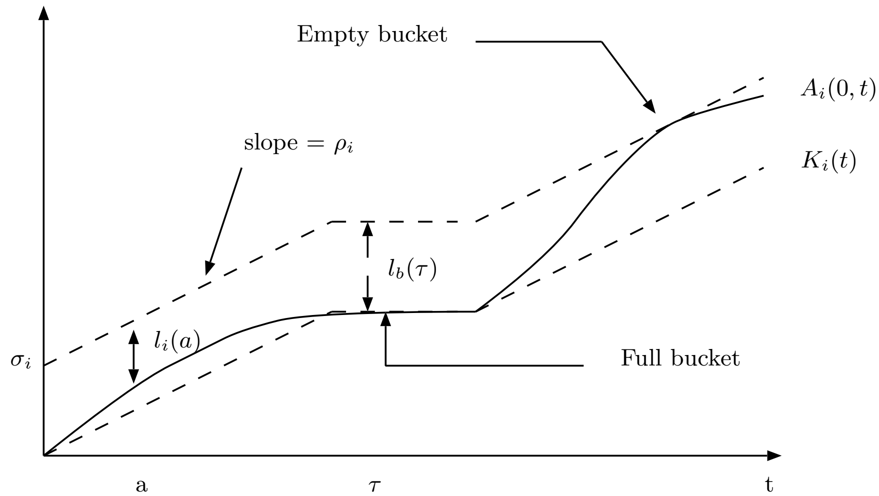

Figure 2: $A_i(t)$ and $l_i(t)$.

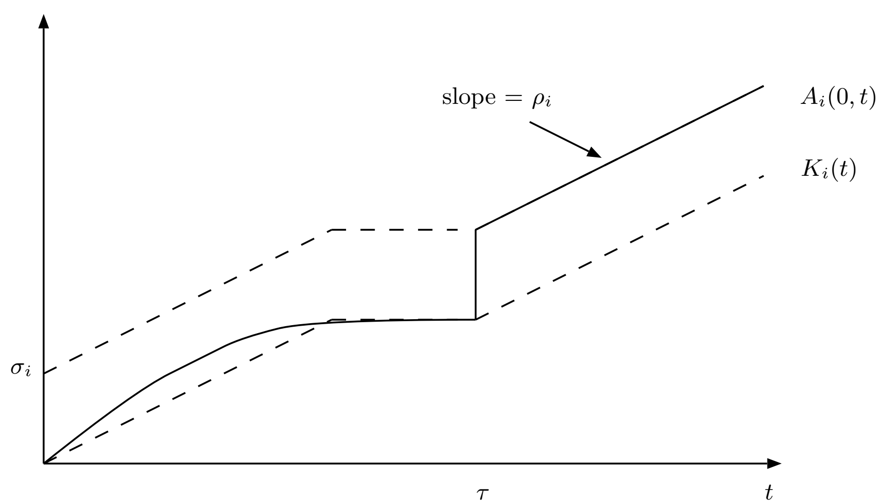

Figure 3: A session i arrival process that is greedy from time τ.

to describe traffic that enters the network. Token is generated at constant rate, ρ, and packets can be admitted into the network only after removing the required amount of token from the token bucket. There is no bound on the number of packets that can be buffered, but the token bucket contains at most σ bits worth of token. In addition to securing the required amount of token, traffic is constrained to leave the bucket at a maximum rate of $C > \rho$.

The constraint imposed by the leaky bucket is as follows. If $A_i(\tau, t)$ is the amount of session i flow that leaves the leaky bucket and enters the network in an interval $(\tau, t]$, then

$$A_i(\tau, t) \leq \min\{C_i(t-\tau), \rho_i(t-\tau) + \sigma_i\}, \ \forall t \geq \tau \geq 0, \tag{3}$$

for every session i, and we say that session i conforms to (σ_i, ρ_i, C_i), or $A_i \sim (\sigma_i, \rho_i, C_i)$. The arrival constraint is attractive since it restricts the amount of traffic in terms of the average sustainable rate (ρ), peak rate (C), and burstiness (σ and C). Fig. 2.2 shows how a fairly bursty source might be characterized using the constraints.

Represent $A_i(0, t)$ as in Fig. 2.2. Let there be $l_i(t)$ bits worth of token in the session i token bucket at time t. We assume that the session starts out with a full bucket of token. If $K_i(t)$ denote the total amount of token accepted at the session i bucket in an interval $(0, t]$, then

$$K_i(t) = \min_{0 \leq \tau \leq t} \{A_i(0, \tau) + \rho_i(t - \tau)\}. \tag{4}$$

Thus for all $\tau \leq t$,

$$K_i(t) - K_i(\tau) \leq \rho_i(t - \tau). \tag{5}$$

We may now express $l_i(t)$ as

$$l_i(t) = \sigma_i + K_i(t) - A_i(0, t). \tag{6}$$

From (5) and (6), we obtain the useful inequality

$$A_i(\tau, t) \leq l_i(\tau) + \rho_i(t - \tau) - l_i(t). \tag{7}$$

In this thesis we assume that $C_i = \infty$, because this is the easiest case to visualize (We do not have to worry about the input links.), and it bounds the performance of the finite capacity case. If

$$A_i(\tau, t) \leq \rho_i(t - \tau) + \sigma_i, \ \forall t \geq \tau \geq 0, \tag{8}$$

we say that session i conforms (σ_i, ρ_i) or $A_i \sim (\sigma_i, \rho_i)$.

Finally, we introduce "greedy regime". Session i is called greedy start at time τ if

$$A_i(\tau, t) = \rho_i(t - \tau) + l_i(\tau), \ \forall t \geq \tau.$$

In terms of the leaky bucket, this means that the session uses as much token as possible for all time $t \geq \tau$ (Fig. 2.2). Let $\hat{A}$ denote the arrival process that all sessions greedily start at time 0. We will call this arrival process all-greedy arrival process at time 0 or all greedy regime at time 0. Let $\hat{A}_i(\tau, t)$ denote the amount of session i flow in an interval τ, t under $\hat{A}$. We then have

$$\hat{A}_i(0, t) = \rho_i t + \sigma_i, \ \forall t \geq 0. \tag{9}$$

3 Analysis of a single GPS server

In this section, we show bounds of backlog and delay for each session under the assumption that a service process $r(t)$ is given. To obtain those bounds, we define a *fictitious worst-case service process*, $\hat{r}(t)$ for the given service process, $r(t)$. We show the bounds under $\hat{r}(t)$ are greater than those under $r(t)$ and those bounds are "fictitiously" achieved under a *fictitious worst-case regime*, which is an extension of "all-greedy regime" in the constant rate case [1]. We also show a way to characterize the burstiness of the output traffic for every session i, which will especially useful in our analysis of GPS networks. Finally, we relate the results of GPS to those of PGPS.

We analyze a single GPS server that serves N leaky-bucket constrained sessions $(1, \ldots, N)$. Let $A_i(\tau, t)$ denote the amount of session i flow that enters the network in an interval $(\tau, t]$. We then have

$$A_i(\tau, t) \leq \rho_i(t - \tau) + \sigma_i, \ \forall t \geq \tau \geq 0. \tag{10}$$

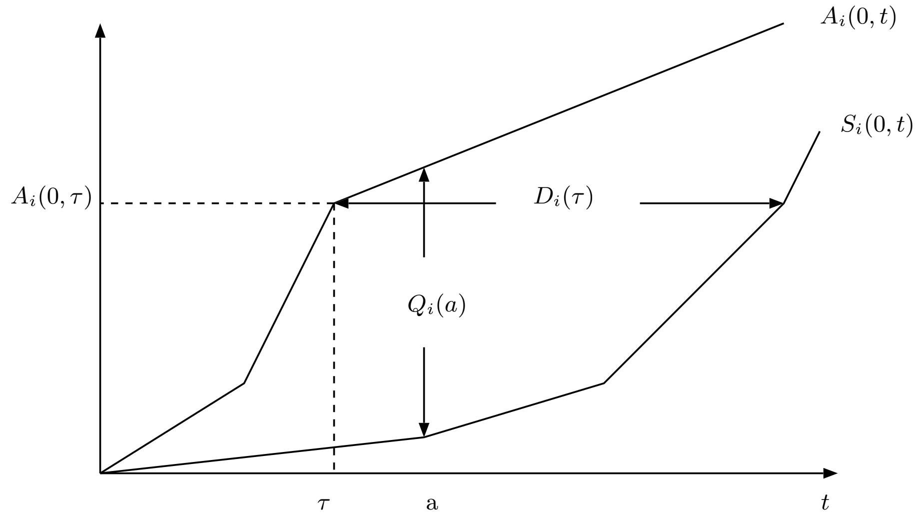

Figure 4: $A_i(0,t)$, $S_i(\tau,t)$, $Q_i(t)$, and $D_i(t)$.

Let $r(t)$ denote the service process. Suppose $r(t)$ satisfies

$$\lim_{t\to\infty} \frac{R(\tau,\tau+t)}{t} > \sum_{j=1}^{N} \rho_j, \;\; \forall \tau \geq 0, \tag{11}$$

and

$$r(t) \leq r, \;\; \forall t \geq 0. \tag{12}$$

(11) and (12) mean that the average service rate is greater than total average sustain rate and the maximum service rate is upper-bounded by r, respectively. Note that the constant rate case, i.e., $r(t) = r$, satisfies (11) and (12). We assume that the server is empty at time 0.

We now define backlog and delay for each session. Note that $S_i(0,t)$ is continuous and nondecreasing for all t. (see Fig. 3.) The session i backlog at time τ, $Q_i(\tau)$, is defined to be

$$Q_i(\tau) = A_i(0,\tau) - S_i(0,\tau).$$

The session i delay at time τ is denoted by $D_i(\tau)$, which is the amount of time that it would take for the session i backlog to clear if no session i flow arrived after time τ. Thus we have

$$D_i(\tau) = \inf\{t \geq \tau : S_i(0,t) = A_i(0,\tau)\} - \tau.$$

From Fig. 3, we see that $D_i(\tau)$ is the horizontal distance between curves $A_i(0,t)$ and $S_i(0,t)$ at the ordinate value of $A_i(0,\tau)$.

We are interested in obtaining the upper bound of the backlog and delay over all time and all arrival process, given a service process $r(t)$. We define the maximum backlog and delay for session i:

$$Q_i^* = \max_A \max_{\tau\geq 0} Q_i(\tau), \tag{13}$$

$$D_i^* = \max_A \max_{\tau\geq 0} D_i(\tau). \tag{14}$$

The problem we will solve in the following is to obtain the upper bound of Q_i^* and D_i^* for every session i and any arrival process, given weights $\phi_1, \ldots \phi_N$ and service process $r(t)$ for a GPS server and (σ_j, ρ_j), $j = 1, \ldots, N$.

3.1 Definition and preliminary results

In this subsection, we discuss a GPS server for an arbitrary arrival process that satisfies (10). We define σ_i^τ for session i and time $\tau \geq 0$ as

$$\sigma_i^\tau = Q_i(\tau) + l_i(\tau). \tag{15}$$

Thus σ_i^τ denotes the sum of the amount of token left in the bucket and the session backlog at time τ. From (7) and (15), we have

$$Q_i(\tau) + A_i(\tau, t) - Q_i(t) \leq \sigma_i^\tau - \sigma_i^t + \rho_i(t - \tau).$$

Note that

$$S_i(\tau, t) = Q_i(\tau) + A_i(\tau, t) - Q_i(t).$$

Thus we establish the following lemma:

Lemma 3.1 *For every session i and in any interval $[\tau, t]$:*

$$S_i(\tau, t) \leq \sigma_i^\tau - \sigma_i^t + \rho_i(t - \tau). \tag{16}$$

Note that this lemma is an extension of Lemma 2 in [1].

Define a system busy period as the maximal interval B such that for any interval $[\tau, t] \in B$:

$$\sum_{i=1}^{N} S_i(\tau, t) = R(\tau, t).$$

We show an important proposition with regard to a system busy period.

Proposition 3.1 *For any service process $r(t)$, the length of a system busy period is upper-bounded.*

(Proof)

Let τ denote the time when a system busy period starts. We denote the length of the system busy period as $t^*(\tau)$. Suppose each session has sent any traffic before time τ and is greedy starting from τ. Let $\hat{t}^*(\tau)$ denote the length of the system busy period that starts at τ, i.e.,

$$\hat{t}^*(\tau) = \inf\{\ t \mid R(\tau, \tau + t) > \sum_{1 \leq j \leq N} \rho_j t + \sigma_j\}. \tag{17}$$

Since $\hat{t}^*(\tau) > t^*(\tau)$, we will prove $\hat{t}^*(\tau) < \infty$. From (11), for any small $\epsilon > 0$, there exists $M < \infty$ such that for all $t \geq M$. Thus we have

$$\frac{R(\tau, \tau + t)}{t} \geq \sum_{1 \leq j \leq N} \rho_j + \epsilon.$$

Define $\tilde{t}(\tau)$ as $\max\{M,\ \sigma_j/\epsilon\}$. For $t \geq \tilde{t}(\tau)$,

$$\begin{aligned} R(\tau, \tau + t) &\geq \sum_{1 \leq j \leq N} \rho_j t + \epsilon t && (18) \\ &\geq \sum_{1 \leq j \leq N} \rho_j t + \sigma_j. && (19) \end{aligned}$$

We then have

$$\hat{t}^*(\tau) \leq \tilde{t}(\tau) < \infty. \tag{20}$$

This argument holds for any τ. We define $\hat{t}^*$ as

$$\hat{t}^* = \max_{\tau \geq 0}\ \hat{t}^*(\tau). \tag{21}$$

Note that $\hat{t}^*$ is the upper-bound of the length of the system busy period for any arrival process, i.e., $t^*(\tau) \leq \hat{t}^*$ for any τ and $\hat{t}^*$ is finite from (20). □

Since the system is work conserving, if $B = [t_1, t_2]$, then $\sum_{i=1}^{N} Q_i(t_1) = \sum_{i=1}^{N} Q_i(t_2) = 0$. We now define a session i busy period as the maximal interval B_i in a single system busy period, such that for all $\tau, t \in B_i$:

$$\frac{S_i(\tau,t)}{\phi_i} \geq \frac{S_j(\tau,t)}{\phi_j},\ j = 1, \ldots, N.$$

In the next lemma, we show an useful inequality.

Lemma 3.2 *Assume that an interval $[\tau, t]$ is contained in a session p busy period. For any subset $\mathcal{M}$ of m sessions, $1 \leq m \leq N$ and any time $t \geq \tau$:*

$$S_p(\tau,t) \geq \frac{R(\tau,t) - \sum_{j \notin \mathcal{M}} \{\rho_j (t-\tau) + \sigma_j^{\tau}\}}{\sum_{j \in \mathcal{M}} \phi_j} \phi_p. \tag{22}$$

(Proof)

By definition of GPS,

$$S_i(\tau,t) \leq \frac{\phi_i}{\phi_p} S_p(\tau,t),\ i = 1, 2, \ldots, N.$$

From (16) and $\sigma_i^t \geq 0$,

$$S_i(\tau,t) \leq \sigma_i^{\tau} + \rho_i (t-\tau),\ i = 1, 2, \ldots, N.$$

We then have

$$S_i(\tau,t) \leq \min\{\sigma_i^{\tau} + \rho_i(t-\tau),\ \frac{\phi_i}{\phi_p} S_p(\tau,t)\},\ i = 1, 2, \ldots, N. \tag{23}$$

Since the system is busy in $[\tau, t)$, the server serves exactly $R(\tau, t)$ units of traffic in $[\tau, t)$. Thus we have

$$\begin{aligned} R(\tau,t) &\leq \sum_{j=1}^{N} \min\{\sigma_j^{\tau} + \rho_j(t-\tau), \frac{\phi_j}{\phi_p} S_p(\tau,t)\} && (24) \\ &\leq \sum_{j \in M} \sigma_j^{\tau} + \rho_j(t-\tau) + \sum_{j \in M} \frac{\phi_j}{\phi_p} S_p(\tau,t). && (25) \end{aligned}$$

Rearraging the terms yields the result. □

Note that we can derive Lemma 6 in [1] by replacing $R(\tau, t)$ with $(t - \tau)$, i.e., assuming $r(t) = 1$.

3.2 Fictitious worst-case regime bound

We define a *fictitious worst-case service process.* Suppose $r(t)$ satisfies (11) and (12). $\hat{r}(t)$ is called a fictitious worst-case service process for $r(t)$ if $\hat{r}(t)$ satisfies the followings.

- $\hat{r}(t)$ satisfies (11) and (12).
- For any time $\tau \geq 0$,

  $$\hat{R}(0,t) \leq R(\tau, \tau + t),\ 0 < t < \hat{t}^*(\tau), \tag{26}$$

  where $\hat{R}(\tau,t) = \int_{\tau}^{t} \hat{r}(x)dx$ and $\hat{t}^*(\tau)$ is defined in (17).
- $\hat{r}(t)$ is monotone nondecreasing for $t \geq 0$.

From Proposition 3.1, we establish the following result.

Proposition 3.2 *Suppose a service process $r(t)$ satisfies* (11) *and* (12)*. There exists at least one fictitious service process $\hat{r}(t)$ for $r(t)$.*

(Proof)

We choose $\hat{r}(t)$ such that

$$\hat{r}(t) = \begin{cases} 0, & \text{if } \ 0 \ \leq \ t \ \leq \ \hat{t}^* - \frac{\sum_{j=1}^{N} \rho_j \hat{t}^* + \sigma_j}{r}, \\ r, & \text{otherwise,} \end{cases}$$

i.e.,

$$\hat{R}(0,t) = [r(t-\hat{t}^*) + \sum_{1\le j\le N} \sigma_j + \rho_j \hat{t}^*]^+, \tag{27}$$

where $[x]^+$ stands for $\max\{0,x\}$ and $\hat{t}^*$ is defined in (21). This function satisfies first and third conditions. Suppose there exists t ($\tau < x < \hat{t}^*$) which does not satisfy inequation (26) and each session has sent any traffic before time τ and is greedy starting from τ. If the server works at maximal rate r from time t, the busy period does not terminate in $\hat{t}^*$. This contradicts the definition of $\hat{t}^*$. □

We call the case where the arrival process is $\hat{A}$ (all-greedy regime from time 0) and the service process is $\hat{r}(t)$ as a *fictitious worst-case regime* from time 0. We now define $\hat{S}_i(\tau,t)$ as the amount of service which session i receives between $[\tau,t]$ under the fictitious worst-case regime from time 0. In Lemma 3.3, we show that the amount of service in $(0,t]$ can be computed under the fictitious worst-case regime.

Lemma 3.3 *Let $\hat{B}(t)$ denote the set of sessions that are busy at time t under a fictitious worst-case regime. We then have*

$$\hat{S}_i(0,t) = \frac{\hat{R}(0,t) - \sum_{j\notin \hat{B}(t)}\{\rho_j t + \sigma_j\}}{\sum_{j\in\hat{B}(t)} \phi_j}\phi_i. \tag{28}$$

(Proof)
See Appendix A.1. □

In the next lemma we establish the relationship between S_i and $\hat{S}_i$.

Lemma 3.4 *Suppose that time t is contained in a session p busy period that begins at τ. We then have*

$$S_i(\tau,t) \ge \hat{S}_i(0,t-\tau). \tag{29}$$

(Proof)
See Appendix A.2. □

Note that Lemmas 3.3 and 3.4 are extensions of Lemma 6-(ii) and 10 in [1], respectively, when $r(t)=1$. These lemmas are very important to analyze backlog and delay. We now define $\hat{Q}_i(t)$ (resp. $\hat{D}_i(t)$) to be backlog (resp. delay) of session i at time t under a fictitious worst-case regime from time 0, i.e.,

$$\hat{Q}_i^* = \max_{0\le t} \hat{Q}_i(t),$$
$$\hat{D}_i^* = \max_{0\le t} \hat{D}_i(t).$$

The following theorem is the main result in this section. The backlog and delay bounds for a given service process is upper-bounded by those under its fictitious worst-case regime

Theorem 3.1 *For any service process that satisfies* (11) *and* (12) *and any session i, Q_i^* and D_i^* are upper bounded by $\hat{Q}_i^*$ and $\hat{D}_i^*$, respectively, i.e.,*

$$Q_i^* \le \hat{Q}_i^*,$$
$$D_i^* \le \hat{D}_i^*.$$

(Proof)
See Appendix A.3. □

Finally, we focus on determining, for every session i, the least quantity σ_i^{out} such that

$$S_i \sim (\sigma_i^{out}, \rho_i, r).$$

This definition of output burstiness is due to Cruz [5]. By characterizing in this manner, we can begin to analyze the networks of GPS servers. There is a convenient relationship between σ_i^{out} and Q^* in the constant service rate (Lemma 12 in [1]). This relationship also holds in the variable service rate case.

Lemma 3.5

$$\sigma_i^{out} = Q_i^*.$$

(Proof)

The argument in the proof of Lemma 12 of [1] holds in the variable rate case. See [1]. □

From this Lemma and Theorem 3.1, we have the following result.

Corollary 3.1

$$\sigma_i^{out} \leq \hat{Q}_i^*. \tag{30}$$

A Fluctuation-Constrained GPS Server

We now consider a fluctuation-constrained GPS server, i.e., we assume a *convex fluctuation constraint* instead of (11). Let $r(t)$ denote the instantaneous output transmission capacity of a variable-rate server. The server is said to be "fluctuation constrained" [7], $r \sim (\delta, \mu)$, if:

$$\int_\tau^t r(x)dx \geq \{\mu(t-\tau) - \delta\}^+. \tag{31}$$

This constraint is valid in considering the following situation: there are two classes served by a server. Sessions in class 1 are served in GPS discipline and other sessions (class 2) are served in FCFS. The service rate is r (constant). Each session in class 1 is given priority to ones in class 2, i.e., a packet in the queue of class 2 does not begin to be served until there is no packet in class 1. Suppose that the service of any packet is not interrupted. The instantaneous output transmission capacity of the server for class 1, $r_1(t)$ is fluctuated constrained, $r_1 \sim (L_{\max}, r)$, where $L_{\max}$ is the maximum length of a packet of class 2.

We will extend this constraint. Let $f(t)$ denote non-decreasing convex function for $t \geq 0$ and $f(0) \leq 0$. A server is said to be *convex fluctuation constrained*, $r(t) \sim f(t)$, if

$$\int_\tau^t r(x)dx \geq f(t-\tau)^+, \ \forall\tau. \tag{32}$$

Note that this constraint agrees with (31) when $f(t) = \mu t - \delta$.

We now consider a GPS server that serves N sessions $(1, \ldots, N)$. The sessions are leaky-bucket constrained, $A_i \sim (\sigma_i, \ \rho_i)$ $(i = 1, \ldots, N)$ and its service process $r(t)$ is convex fluctuation constrained, $r(t) \sim f(t)$ and upper-bounded by r, $r(t) \leq r$ for all $t \geq 0$. We assume that the server is empty at time 0. Let $Q_i^{\max}$ and $D_i^{\max}$ denote the maximum backlog and delay, respectively, for session i over all time t, arrival process A, and service process $r(t)$, respectively, i.e.,

$$Q_i^{\max} = \max_{r(t)} \max_A \max_{\tau \geq 0} Q_i(\tau) \tag{33}$$

$$D_i^{\max} = \max_{r(t)} \max_A \max_{\tau \geq 0} D_i(\tau). \tag{34}$$

Note that we can choose $f(t)^+$ as $\hat{R}(0,t)$ for any $r(t)$. So, we have the next result from Theorem 3.1.

Theorem 3.2 *If there exists t^* such that $f(t^*) \geq \sum_{j=1}^N \rho_j t + \sigma_j$, for every session i, $Q_i^{\max}$ and $D_i^{\max}$ are achieved (not necessarily at the same time) when the arrival process is $\hat{A}$ and the service process is $f(t)^+$.*

Thus we can compute $Q_i^{\max}$ and $D_i^{\max}$ in a way similar to [12].

3.3 Analysis of a PGPS server

A problem with GPS is that it is an idealized discipline that does not transmit packets as entities. It assumes that the server can serve multiple sessions simultaneously and that traffic is infinitely divisible. Packet-by-packet GPS (PGPS) is a simple packet-by-packet transmission scheme that is an excellent approximation to GPS even when the packets are of variable length. We will adopt the convention that a packet arrived only after its last bit has arrived. In this section we study the relationship between PGPS and GPS discipline under the same arrival and service processes in the variable rate case.

Let d_p denote the time when packet p will depart under GPS. A good approximation scheme is work-conserving scheme that serves packets in increasing order of d_p. PGPS server picks the first packet that would complete service in the GPS simulation if no additional packets arrived after time τ [1]. In the variable service rate case, this PGPS scheme is valid and we have the same result as in [1].

Lemma 3.6 *Let p and p' denote packets in a GPS system at time τ, and suppose that packet p completes service before packet p' if there are no arrival after time τ. Packet p will also complete service before packet p' for any arrival process after time τ.*

(Proof)
See the proof of Lemma 1 in [1]. □

Let $\tilde{d}_p$ denote the time at which packet p departs under PGPS. We show that

Theorem 3.3 *For any packet p,*

$$\int_{d_p}^{\tilde{d}_p} r(t)dt \leq L_{\max},$$

where $L_{\max}$ represents the maximum packet length.

(Proof)
See Appendix A.4. □

Let $S_i(\tau, t)$ and $\tilde{S}_i(\tau, t)$ denote the amount of session i traffic (in bits) served under GPS and PGPS in theinterval $[\tau, t]$.

Theorem 3.4 *For all time t and session i:*

$$S_i(0,t) - \tilde{S}_i(0,t) \leq L_{\max}.$$

(Proof)
See Appendix A.5.

□

Let $\tilde{Q}_i(t)$ and $Q_i(t)$ denote the session i backlog at time t under PGPS and GPS, respectively. It then immediately follows from Theorem 3.4 that

Corollary 3.2 *For all times t and session i*

$$\tilde{Q}_i(t) - Q_i(t) \leq L_{\max}.$$

Virtual Time Implementation of PGPS

Virtual time implementation of PGPS for a constant rate is showed in [1] and for a variable rate is in [13]. In this section we introduce definition of virtual time for variable rate in [13].

We consider arrivals and departures from the GPS server as events. Let t_j denote the time when the j^{th} event occurs (simultaneous event are ordered arbitrarily). We denote the time of the first arrival of a system busy period by $t_1 = 0$. Suppose that, the set of sessions that are busy in the interval (t_{j-1}, t_j) is fixed for each $j = 2, 3 \ldots$, and we may denote this set as B_j. Virtual time $V(t)$ is defined to be zero for all times when the server is idle. Consider any system busy period, and assume that it begins at time zero. $V(t)$ then evolves as follows:

$$\begin{aligned} V(0) &= 0, \\ V(t_{j-1} + \tau) &= V(t_{j-1}) + \frac{R(t_{j-1}, t_{j-1} + \tau)}{\sum_{i \in B_j} \phi_i}, \quad for\ \tau \leq t_j - t_{j-1},\ (j = 2, 3, \ldots). \end{aligned} \tag{35}$$

The rate of change of V, namely $\frac{\delta V(t_{j-1}+\tau)}{\delta \tau}$, is equal to $\frac{r(t_{j-1}+\tau)}{\sum_{i \in B_j} \phi_i}$ and each backlogged session i receives service at rate $\phi_i \frac{\delta V(t_{j-1}+\tau)}{\delta \tau}$. Thus V can be interpreted as increasing at the marginal rate at which backlogged sessions receive service.

Suppose that the k^{th} session i packet arrives at time a_i^k and has length L_i^k. We denote the virtual times at which this packet begins and completes service by S_i^k and F_i^k, respectively. Defining $F_i^0 = 0$ for all i, we have

$$\begin{aligned} S_i^k &= \max\{F_i^{k-1}, V(a_i^k)\}. \\ F_i^k &= S_i^k + \frac{L_i^k}{\phi_i}. \end{aligned} \tag{36}$$

In the constant rate case, there are three attractive properties of the virtual time interpretation from the standpoint of implementation. These properties are valid in the variable case, too. First, the virtual time finishing times can be determined at the packet arrival time. Second, packets are served in order of virtual time finishing time. Finally, we need only update virtual time when there are events in the GPS system. However, the price to be paid for these advantages is some overhead in keeping track of sets B_j and $R(t_{j-1}, t_j)$, which is essential in updating of virtual time.

Given this mechanism for updating virtual time, PGPS for variable rate is defined as follows: When a packet arrives, the virtual time is updated and the packet is stamped with its virtual time finishing time. The server is work conserving and serves packets in an increasing order of the time-stamp.

However, PGPS is not feasible for high speed networks [13], because

- it may not be possible to accurately estimate $r(t)$ due to the unpredictable and multiple time-scale variation in VBR video bit rate and
- this would make the computation of the virtual time $V(t)$ more expensive.

To overcome these difficulties, some fair queueing disciplines are formulated (e.g. [13]).

4 Analysis of GPS networks

In this section, we analyze a GPS network that serves N sessions. We first explain a model of the GPS network. We show the stability of Consistent Relative Session Treatment (CRST) networks. We then show bounds of backlog and delay for each session, when service processes of all nodes and internal traffic are given. Finally, we relate the results of GPS to those of PGPS.

4.1 Preliminaries

A network is modeled as a directed graph where nodes represent switches and arcs represent links. A route is a path in the graph and the path taken by session i is defined as $P(i)$. Let $P(i,k)$ denote the k^{th} node in $P(i)$ and K_i denote the total number of nodes in $P(i)$. The rate of the link associated with server m at time t is denoted by $r^m(t)$. The amount of session i traffic that enters the network in an interval $[\tau, t]$ is given by $A_i(\tau, t)$. We denote the amount of session i traffic served at node $P(i,k)$ in the same interval $[\tau, t]$ by $S_i^{(k)}(\tau, t),\ k = 1, \ldots, K_i$. Thus $S_i^{(K_i)}$ represents the amount of traffic that leaves the network. We characterize the service process by "pseudo" leaky bucket parameters $\sigma_i^{(k)}$ and ρ_i, so that

$$S_i^{(k)}(\tau, t) \leq \sigma_i^{(k)} + \rho_i (t - \tau),\ \forall t \geq \tau \geq 0,$$

i.e., $S_i^{(k)} \sim (\sigma_i^{(k)}, \rho_i)$. Often, we will analyze a particular server, m. Define $I(m)$ as the set of sessions that are served by server m. For every session $i \in I(m)$, let $A_i^m \sim (\sigma_i^m, \rho_i)$ and $S_i^m \sim (\sigma_i^{m,out}, \rho_i)$ denote the arrival process and the departure process at that node, respectively. The weight of session i at node m is denoted by ϕ_i^m.

Suppose arrival processes of all sessions and service processes of all nodes are given. We define $Q_i^{(k)}(t)$ as the session i backlog at node $P(i,k)$ at time t. Similarly, let $Q_i^m(t)$ denote the session i backlog at node $m \in P(i)$. Thus if $m = P(i,k)$, then

$$Q_i^{(k)}(t) = Q_i^m(t) = A_i^m(0,t) - S_i^m(0,t).$$

Define the total session i backlog at t as

$$Q_i(t) = \sum_{k=1}^{K_i} Q_i^{(k)}(t).$$

Also, let $D_i(t)$ denote the time spent in the network by session i flow that arrives at time t. A network is called stable if $D_i(\tau) < \infty$ or $Q_i(\tau) < \infty$ for any session i and any time $\tau \geq 0$.

We now introduce Assumptions 4.1.

Assumptions 4.1 *We assume the followings.*

(A) *Each session i is leaky bucket constrained.*

$$A_i(\tau, t) \le \rho_i(t-\tau) + \sigma_i, \ \forall t \ge \tau \ge 0.$$

(B) *The system is empty at time* 0. *For any session i,*

$$Q_i(0) = 0.$$

(C) *For each node m and all $\tau \ge 0$,*

$$\lim_{t\to\infty} \frac{R^m(\tau, \tau+t)}{t} > \sum_{j=1}^{N} \rho_j,$$

where $R^m(\tau, t) = \int_\tau^t r^m(x)dx$.

(D) *For each node m, the maximum service rate is upper-bounded by r^m.*

$$r^m(t) \le r^m, \ \forall t \ge 0.$$

We are interested in obtaining the upper bound of backlog and delay over all time and over all arrival processes, when service processes of all nodes are given. We define the maximum backlog and delay for session i:

$$Q_i^* = \max_A \max_{\tau\ge 0} Q_i(\tau), \tag{37}$$

$$D_i^* = \max_A \max_{\tau\ge 0} D_i(\tau). \tag{38}$$

We now define a system (resp. session i) busy period in the network as the maximal interval B (resp. B_i) such that for every $\tau \in B$ (resp. B_i), there is at least one server that is in a system (resp. session i) busy period at time τ.

Fictitious Worst-case Bounds for a single Server

There are two steps to provide bounds on delay and backlog. The first step consists of characterizing internal traffic, so that at each node m and for $i \in I(m)$ we have σ_i^m such that $A_i^m \sim (\sigma_i^m, \rho_i)$. In the second step, the internal characterization is used to analyze the session i route for delay and backlog. According to [2], we calculate upper bounds on the minimum value $\sigma_i^{m,out}$ such that $S_i^m \sim (\sigma_i^{m,out}, \rho_i)$ for each node m. Suppose that for every $j \in I(m)$, $A_j^m \sim (\sigma_j^m, \rho_i)$ is given. In section 3, it was shown that upper bounds of backlog and delay of node m for a given service process $r^m(t)$ are achieved under a fictitious worst-case regime. Let $\hat{A}_i^m$ denote the resulting session i arrival process for all $i \in I(m)$ and $\hat{S}_i^m$ denote the resulting service function at node m. Recall $\hat{S}_i^m(0,t)$ is convex in t. From Lemma 3.5, we can find the smallest value $\hat{\sigma}_i^{m,out}$ such that $\hat{S}_i^m \sim (\hat{\sigma}_i^{m,out}, \rho_i)$. From the discussion above,

$$\hat{\sigma}_i^{m,out} \ge \sigma_i^{m,out}. \tag{39}$$

Thus we may bound the burstiness of S_i^m by $\hat{\sigma}_i^{m,out}$.

4.2 GPS networks under consistent relative session treatment

In this subsection we show that a CRST network is stable if Assumptions 4.1 hold. We will provide an algorithm for characterizing internal traffic for every session in a way similar to [2]. Session j is said to impede session i at a node m if $\frac{\phi_i^m}{\phi_j^m} < \frac{\rho_i}{\rho_j}$. Note that for any two sessions, i and j, that contend for a node m, either session i impedes session j or vice-verse, unless $\frac{\phi_i^m}{\phi_j^m} = \frac{\rho_i}{\rho_j}$. Constant Relative Session Treatment (CRST) GPS assignment is one for which there exists a strict ordering of sessions such that for any two sessions i and j, if session i is less than session j in the ordering, then session i does not impede session j at any node of the network. All sessions of a CRST network can be partitioned into nonempty class $H_1, \ldots, H_L$, such that sessions in H_k are impeded only those in H_l, $l < k$. If two sessions i, j are in the same class, their routes are either edge disjoint or $\frac{\phi_i^m}{\phi_j^m} = \frac{\rho_i^m}{\rho_j^m}$, at every node, m, that is common to the routes of sessions i and j. Clearly, each session $j \in H_1$ is not impeded by any other sessions.

Lemma 4.1 *For any session $j \in H_1$, there exists $\hat{t}_j^m < \infty$ such that :*

$$\hat{t}_j^m = \inf\{ \ t \mid \frac{\phi_j}{\sum_{i \in I(m)} \phi_i} \hat{r}(t) > \rho_j \}.$$

(Proof)

See Appendix B.1. □

Under $\hat{A}^m$ and $\hat{r}(t)$, the guaranteed backlog cleaning rate of session $j \in H_1$ exceeds ρ_j after $\hat{t}_j^m$. Thus we have

$$\hat{\sigma}_j^{m,out} = \sigma_j^m + \rho_j \hat{t}_j^m.$$

From (39), we can upper-bound internal traffic of all the sessions in H_1. We now introduce Lemma 3 and 4 in [2], because these lemmas hold in the variable rate case, too. They will be crucial to us in continuing the process to the sessions which belongs to the higher indexed classes.

Lemma 4.2 (Lemma 3 in [2]) *Suppose sessions i and j contend for a node m and that session j does not impede session i. The value of $\hat{\sigma}_i^{m,out}$ is independent of the value of σ_j^m.*

Lemma 4.3 (Lemma 4 in [2]) *Suppose sessions $i \in I(m)$ for node m and that for every session $j \in I(m)$ that can impede i, σ_j^m is bounded. $\sigma_i^{m,out}$ must be bounded as well.*

(Proof)

See the proofs of Lemmas 3 and 4 in [2]. □

These lemmas can be used to sequentially characterize internal traffic of the sessions in classes $H_2, H_3, \ldots, H_L$. We can use the same procedure as in [2].

– Compute $H_1, \ldots, H_L$.
– $k = 1$
  While $k \leq L$, for each session $i \in H_k$
    For $p = 1$ to K_i
      $m = P(i, p)$
      Compute $\hat{\sigma}_j^m$ given:
        $\sigma_j^m = \hat{\sigma}_j^m$ for all sessions j that impede i at m (computed earlier steps)
        σ_i^m as computed earlier.
        $\sigma_j^m = 0$ for all sessions j that do not impede i at m
.   Set $\sigma_i^{(p)} = \hat{\sigma}_j^{m,out}$.
– $k = k + 1$.

From (39), we have bounds on σ_i^m for every session i and node $m \in P(i)$. Thus we establish the following theorem.

Theorem 4.1 *A CRST GPS network is stable if* Assumptions 4.1 *hold.*

4.3 Computing delay and backlog bounds for stable networks with known internal burstiness and service processes

We consider a stable GPS system whose internal traffic burstiness can be known. We assume that service processes of all nodes are given. We are interested in computing bounds of end-to-end backlog and delay for each session. Note that the maximum backlog and delay at a single node of the network can be upper-bounded by applying a fictitious worst-case regime bound. We can obtain the end-to-end bounds for session i by adding the bounds of each node $m \in P(i)$ (called Additive Method [6]). However, these bounds are loose. The problem is that we are ignoring strong dependencies among the queues at the nodes in $P(i)$ [2]. So, we treat the session route as a whole. We define a *fictitious universal service curve*, which is an extension of a "universal service curve" in the constant rate case [2]. Moreover we show that the bounds are "fictitiously" achieved under a *staggered fictitious worst-case regime.*

To simplify notations, we focus on a particular session, i, that follows the route $1, 2, \ldots, K$. Fig. 5 illustrates the system to be analyzed. This figure is the same as Fig. 3 in [1].

We now assume independent sessions relaxation.

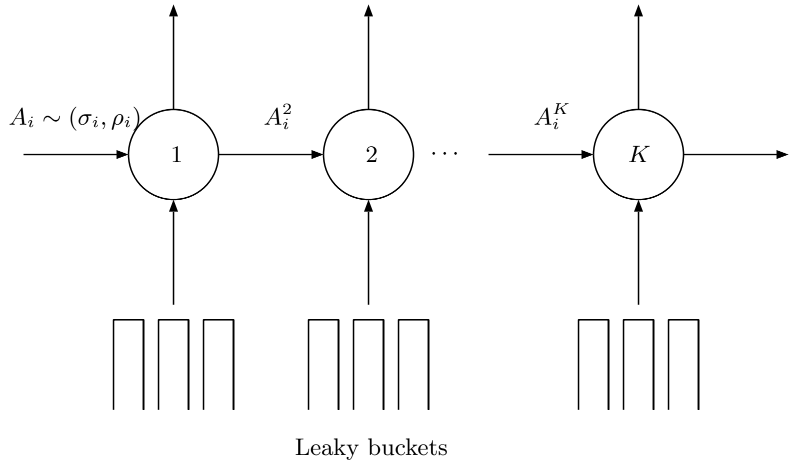

Figure 5: Analyzing the session i route as a whole, under the independent sessions relaxation. Session i traffic enters the network so that it is consistent with (σ_i, ρ_i), and $A_i^m = S_i^{m-1}$ for $m = 2, 3, \ldots, K$. The independent sessions at node m are free to send traffic in any manner as long as $A_j^m \sim (\sigma_i^m, \rho_i)$ for every session $j \in I(m) - \{i\}, m = 1, \ldots, K$.

- The sessions $j \in I(m) - \{i\}$ (for $m = 1, \ldots, K$) are free to send traffic in any manner as long as $A_j^m \sim (\sigma_j^m, \rho_j)$.
- Session i traffic is constrained to flow along its route so that $A_i^m = S_i^{m-1}$.

The value of D_i^* and Q_i^*, which hold under the independent sessions relaxation, must be upper bounds on the true values of these quantities [2]. Thus we will obtain the upper bounds for D_i^* and Q_i^* that hold under the independent sessions relaxation.

Fictitious Universal Service Curve

A fictitious universal curve of session i is easily constructed by applying a fictitious worst-case bound at each node of session i. Intuitively, the value of the fictitious universal curve at time t is a bound on the amount of flow that can traverse the network in the first t time unit of a network session i busy period.

To simplify notations, we will focus on a session i such that $P(i) = (1, 2, \ldots, K)$. The functions $\hat{S}_i^1, \ldots, \hat{S}_i^K$ can be obtained from the internal traffic characterization of section 3. We now define U_i, as a session i fictitious universal service curve, i.e.,

$$U_i(t) = \min\{G_i^K(t), \hat{A}_i(0,t)\}. \tag{40}$$

The curve G_i^k is defined as

$$G_i^k(t) = \begin{cases} \hat{S}_i^1(0,t), \text{ if } \ k = 1, \\ \min_{\tau \in [0,t]}\{G_i^{k-1}(\tau) + \hat{S}_i^k(0, t-\tau)\}, \ t - \tau \leq t_k^B, \text{ otherwise,} \end{cases}$$

where t_m^B represents the duration of a session i busy period at m under a fictitious worst-case regime. For $t \geq \sum_{m=1}^k t_m^B$, we set

$$G_i^k(t) = G_i^k(\sum_{m=1}^k t_m^B) + \hat{A}_i(\sum_{m=1}^k t_m^B, t). \tag{41}$$

We now expand the recursion in terms of $\tau_1, \ldots, \tau_k$, where τ_m corresponds to the minimizing value for node m. Clearly, $\tau_1 = 0$ and define $\tau_{k+1} = t$. Note that $\tau_{m+1} - \tau_m \leq t_m^B$ for each $m = 1, \ldots, k$. We then have

$$\begin{aligned} G_i^k &= \min_{\tau_k \in [0,t]} \min_{\tau_{k-1} \in [0,\tau_k]} \cdots \min_{\tau_2 \in [0,\tau_3]} \sum_{m=1}^k \hat{S}_i^m(0, \tau_{m+1} - \tau_m) \\ &= \min_{0 \leq \tau_2 \leq \ldots \tau_k \leq t} \sum_{m=1}^k \hat{S}_i^m(0, \tau_{m+1} - \tau_m). \end{aligned} \tag{42}$$

In the next Lemma we show that $G_i^k(t)$ must meet $\hat{A}_i(0,t)$ at some time before $\sum_{m=1}^k t_m^B$.

Lemma 4.4 *If* Assumptions 4.1 *hold, we have*

$$G_i^k(\sum_{m=1}^k t_m^B) \geq \hat{A}_i(0, \sum_{m=1}^k t_m^B).$$

(Proof)
See Appendix B.2. □

From (41), we must have $G_i^k(t) \geq \hat{A}_i(0,t)$, for any $t \geq \sum_{m=1}^k t_m^B$. There exists $B_k \leq \sum_{m=1}^k t_m^B$ such that

$$G_i^k(t) \begin{cases} < & \hat{A}_i(0,t), \text{ if } t < B_k, \\ = & \hat{A}_i(0,t), \text{ if } t = B_k, \\ > & \hat{A}_i(0,t), \text{ otherwise } t > B_k. \end{cases}$$

Thus we have

$$U_i(t) = \begin{cases} G_i^K(t), & t \leq B_K, \\ \hat{A}_i(0,t), & t > B_K. \end{cases}$$

Having defined U_i, we now relate it to the session i departure from the network. In the following lemma, we establish a relationship between S_i^m and G_i^m.

Lemma 4.5 *Suppose the session i arrival process, A_i, and service processes of each node m, $r^m(t)$ $(m = 1, \ldots K)$, are given. Consider some time τ such that $Q_i(\tau) = 0$. For each m and any $t > \tau$,*

$$S_i^m(\tau,t) \geq \min_{V \in [\tau,t]} \{A_i(\tau,V) + G_i^m(t-V)\}. \tag{43}$$

(Proof)
See Appendix B.3. □

Thus we have the following theorem.

Theorem 4.2 *For every session i:*

$$Q_i^* \leq \max_{\tau \geq 0} \{\hat{A}_i(0,\tau) - G_i^K(\tau)\}, \tag{44}$$

$$D_i^* \leq \max_{\tau \geq 0} \left\{ \min\{ \ t \ | G_i^K(t) = \hat{A}_i(0,\tau)\} - \tau \right\}. \tag{45}$$

(Proof)
See Appendix B.4. □

(K,t)-staggered Fictitious Worst-case Regime

In this subsection we make clear the relationship between a staggered fictitious worst-case regime and a session i universal service curve. As in the previous sections, we will focus on a staggered fictitious worst-case regime with respect to session i and assume that $P(i) = \{1, 2, \ldots, K\}$. Any staggered fictitious worst-case regimes can be characterized by a vector,

$$(T_1, \ldots, T_K), \ T_1 \leq T_2 \leq \ldots \leq T_K.$$

This vector means that node j works at rate r^j in the interval $[T_1, T_j)$, all independent sessions at node j do not send any traffic in the same interval, and they simultaneously start a fictitious worst-case regimes at time T_j.

A (K,t)-staggered fictitious worst-case regime, $t \leq B_K$, is the staggered fictitious worst-case regime characterized by $(0, T_2, \ldots, T_K)$ such that

$$\sum_{k=1}^K \hat{S}_i^k(0, T_{k+1} - T_k) = G_i^K(t), \tag{46}$$

where $T_1 = 0, \ T_{K+1} = t$ and $T_{k+1} - T_k \leq t_k^B$ for $k = 1, \ldots, K$. Note that

- Since $t \leq B_K$, $G_i^K(t) = U_i(t)$.
- For each $k = 1, 2, \ldots, K-1$, the staggered fictitious worst-case regime defined by $(0, T_2, \ldots, T_k)$ describes a (k, T_{k+1})-staggered fictitious worst-case regime.

Comparing (46) with (42) it is clear that $(T_2, \ldots, T_K)$ is a minimizing vector in (42). Thus the universal service curve can be used to determine $(T_2, \ldots, T_K)$. Thus we have the following theorem.

Theorem 4.3 *For any (K, t)-staggered fictitious worst-case regime:*

$$S_i^K(0, t) = G_i^K(t).$$

(Proof)
See Appendix B.5. □

Let $\hat{Q}_i(\tau)$ and $\hat{D}_i(\tau)$ denote the session i backlog and delay at τ under a (K, t)-staggered fictitious worst-case regime for given service processes. From Theorems 4.2 and 4.3, we have the following theorem.

Theorem 4.4 *Suppose the independent sessions relaxation. Under (K, t)-staggered fictitious worst-case regimes,*

$$\begin{aligned}
\max_{\tau \geq 0} \hat{Q}_i(\tau) &= \max_{\tau \geq 0} \{\hat{A}_i(0, \tau) - G_i^K(\tau)\}, \\
\max_{\tau \geq 0} \hat{D}_i(\tau) &= \max_{\tau \geq 0} \left\{ \min\{\ t\ | G_i^K(t) = \hat{A}_i(0, \tau)\} - \tau \right\}.
\end{aligned}$$

A Convex Fluctuation Constrained GPS Network

We now consider a GPS network that satisfies Assumptions 4.1-(A), (B), and (D). Moreover, we assume that its internal traffic can be known and also that the GPS servers in the network are convex-fluctuation-constrained, instead of Assumptions (4.1)-(C). Suppose $f^m(t)$ is non-decreasing convex function for $t \geq 0$ and $f(0) \leq 0$ for each node m. We assume $r^m(t) \sim f^m$. Let $Q_i^{\max}$ and $D_i^{\max}$ denote the maximum backlog and delay for session i over all time t, arrival process A, and each service process $r^m(t)$, respectively, i.e.,

$$Q_i^{\max} = \max_{r^m(t)\ m \in P(i)} \max_A \max_{\tau \geq 0} Q_i(\tau), \tag{47}$$

$$D_i^{\max} = \max_{r^m(t)\ m \in P(i)} \max_A \max_{\tau \geq 0} D_i(\tau). \tag{48}$$

Note that for each node m, we can choose $f^m(t)^+$ as a fictitious worst-case service process, $\hat{R}^m(0, t)$, for any service process, $r^m(t)$. Thus we can compute $Q_i^{\max}$ and $D_i^{\max}$ from Theorems 4.2 and 4.3.

4.4 Analysis of PGPS networks

We analyze a PGPS network in a way similar to [2]. When packet sizes are not negligible, there are two effects to consider. First, packets must be served non-preemptively, i.e., once a server has begun serving a packet, it must continue to do so until completion. Secondly, no packet is eligible for service until its last bit has arrived, since in most networks with heterogeneous link speeds, packets are not transmitted until they have completely arrived. We assume that service is not virtual cut-through. If $m-1$ and m are successive nodes on a session i's route, we cannot assume, as we did in the previous section, that $S_i^{m-1} = A_i^m$. In fact, for $P(i) = \{1, 2, \ldots, K_i\}$:

$$S_i^{m-1}(\tau, t) \geq A_i^m(\tau, t) \geq S_i^{m-1}(\tau, t) - L_i, m = 2, \ldots, K_i, \tau < t, \tag{49}$$

where $L_i \leq \sigma_i$ represents the maximum packet size for each session i. This effect is illustrated in Fig. 6, which is the same figure as Fig. 8 in [2]. Note that since the GPS server does not begin serving a packet until its last bit has arrived, it "sees" the arrivals as a series of impulses, such that the height of each impulse is at most L_i. However, since we are not assuming any peak rate constraint in the input characterizations, A_i^m is consistent with $(\sigma_i^m + L_i, \rho_i)$.

According to [2], we first enforce the non-virtual cut-through effect, but allow preemptive service, and then incorporate the effects of non-preemptive service.

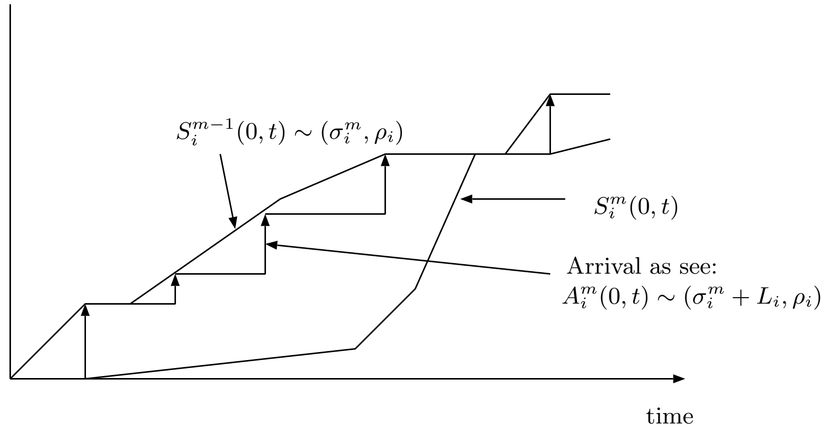

Figure 6: A system in which the packet sizes are non-negligible, $A_i^m(0,t)$ represents the cumulative arrivals seen by server m. The length of each impulse of $A_i^m(0,t)$ is bounded by L_i, the maximum packet size for session i. Since $L_i \leq \sigma_i$, it can be seen from the figure that $A_i^m \sim (\sigma_i^m + L_i, \rho_i)$.

Noncut-Through GPS

To analyze networks of GPS servers with non-negligible packet sizes, we follow the same steps as we did in section 4.3, i.e., we first characterize internal traffic in terms of leaky bucket parameters, and then bound the worst-case delay and backlog for each session by analyzing its route as a whole. To incorporate effects of packet lengths, we stipulate that (49) holds. Consider a GPS network with CRST assignments. Internal traffic can be characterized using essentially the same procedure as in section 4.3 to compute the fictitious worst-case bounds. To analyze the session i route, we proceed as follows. We define $\hat{S}_i^m$ as the session i output at node m under a fictitious worst-case regime. Thus the session i fictitious universal service curve is computed.

In what follows we assume for simplicity in description that $P(i) = \{1, 2, \ldots, K\}$.

Lemma 4.6 *Consider some time τ such that $Q_i(\tau) = 0$. For each m $(1 \leq m \leq K)$ and each $t > \tau$,*

$$S_i^m(\tau, t) \geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(0, t - V)\} - mL_i. \tag{50}$$

(Proof)
See Appendix B.6. □

We now define Q_i^* and D_i^* as the maximum session i packet backlog and delay, respectively. From Lemma 4.6, we have the following theorem.

Theorem 4.5 *For every session i:*

$$Q_i^* \leq \max_{\tau \geq 0} \{\hat{A}_i(0, \tau) - G_i^K(\tau)\} + KL_i, \tag{51}$$

$$D_i^* \leq \max_{\tau \geq 0} \Big\{ \min\{t \mid G_i(t) = \hat{A}_i(0, \tau) + (K-1)L_i\} - \tau \Big\}. \tag{52}$$

(Proof)
See Appendix B.7. □

Nonpreemptive Service: PGPS

Suppose a network of PGPS servers is given, where the assignments of the ϕ_i's meet the CRST requirements of section 4.2. Consider $j \in H_1$ and let $P(j) = \{1, 2, \ldots, K_j\}$. We know from Corollary 3.2 that

$$\tilde{Q}_j^1(\tau) - Q_j^1(\tau) \leq L_{\max}, \tag{53}$$

for all τ where $\tilde{Q}_j^m(\tau)$ and $Q_j^m(\tau)$ represent the session i backlogs at node m, under PGPS and GPS, respectively. Thus we have

$$\tilde{Q}_j^{1,*} - Q_j^{1,*} \leq L_{\max}. \tag{54}$$

From (54) and Lemma 3.5, we can characterize internal traffic at each node in $P(j)$ from the same procedure as in the previous section. If applying a fictitious worst-case bound to a node yields a bound of $\sigma_j^{out,m}$, then the bound on this quantity under PGPS is just $\sigma_j^{out,m} + L_{max}$. Thus we can characterize the internal traffic at each node in $P(j)$ under PGPS.

The next step is the analysis of delay along the session i route. We have the following theorem:

Theorem 4.6 *Suppose Assumptions* 4.1 *hold. For each session i:*

$$\tilde{D}_i^* \leq \max_{\tau \geq 0} \min\{t : G_i^K(t) = \hat{A}_i(0,\tau) + (K-1)L_i\} - \tau + \sum_{m=1}^{K} \inf\{\ t \mid \hat{R}^m(0,t) \geq L_{\max}\}.$$

(Proof)
See Appendix B.8. □

5 Statistical analysis of GPS networks

In this section, we first analyze a single GPS server whose incomming traffic and service rate are consistent with exponentially bounded burstiness (E. B. B.) process and exponentially bounded fluctuation (E. B. F.) process, respectively. Next, we show the stability of CRST networks with an arbitrary topology.

5.1 Preliminaries

We consider a single GPS server that serves N sessions. For $1 \leq i \leq N$, we define A_i and S_i as the arrival process for session i and the corresponding service process, respectively. For any $\tau \leq t$, let $A_i(\tau,t)$ denote the amount of traffic from session i that arrives during the interval $[\tau,t]$, and $S_i(\tau,t)$ denote the amount of service session i received during the same period. The session i backlog at time t, denoted by $Q_i(t)$, is given by $Q_i(t) = \sup_{\tau \leq t}\{A_i(\tau,t) - S_i(\tau,t)\}$. The delay experienced by session i traffic arriving at time t is denoted by $D_i(t)$. The interval of a session i busy period is denoted by B_i and that of the system busy period is denoted by B. We use E. B. B. process and E. B. F. process, which are introduced in [11] and [7] to model source traffic and a fluctuated server, respectively. We assume that session i arrival process is a $(\rho_i, \Lambda_i, \alpha_i)$-E. B. B. process, i.e., for any τ and t, $A_i(\tau,t)$ has the following property: for any $x \geq 0$,

$$Pr(A_i(\tau,t) > \rho_i(t-\tau) + x) < \Lambda_i e^{-\alpha_i x}. \tag{55}$$

We will call ρ_i the long-term upper rate of the arrival process, Λ_i the prefactor, and α_i the decay rate of the exponential decay function. Let $r(t)$ denote the service rate of the server at time t. We assume that the service process is a (r, M, β)-E. B. F. process, i.e., for any τ and t, $r(t)$ has the following property: for any $x \geq 0$,

$$Pr(\int_{\tau}^{t} r(x)dx < r(t-\tau) - x) < Me^{-\beta x}. \tag{56}$$

We will call r the long-term lower rate of the service process, M_i the prefacter, and β_i the decay rate of the exponential decay function. As a necessary stability condition, we require that $\sum_{i=1}^{N} \rho_i < r$. To simplify notations, we define $R(\tau,t) = \int_{\tau}^{t} r(x)dx$.

A corresponding concept of these processes is the concept of an exponentially bounded (E. B.) process. In [11], stochastic process $X(t)$ is called an (α, Λ)-E. B. process if for any t and any $x \geq 0$,

$$Pr\left(X(t) \geq x\right) \leq \Lambda e^{-\alpha x}. \tag{57}$$

In [1], [2], Parakh and Gallager showed that, given $\sum_{i=1}^{N} \rho_i < r$, there exists an ordering among the sessions such that, after relabeling the sessions,

$$\rho_i < \frac{\phi_i}{\sum_{j=i}^{N} \phi_j}(r - \sum_{j=1}^{i-1} \rho_j), \quad 1 \leq i \leq N. \tag{58}$$

Such an ordering is called a “feasible ordering” with respect to $\{\rho_i\}_{1 \leq i \leq N}$, $\{\phi_i\}_{1 \leq i \leq N}$, and r. In general, there are many feasible orderings associated with these factors.

5.2 Sample path behavior of a single GPS server

We will use "Decomposition"[4] to study sample path behavior of sessions. By abuse of notation, let A_i and $r(t)$ also denote a sample path of a random arrival process A_i and $r(t)$ respectively, so $A_i(\tau, t)$ denotes the amount of traffic from session i during the time interval $[\tau, t]$ and $r(t)$ denotes the rate of the server at time t on these sample paths. Similarly, we will use S_i, Q_i, and D_i to denote the corresponding sample paths of the corresponding random process S_i, Q_i, and D_i.

Imagine that we divide a GPS server into a set of N fictitious servers. Let $r_i(t)$ denote the service process of each server i for $1 \le i \le N$. See Fig. 7. This figure is almost the same as Fig. 1 in [4]. We set

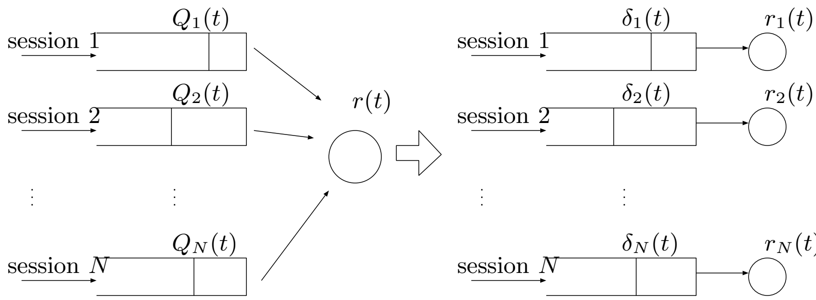

Figure 7: Decomposition of a GPS Server.

$r_i(t) = \frac{r_i}{r} r(t)$, where $r > \sum_{i=1}^N r_i$. Note that $r_i(t)$ is $(r_i, M_i, \frac{\beta r}{r_i})$-E. B. F. process. We now define $M_i = M$, $\beta_i = \frac{\beta r}{r_i}$. Thus $r_i(t)$ is (r_i, M_i, β_i)-E. B. F. process. For each session i in the decomposed system, let $\delta_i(t)$ denote its backlog at time t. We want to bound the actual session i backlog $Q_i(t)$ of the real-GPS system in terms of the fictitious session i backlog $\delta_i(t)$'s of the imaginary decomposed system. We now should decide r_i. Obviously we must have $\sum_{i=1}^N r_i(t) \le r(t)$ and $\rho_i < r_i$ for δ to be well-defined. We need to impose an additional relation on the r_i's in order to reflect the GPS scheduling discipline, corresponding to relation (58). We choose a set of r_i's such that, after some relabeling of the sessions,

$$\rho_i \;< r_i \;\le\; \frac{\phi_i}{\sum_{j=i}^N \phi_j}(r - \sum_{j=1}^{i-1} r_j), \;\; 1 \le i \le N. \tag{59}$$

As long as $\sum_{i=1}^N \rho_i < \sum_{i=1}^N r_i < r$, such a feasible ordering of the sessions always exists [1]. Without loss of generality, we assume that $1, 2, \ldots, N$ is a feasible ordering of the N sessions with respect to $\{r_i\}_{1 \le i \le N}$. From the study of G/G/1 queue, it is well known that $\delta_i(t)$ can be expressed in the form,

$$\delta_i(t) = \sup_{s \le t}\{A_i(s,t) - R_i(s,t)\}. \tag{60}$$

Clearly, $\delta_i(t) \ge 0$ for any t. For $\tau \le t$, applying (60) to $\delta_i(\tau)$ and $\delta_i(t)$, we can derive the following useful inequalities:

$$\begin{aligned} A_i(\tau,t) &\le R_i(\tau,t) + \delta_i(t) - \delta_i(\tau) & (61)\\ &\le R_i(\tau,t) + \delta_i(t). & (62) \end{aligned}$$

For any t, let $\eta_i(t) = Q_i(t) - \delta_i(t)$. As $S_i(\tau,t) = A_i(\tau,t) + Q_i(\tau) - Q_i(t)$, from (62) we have

$$\begin{aligned} S_i(\tau,t) &\le R_i(\tau,t) + \eta_i(\tau) - \eta_i(t) & (63)\\ &\le R_i(\tau,t) + \eta_i(\tau) + \delta_i(t), & (64) \end{aligned}$$

where the last inequality follows from $Q_i(t) \ge 0$.

We now state an important fact.

Lemma 5.1 *For each session i and any t,*

$$\sum_{j=1}^{i} Q_j(t) \le \sum_{j=1}^{i} \delta_j(t). \tag{65}$$

(Proof)
See appendix C.1. □

This lemma says that on a sample path basis, the sum of the actual backlogs of the first i sessions of a feasible ordering in the real GPS system is upper-bounded by the sum of the fictitious backlogs of the corresponding sessions in the imaginary decomposed system. We can also bound the actual backlog of each individual session of the real GPS system in terms of the $\delta_i(t)$'s, but first we need to establish a lower bound on the session i service function S_i, when it is in a session busy period. This is stated in Lemma 5.2. The bounds on individual sessions are then given in Lemma 5.3.

Lemma 5.2 *For any t, let τ denote the beginning of a session i busy period that constraints t. Thus*

$$S_i(\tau,t) \geq R_i(\tau,t) - \frac{\phi_i}{\sum_{j=i}^{N}\phi_j}\sum_{j=1}^{i-1}\delta_j(t). \tag{66}$$

(Proof)
See Appendix C.2. □

Lemma 5.3 *For any t,*

$$Q_i(t) \leq \delta_i(t) + \frac{\phi_i}{\sum_{j=i}^{N}\phi_j}\sum_{j=1}^{i-1}\delta_j(t). \tag{67}$$

(Proof)
See Appendix C.2. □

For $\tau \leq t$, $S_i(\tau,t) \leq A_i(\tau,t) + Q_i(\tau)$, and therefore applying Lemma 5.3 to $Q_i(\tau)$ yields the following.

Lemma 5.4 *For any $\tau \leq t$,*

$$S_i(\tau,t) \leq A_i(\tau,t) + \delta_i(t) + \frac{\phi_i}{\sum_{j=i}^{N}\phi_j}\sum_{j=1}^{i-1}\delta_j(t).$$

Note that these three lemmas correspond to the results in [4].

5.3 Statistical analysis of a single GPS Server

In the previous section we have established several useful relations on backlog on a sample path basis. In this section, we will use E. B. B. process and E. B. F. process to establish upper bounds on the tail distributions of backlog and delay for each session. From Lemmas 5.3 and 5.5, we see that in order to bound the tail distributions of $Q_i(t)$ we only need to bound $\delta_i(t)$ for any t. We can use the result in [7], which provides an upper bound on $\delta_i(t)$.

Lemma 5.5 *For any $x > 0$,*

$$Pr(\delta_i(t) \geq x) \leq \frac{(\Lambda_i + M_i)e^{\gamma_i\rho_i\xi}}{1 - e^{-\gamma_i\epsilon_i\xi}}e^{-\gamma_i x}, \tag{68}$$

where $\frac{1}{\gamma_i} = \frac{1}{\alpha_i} + \frac{1}{\beta_i}$, $\epsilon_i = r_i - \rho_i$, and $\xi > 0$ is arbitrary discritization parameter consistent with $\frac{(\Lambda_i+M_i)e^{\gamma_i\rho_i\xi}}{1-e^{-\gamma_i\epsilon_i\xi}} > 1$.

(Proof)
See Appendix C.4. □

We will choose $\xi = 1$ in the rest of the thesis. We are interested in bounding the moment generating function of $\delta_i(t)$, i.e., $E[e^{\theta\delta_i(t)}]$ for some θ. By modifying the proof of Lemma 5 in [4], we can establish the following result.

Lemma 5.6 *For* $0 < \theta < \min\{\alpha_i, \beta_i\}$,

$$E[e^{\theta\delta_i}] \quad \leq \quad \frac{e^{\theta(\hat{\sigma}(\theta)+\hat{\eta}(\theta)+\rho_i)}}{1-e^{\theta\epsilon_i}}, \tag{69}$$

where

$$\eta_i(\theta) = \frac{1}{\theta}\log(1+\frac{M_i\theta}{\beta_i-\theta}) \text{ and } \hat{\sigma}_i(\theta) = \frac{1}{\theta}\log(1+\frac{\theta\Lambda_i}{\alpha_i-\theta}).$$

(Proof)
See Appendix C.5. □

For each i, we define $\psi_i = \frac{\phi_i}{\sum_{j=i}^N}\phi_j$. Using Chernoff's Bound,

$$Pr(Q_i(t) \geq q) \leq Pr(\delta_i(t) + \psi_i \sum_{j=1}^{i-1} \delta_j(t) \geq q) \tag{70}$$

$$\leq E[e^{\delta_i(t)+\psi_i\sum_{j=1}^{i-1}\delta_j(t)}]e^{-\theta q}. \tag{71}$$

Because $\delta_i(t)$ is not independent, we will use Hölder's inequality. For $1 \leq i \leq N$, let $\{p_j\}_{1\leq j\leq i}$ be such that $p_j > 1$ and $\sum_{j=1}^i 1/p_j = 1$. We then have

$$E[e^{\delta_i(t)+\psi_i\sum_{j=1}^{i-1}\delta_j(t)}] \leq E[\exp(p_i\theta\delta_i(t))]^{1/p_i} \prod_{j=1}^{i-1}(E[\exp(p_j\psi_i\theta\delta_j(t))])^{1/p_j}. \tag{72}$$

We now define $C_i(\theta) = \hat{\sigma}_i(\theta) + \hat{\eta}_i(\theta)$. Thus we have

$$E[\exp(p_j a\theta\delta_j(t))]^{1/p_j} \leq \frac{\exp(\theta a\rho_j + C_j(P_j a\theta))}{(1-e^{p_j a\theta\epsilon_j})^{1/p_j}} \leq \frac{\exp(\theta a\rho_j + C_j(P_j a\theta))}{(1-e^{p_j a\theta\epsilon_j})}. \tag{73}$$

From (71), (72), and (73), we have Theorem 5.1.

Theorem 5.1 *For* $q > 0$ *and* $\theta < \min_{1\leq j\leq i}\{\alpha_j/p_j, \beta_j/p_j\}$,

$$Pr(Q_i(t) \geq q) \leq \Lambda_i^{out} e^{-\theta q},$$

where

$$\Lambda_i^{out} = \frac{\exp(\theta[C_i(p_i\theta) + \rho_i + \psi_i\sum_{j=1}^{i-1}(C_j(p_j\psi_i\theta) + \rho_j)])}{(1-e^{-p_i\theta\epsilon_i})\prod_{j=1}^{i-1}(1-e^{-p_j\psi_i\theta\epsilon_i})}.$$

Similarly, we have the following theorem from Lemma 5.4.

Theorem 5.2

$$Pr(S_i(\tau,t) \geq \rho_i(t-\tau) + x) \leq \Lambda_i^{out} e^{-\theta x}, \ x > 0.$$

Finally, we obtain the tail distribution of the session i delay. Let τ denote the time when the session i busy period which contains the time t starts.

$$Pr\left(D_i(t) > d\right) \quad = \quad Pr\left(A_i(\tau,t) - S_i(\tau,t+d) > 0\right) \tag{74}$$

$$\leq \quad Pr\left(A_i(\tau,t) - R_i(\tau,t+d) + \frac{\phi_i}{\sum_{j=i}^N \phi_j}\sum_{j=1}^{i-1}\delta_j(t+d) > 0\right), \tag{75}$$

where the inequality in (71) follows from Lemma 5.2. Note that $\delta_i(t+d) = A_i(\tau,t+d) - R_i(\tau,t+d)$. We then have

$$Pr\left(D_i(t) > d\right) \quad \leq \quad Pr\left(\delta_i(t+d) + \frac{\phi_i}{\sum_{j=i}^N \phi_j}\sum_{j=1}^{i-1}\delta_j(t+d) > A_i(t,t+d)\right) \tag{76}$$

$$\leq \quad \Lambda_i^{out} E[\exp(-p_{i+1}\theta A_i(t,t+d))]^{1/p_{i+1}} \tag{77}$$

$$= \quad \Lambda_i^{out} e^{-\theta\hat{\sigma}_i(p_{i+1}\theta)} e^{-\theta\rho_i d}. \tag{78}$$

We have the following result.

Theorem 5.3 *For $d > 0$ and $\theta < \min_{1\le j\le i}\{\alpha_j/p_j, \beta_j/p_j\}$,*

$$Pr(D_i(t) \ge d) \le \Lambda_i^{out'} e^{-\theta\rho_i d},$$

where

$$\Lambda_i^{out'} = \frac{\exp(\theta[C_i(p_i\theta) + \rho_i + \psi_i \sum_{j=1}^{i-1}(C_j(p_j\psi_i\theta) + \rho_j - \hat{\sigma}_i(p_{i+1}\theta))])}{(1 - e^{-p_i\theta\epsilon_i})\prod_{j=1}^{i-1}(1 - e^{-p_j\psi_i\theta\epsilon_i})}.$$

Note that backlog, delay, and output processes for each session are E. B. processes.

5.4 Statistical analysis of GPS networks

Feasible Partition

We use "feasible partition"[4] to show the stability of CRST network. We observe that the backlog and delay bounds of the i^{th} session in a feasible ordering with respect to $\{r_i\}_{1\le i\le N}$ depend only on sessions $1, \dots, i-1$ and not on sessions $i+1, \dots, N$. Note that the feasible ordering is defined with respect to $\{r_i\}_{1\le i\le N}$ we choose. There are many different ways to choose $\{r_i\}_{1\le i\le N}$ that satisfy (59). Furthermore, even for a given $\{r_i\}_{1\le i\le N}$, there are possibly many different feasible orderings.

Unlike feasible ordering, with respect to the given parameters $\{\phi_i\}_{1\le i\le N}$ and $\{\rho_i\}_{1\le i\le N}$, there is a unique feasible partition of the N sessions. This feasible partition, denoted by $\mathcal{H} = \{H_l\}_{1\le l\le L}$, where H_l, $1 \le l \le L$ are disjoint and $H_1 \cup \cdots \cup H_L = \{1, \dots, N\}$, is defined recursively as follows:

$$i \in H_1 \text{ if } \frac{\rho_i}{\phi_i} < \frac{r}{\sum_{j=1}^{N}\phi_j}, \tag{79}$$

and for $k \ge 1$, if $H_1^k := H_1 \cup \cdots \cup H_k \ne \{1, 2, \dots, N\}$, then H_{k+1} is defined such that

$$i \in H_{k+1} \text{ if } \frac{\rho_i}{\phi_i} < \frac{1}{\sum_{j\ne H_1^k}\phi_j}(r - \sum_{j\in H_1^k}\rho_j). \tag{80}$$

For sessions in H_1, we can prove the following result from Lemmas 5.2, 5.3, and 5.5 without assuming that $A_1, \dots, A_N$ are independent.

Theorem 5.4 *Assume that session i is a session in H_1. For any time $t > 0$, any $q > 0$ and $d > 0$,*

$$Pr\,(Q_i(t) \ge q) \le \frac{(\Lambda_i + M_i)e^{\gamma_i\rho_i}}{1 - e^{-\gamma_i\epsilon_i}} e^{-\gamma_i q}, \tag{81}$$

$$Pr\,(D_i(t) \ge d) \le \frac{(\Lambda_i + M_i)e^{\gamma_i\rho_i}}{1 - e^{-\gamma_i\epsilon_i}} e^{-\gamma_i r_i d}, \tag{82}$$

where $\frac{1}{\gamma_i} = \frac{1}{\alpha_i} + \frac{1}{\beta_i}$.

(Proof)
See Appendix C.6. □

For sessions in partition classes other than H_1, we can also obtain their bounds from the statement in this section and the previous section.

Statistical Analysis of CRST GPS Network

We now show the stability of CRST networks. Consider a GPS network consisting of M nodes labeled $m = 1, \dots, M$. The traffic source for session i and the fluctuation of node m are modeled as $(\rho_i, \Lambda_i, \alpha_i)$-E. B. B. process and (r^m, M_i, β_i)-E. B. F. process, respectively. The weight for a session i and the set of sessions present at node m are denoted by ϕ_i^m and $I(m)$, respectively. For $1 \le i \le N$, let $A_i(\tau, t)$ denote the amount of arrival traffic into the network in $[\tau, t]$. The route traversed by session i is denoted by $P(i)$ and the k^{th} node in $P(i)$ by $P(i, k)$. Let K_i denote the total number of nodes in $P(i)$. For $1 \le k \le K_i$, we denote the session i arrival process at the k^{th} node of its route by $A_i^{(k)}$ and the session i departure by $S_i^{(k)}$. Note that we have $A_i^{(1)} = A_i$ and $A_i^{(k+1)} = S_i^k$. $S_i^{(K_i)}$ describes session i traffic that leaves the network. Similarly, denote the session i backlog (resp. delay) at node $P(i, k)$ at time t by $Q_i^{(k)}(t)$ (resp.

$D_i^{(k)}(t)$), the total amount of session i traffic queued in the network at time t by $Q_i^{net}(t)$, and the session i end-to-end delay at time t by $D_i^{net}(t)$. A network system (resp. session i) busy period is defined to be the maximal interval B (resp. B_i) such that for every $\tau \in B$ (resp. $\tau \in B_i$), there is at least one server that is in a system (resp. session i) busy period at time τ.

We say a GPS network is stable if, for each session i in the network,

$$\lim_{q\to\infty} Pr\left(Q_i^{net}(t) > q\right) = 0, \text{ or} \tag{83}$$

$$\lim_{d\to\infty} Pr\left(D_i^{net}(t) > d\right) = 0. \tag{84}$$

If at every node of the network, the backlog (or the delay) process for each session at the node is an E. B. process, then clearly the network is stable. If we have an acyclic GPS network with E. B. B. arrival processes, the network can be easily shown to be stable from the input-output relation established in the previous section. On the other hand, for a cyclic network, stability is generally much harder to establish. We will show that under CRST networks with arbitrary topology is stable if $\sum_{i\in I(m)} \rho_i < r^m$ for all nodes m in the network.

In this section we define CRST more weakly than in section 4. For each node m, $m = 1, \ldots, M$, let $\{\phi_i^m\}_i \in I(m)$ denote the GPS assignment for the sessions at node m and $\mathcal{H}^m = \{H_l^m\}_{1\le l\le L_m}$ denote the induced feasible partition of the sessions at node m. We say this collection of GPS assignments $\{\phi_i^m : i \in I(m), 1 \le m \le M\}$ for the network is a CRST GPS assignment if there is a partition $\mathcal{H} = \{H_l\}_{1\le l\le L}$ of the N sessions in the network such that $\mathcal{H}$ is consistent with $\mathcal{H}^m$ at each node m. This means that, for any $i, j \in I(m)$, if $i \in H_l$ and $j \in H_k$ such that $l < k$, then $i \in H_{l'}^m$ and $j \in H_{k'}^m$ such that $l' \le k'$. We call $\mathcal{H}$ the CRST partition of the sessions in the network. Networks with CRST GPS assignment are called CRST GPS networks. It is worth pointing out that CRST is a condition imposed on the global GPS assignments, not on the topology of the network.

The stability of CRST GPS networks can be established from the results obtained for the single node case. Let $\mathcal{H} = \{H_l\}_{1\le l\le L}$ denote the CRST partition of the N sessions in the network. From the statistical analysis of a single GPS server, we see that for a session $i \in H_l$, at any node m along its route, bounds on the distribution of session i backlog $Q_i^m(t)$ and delay $D_i^m(t), t \ge 0, 1 \le m \le N$, depend only on the sessions at node m that are in $H_k, k < l$, and not on the sessions at node m that are in $H_k, k \ge l$. The same statement also holds for the characterization of the session i output process S_i^m at node m. This suggests a recursive procedure for computing backlog and delay bounds and characterizing the output process for each session at any node along its route. For sessions in H_1, the performance can be obtained independently along their routes by applying the input-output relations and the bounds established for the single node case. For any $k, 2 \le k \le N$, once the output process characterization has been derived for each session in $H_l, l < k$, at every node of its route, the bounds on backlog and delay distribution and the output process characterization for sessions in H_k can be derived at any node along their routes. Finally end-to-end delay can be computed from per-node bounds along the session routes using probability union bounds. Therefore, we have the following theorem.

Theorem 5.5 *Suppose each session i in a CRST GPS network is a (ρ_i, A_i, α_i)-E. B. B. process and each server m is a (r^m, B^m, β^m)-E. B. F. process. The network is stable if $\sum_{i\in I(m)} \rho_i < r^m$ at each node m.*

6 Conclusion

We analyzed variable service rate GPS networks in the two settings. In the deterministic setting, we show bounds of each session's delay and backlog in a single server, given its service process. We then show the stability of a CRST network and bounds of each session' delay and backlog in a network. In the stochastic setting, we show statistical bounds on the distributions of backlog and delay of each session for a single server and a CRST network, when arrival and service processes are E. B. B. and E. B. F. processes, respectively.

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Appendix

A Proofs in section 3

Preliminary

We now introduce "feasible ordering" [1]. Consider m leaky-bucket constrained sessions. Let ρ_i and ϕ_i denote the sustain average rate and the weight of each session i. $1,\ldots,m$ is called a feasible ordering for service rate r if

$$\rho_i \le \frac{\phi_i}{\sum_{j=i}^{N}\phi_j}(r-\sum_{j=1}^{i-1}\rho_j),\ i=1,\ldots,m. \tag{85}$$

There exists at least one feasible ordering if $r > \sum_{j=1}^{m}\rho_j$ [1]. Next, we define $V_r(\tau,t)$ as

$$V_r(\tau,t)=\int_{\tau}^{t}\frac{r-r(x)}{r}dx.$$

From this definition, we easily derive the following results.

$$V_r(\tau,t)=\frac{r-\bar{r}}{r}(t-\tau)+\frac{\bar{r}}{r}V_{\bar{r}}(\tau,t). \tag{86}$$

$$\int_{\tau}^{t} r(x)dx = r\left(t-\tau-V_r(\tau,t)\right). \tag{87}$$

An important inequality

Lemma A.1 *Suppose $1,2,\ldots,m$ is a feasible ordering for r and session p $(1\le p\le m)$ is busy in an interval $[\tau,t)$. Define x to satisfy*

$$S_p(\tau,t)=\rho_p\left(t-\tau-V_r(\tau,t)\right)-x. \tag{88}$$

The following inequality holds.

$$\sum_{i=1}^{p-1}S_i(\tau,t) > \sum_{i=1}^{p-1}\left(t-\tau-V_r(\tau,t)\right)\rho_i + x\sum_{k=p}^{N}\frac{\phi_k}{\phi_p}.$$

(Proof)

For compactness of notation, let $\phi_{ij}=\frac{\phi_i}{\phi_j}, \forall i,j$. Because of the feasible ordering,

$$S_p(\tau,t) \quad < \quad (t-\tau-V(\tau,t))\left(\frac{r-\sum_{i=1}^{p-1}\rho_i}{\sum_{i=p}^{N}\phi_{ip}}\right)-x. \tag{89}$$

$$\Rightarrow \sum_{i=p}^{N}\phi_{ip}S_p(\tau,t) \quad < \quad (t-\tau-V(\tau,t))\left(r-\sum_{i=1}^{p-1}\rho_i\right)-\sum_{i=p}^{N}\phi_{ip}x. \tag{90}$$

Since $[\tau,t]$ is in a system busy period and $S_i(\tau,t)\le\phi_{ip}S_p(\tau,t)$ for all i, we have

$$\sum_{i=p}^{N}S_i(\tau,t)=r\left(t-\tau-V(\tau,t)\right)-\sum_{i=1}^{p-1}S_i(\tau,t).$$

We then have

$$r(t-\tau-V(\tau,t))-\sum_{i=1}^{p-1}S_i(\tau,t) \quad < \quad (t-\tau-V_r(\tau,t))(r-\sum_{i=1}^{p-1}\rho_i)-x\sum_{k=p}^{N}\phi_{kp}.$$

$$\Rightarrow \sum_{i=1}^{p-1}S_i(\tau,t) \quad > \quad \sum_{i=1}^{p-1}(t-\tau-V_r(\tau,t))\rho_i+x(\sum_{k=p}^{N}\frac{\phi_k}{\phi_p}).$$

□

Thus we have the following results.

Corollary A.1 *Suppose $1, 2, \ldots, m$ is a feasible ordering for r and session p ($1 \le p \le m$) is busy in an interval $[\tau, t)$. If $S_p(\tau, t) \le \rho_p(t - \tau - V_r(\tau, t))$:*

$$\sum_{i=1}^{p-1} S_i(\tau, t) \quad > \quad \sum_{i=1}^{p-1} (t - \tau - V_r(\tau, t))\rho_i. \tag{91}$$

Lemma A.2 *Suppose $1, 2, \ldots, p$ is a feasible ordering for r and session p is busy in an interval $[\tau, t)$. If $S_p(\tau, t) \le \rho_p(t - \tau - V_r(\tau, t))$:*

$$\sum_{i=1}^{p} \sigma_i^t \le \sum_{i=1}^{p} \sigma_i^\tau + \rho_i V_r(\tau, t).$$

(Proof)

From Corollary 3.1, for every i,

$$\begin{aligned} S_i(\tau, t) &\le \sigma_i^\tau - \sigma_i^t + \rho_i(t - \tau) \\ &\le \sigma_i^\tau - \sigma_i^t + \rho_i(t - \tau - V(\tau, t)) + \rho_i V(\tau, t). \end{aligned}$$

Summing over i and using LemmaA.1, we have the result. □

If we choose τ to be the beginning of a session p busy period and $S_p(\tau, t) \le \rho_p(t - \tau - V_r(\tau, t))$, this lemma says that

$$\sigma_p^t + \sum_{k=1}^{p-1} \sigma_k^t \le \sigma_p + \rho_p V_r(\tau, t) + \sum_{k=1}^{p-1} \sigma_k^\tau \rho_k V_r(\tau, t). \tag{92}$$

We now show the important inequality.

Theorem A.1 *Let $1, 2, \ldots, m$ be a feasible ordering for r. For any time $t \ge \tau$*

$$\sum_{i=1}^{p} \sigma_i^t \le \sum_{i=1}^{p} \sigma_i + \rho_i V_r(0, t).$$

(Proof)

We proceed by induction on the index of session p. Basis: $p = 1$. Define τ to be the last time at or before t such that $Q_1(\tau) = 0$. Session 1 is in a busy period in the interval $[\tau, t]$, and we have

$$S_1(\tau, t) \ge \frac{r(t - \tau - V_r(\tau, t))\phi_1}{\sum_{i=1}^{N} \phi_i} > \rho_1(t - \tau - V_r(\tau, t)).$$

From Lemma3.1,

$$\sigma_1^t \le \sigma_1^\tau + \rho_1(t - \tau - V(\tau, t)) - S_1(\tau, t) + \rho_1 V_r(\tau, t) < \sigma_1^\tau + \rho_1 V(\tau, t) \le \sigma_1 + \rho_1 V_r(0, t).$$

This shows the basis.

Inductive step: Assume the hypothesis for $1, 2, \ldots, p-1$ and show it for p. Let τ denote the last time before t such that $Q_p(\tau) = 0$. Note that $\sigma_p^\tau \le \sigma_p$. Now consider two cases:

Case 1 $\sigma_p^t \le \sigma_p^\tau + \rho_p V(\tau, t)$:

By the induction hypothesis,

$$\sum_{k=1}^{p-1} \sigma_k^t \le \sum_{k=1}^{p-1} \sigma_k + \rho_k V(0, t).$$

Thus we have

$$\begin{aligned} \sum_{k=1}^{p} \sigma_k^t &\le \sigma_p^\tau + \rho_p V(\tau, t) + \sum_{k=1}^{p-1} \sigma_k + \rho_k V(0, t) && (93) \\ &\le \sigma_p + \rho_p V(\tau, t) + \sum_{k=1}^{p-1} \sigma_k + \rho_k V(0, t) && (94) \\ &\le \sum_{k=1}^{p} \sigma_k + \rho_k V(0, t). && (95) \end{aligned}$$

we have the result.

Case 2 $\sigma_p^t > \sigma_p^\tau + \rho_p V(0,t)$:

Note that $S_p(\tau,t) < \rho_p(t-\tau-V(\tau,t))$ from Lemma 3.1. From Lemma A.2, we have

$$\begin{aligned} \sigma_p^t + \sum_{k=1}^{p-1} \sigma_k^t &\leq \sigma_p + \rho_p V_r(\tau,t) + \sum_{k=1}^{p-1} \sigma_k^\tau + \rho_k V_r(\tau,t) \qquad (96) \\ &\leq \sum_{k=1}^{p} \sigma_k + \rho_k V(0,t). \qquad (97) \end{aligned}$$

□

A.1 Proof of Lemma 3.3

Note that $\hat{r}(t)$ is non-negative and non-decreasing function. This means $\hat{B}(t_1) \supseteq \hat{B}(t_2)$, if $t_1 \leq t_2$. Note that $i, j \in \hat{B}(t)$ are busy between $[0,t]$, i.e.,

$$\frac{\hat{S}_i(0,t)}{\phi_i} = \frac{\hat{S}_j(0,t)}{\phi_j}. \qquad (98)$$

For $j \in \hat{B}(t)$,

$$\hat{S}_j(0,t) = \rho_j t + \sigma_j. \qquad (99)$$

From (98) and (99), we have

$$\begin{aligned} \sum_{j=1}^{N} S_j(0,t) &= \frac{\hat{S}_i(0,t)}{\phi_i} \sum_{j \in \hat{B}(t)} \phi_j + \sum_{j \notin \hat{B}(t)} \rho_j t + \sigma_j \qquad (100) \\ &= \hat{R}(0,t). \qquad (101) \end{aligned}$$

Rearranging the terms yields the result.

A.2 Proof of Lemma 3.4

We first define $\hat{B}(t)$ as the set of the sessions which are busy at time t under arrival process $\hat{A}$, and service process $\hat{r}(t)$. From Lemma3.2,

$$\begin{aligned} S_p(\tau,t) &\geq \frac{\int_\tau^t r(x)dx - \sum_{j \notin \mathcal{M}} \{\rho_j(t-\tau) + \sigma_j^\tau\}}{\sum_{j \in \mathcal{M}} \phi_j} \phi_p \qquad (102) \\ &= \frac{r(t-\tau-V_r(\tau,t)) - \sum_{j \notin \mathcal{M}} \{\rho_j(t-\tau) + \sigma_j^\tau\}}{\sum_{j \in \mathcal{M}} \phi_j} \phi_p. \qquad (103) \end{aligned}$$

Let $\bar{r} = \hat{r}(t-\tau)$. We consider two cases.

Case 1 $V_r(0,\tau) \leq \frac{r-\bar{r}}{r}\tau$:

From (87),

$$V_{\bar{r}}(0,\tau) \leq 0. \qquad (104)$$

From definition of $\hat{r}(t)$,

$$V_r(\tau,t) \leq \hat{V}_r(0,t-\tau), \qquad (105)$$

where $\hat{V}_r(\tau,t) = \int_\tau^t \frac{r-\hat{r}(x)}{r} dx$. We now denote $\hat{B}(t-\tau)$ as $\mathcal{B}$. We choose $\mathcal{B}$ as $\mathcal{M}$ in (103). Notice that $\bar{\mathcal{B}}$ has feasible ordering for $\bar{r}$. From Lemma 3.3,

$$\begin{aligned} S_p(\tau,t) &\geq \frac{r(t-\tau-\hat{V}_r(0,t-\tau) - \sum_{j \notin \mathcal{B}} \{\rho_j(t-\tau) + \sigma_j\} - \sum_{j \notin \mathcal{B}} \rho_j V_{\bar{r}}(0,\tau)}{\sum_{j \in \mathcal{B}} \phi_j} \phi_p \\ &\geq \frac{r(t-\tau-\hat{V}_r(0,t-\tau)) - \sum_{j \notin \mathcal{B}} \{\rho_j(t-\tau) + \sigma_j\}}{\sum_{j \in \mathcal{B}} \phi_j} \phi_p \\ &= \hat{S}_p(0,t-\tau). \end{aligned}$$

Case 2 $V_r(0,\tau) > r(t-\tau)\tau$:

From (87),

$$V_{\bar{r}}(0,\tau) > 0. \tag{106}$$

Notice that $V_r(\tau,t) = V_r(0,t) - V_r(0,\tau)$. From (103),

$$\begin{aligned}
S_p(\tau,t) &\geq \frac{r(t-\tau-V_r(0,t)+V_r(0,\tau)) - \sum_{j\notin\mathcal{B}}\{\rho_j(t-\tau)+\sigma_j\} - \sum_{j\notin\mathcal{B}}\rho_j V_{\bar{r}}(0,\tau)}{\sum_{j\in\mathcal{B}}\phi_j}\phi_p \\
&= \frac{r(t-\tau) - rV_r(0,t) + (r-\bar{r})\tau + \bar{r}V_{\bar{r}}(0,\tau) - \sum_{j\notin\mathcal{B}}\{\rho_j(t-\tau)+\sigma_j\} - \sum_{j\notin\mathcal{B}}\rho_j V_{\bar{r}}(0,\tau)}{\sum_{j\in\mathcal{B}}\phi_j}\phi_p \\
&= \frac{r(t-\tau) - rV_r(0,t) + (r-\bar{r})\tau + (\bar{r} - \sum_{j\notin\mathcal{B}}\rho_j)V_{\bar{r}}(0,\tau) - \sum_{j\notin\mathcal{B}}\{\rho_j(t-\tau)+\sigma_j\}}{\sum_{j\in\mathcal{B}}\phi_j}\phi_p.
\end{aligned}$$

Notice that $\bar{r} - \sum_{j\notin B}\rho_j > 0$ from definition of $\mathcal{B}$ and feasible ordering, and

$$V_r(0,t) \leq \hat{V}_r(0,t).$$

We then have

$$\begin{aligned}
S_p(\tau,t) &\geq \frac{r(t-\tau) - r\hat{V}_r(0,t) + (r-\bar{r})\tau - \sum_{j\notin\mathcal{B}}\{\rho_j(t-\tau)+\sigma_j\}}{\sum_{j\in\mathcal{B}}\phi_j}\phi_p \\
&\geq \frac{r\left(t-\tau-\hat{V}_r(0,t-\tau)\right) + Y - \sum_{j\notin\mathcal{B}}\{\rho_j(t-\tau)+\sigma_j\}}{\sum_{j\in\mathcal{B}}\phi_j}\phi_p,
\end{aligned}$$

where $Y = r(\hat{V}_r(0,t-\tau) - \hat{V}_r(0,t) + \frac{r-\bar{r}}{r}\tau)$. Because $R(0,t)$ is convex function,

$$\frac{r-\bar{r}}{r} \geq \frac{\hat{V}_r(0,t) - \hat{V}_r(0,t-\tau)}{\tau}.$$

Thus we have

$$S_p(\tau,t) \geq \hat{S}_p(0,t-\tau).$$

□

A.3 Proof of Theorem 3.1

Suppose that for a session i busy period that begins at time τ. We consider $\tilde{t}$ such that

$$Q_p(\tilde{t}) = \max_{t\geq\tau} Q_i(t).$$

From Lemma3.4,

$$\begin{aligned}
Q_p(\tilde{t}) &= A_p(\tau,\tilde{t}) - S_p(\tau,\tilde{t}) && (107)\\
&\leq \hat{A}_p(0,\tilde{t}-\tau) - \hat{S}_p(0,\tilde{t}-\tau) && (108)\\
&= \hat{Q}_p(\tilde{t}). && (109)
\end{aligned}$$

Similarly, let $\tilde{t}$ denote the smallest time in that busy period such that

$$D_p(\tilde{t}) = \max_{t\geq\tau} D_i(t).$$

Definition of delay,

$$A_p(\tau,\tilde{t}) = S_p(\tau,\tilde{t}+D_p(\tilde{t})).$$

From Lemma3.4,

$$\begin{aligned}
& \hat{A}_p(0,t-\tau) - \hat{S}_p(0,t-\tau+D_p(t)) \geq A_p(\tau,t) - S_p(\tau,t+D_p(t)) = 0. && (110)\\
\Rightarrow\ & \hat{S}_p(0,t-\tau+\hat{D}_p(t-\tau)) \geq \hat{S}_p(0,t-\tau+D_p(t)). && (111)\\
\Rightarrow\ & \hat{D}_p(t-\tau)) \geq D_p(t). && (112)
\end{aligned}$$

Thus the maximum backlog and delay are upper-bounded the maximum ones under $\hat{A}$ and $\hat{r}(t)$. We have the results.

A.4 Proof of Theorem 3.3

Since both GPS and PGPS are work-conserving, their busy periods coincide, i.e., the GPS server is in a busy period if and only if the PGPS server is in a busy period. Hence, it suffices to prove the result for each busy period. Consider any busy period and let the time that it begins be time zero. Let $\tilde{p}_k$ denote the k^{th} packet in the busy period to depart under PGPS, and let L_k denote its length. Also, we denote the time when $\tilde{p}_k$ departs under PGPS and the time under GPS by $\tilde{d}_k$ and d_k, respectively. Finally, let a_k denote the time that $\tilde{p}_k$ arrives.

Let m denote the largest integer that satisfies both $0 < m \le k-1$ and $d_m > d_k$. We then have

$$d_m > d_k \ge d_i, \;\; for \; m < i < k. \tag{113}$$

Packet p_m is transmitted before packets $p_{m+1}, \ldots, p_k$ under PGPS but after all these packets under GPS. If no such integer m exists, then $m = 0$.

We now consider the case $m > 0$. For some given service process $r(t)$:

$$\int_{\tilde{s}_m}^{\tilde{d}_m} r(t)dt = L_m. \tag{114}$$

Let $\tilde{s}_k$ denote the time that $\tilde{p}_k$ begins to be served under PGPS. From Lemma3.6,

$$\min\{a_{m+1}, \ldots, a_k\} > \tilde{s}_m. \tag{115}$$

Thus $\tilde{p}_{m+1}, \ldots, \tilde{p}_{k-1}$ arrives after $\tilde{s}_m$ and departs before $\tilde{p}_k$. We then have

$$\int_{\tilde{s}_m}^{d_k} r(t)dt \ge \sum_{i=m+1}^{k} L_i. \tag{116}$$

From (114),

$$\int_{\tilde{d}_m}^{d_k} r(t)dt + L_m \ge \sum_{i=m+1}^{k} L_i. \tag{117}$$

In PGPS, $\tilde{p}_m, \ldots, \tilde{p}_k$ begins to be served in this order. We then have

$$\int_{\tilde{d}_m}^{\tilde{d}_k} r(t)dt = \sum_{i=m+1}^{k} L_i.$$

From (117),

$$\int_{d_k}^{\tilde{d}_k} r(t)dt \le L_m.$$

If $m = 0$, the amount of service in $(0, \tilde{d}_k]$ is $\sum_{i=0}^{k} L_i$ from definition of $\tilde{p}_k$. Suppose $d_k < \tilde{d}_k$. Because $m = 0$, $\tilde{p}_1, \ldots \tilde{p}_{k-1}$ depart before $\tilde{p}_k$ under GPS. This is contradiction. Thus

$$d_k \ge \tilde{d}_k.$$

We have the result.

A.5 Proof of Theorem 3.4

The increase rate of $\tilde{S}_i$ fluctuate between 0 when session i is not being served and $r(t)$ when a session i packet is being transmitted. Since the increase rate of S_i also obey these limits, the difference $S_i(0,t) - \tilde{S}_i(0,t)$ reaches its maximal value when session i packets begin transmission under PGPS. Let $\tilde{s}$ denote some such time, and d and $\tilde{d}$ the time that the packet departs under GPS and PGPS respectively. We denote the length of the packet going into service by L. Since session i packets are served in the same order under both schemes,

$$S_i(0,d) = \tilde{S}_i(0,\tilde{d}). \tag{118}$$

The packet are served between $(\tilde{s}, \tilde{d}]$:

$$\begin{aligned}\tilde{S}_i(\tilde{s}, \tilde{d}) &= L. &(119)\\ \Rightarrow \tilde{S}_i(0, \tilde{d}) &= \tilde{S}_i(0, \tilde{s}) + L. &(120)\end{aligned}$$

Thus we have

$$S_i(0, d) = \tilde{S}_i(0, \tilde{d}) = \tilde{S}_i(0, \tilde{s}) + L. \quad (121)$$

From Theorem3.3,

$$\int_d^{\tilde{d}} r(x)dx \leq L_{\max}. \quad (122)$$

From definition of $\tilde{s}$ and $\tilde{d}$,

$$\int_{\tilde{s}}^{\tilde{d}} r(x)dx = L. \quad (123)$$

From (122) - (123),

$$\int_d^{\tilde{s}} r(x)dx \leq L_{\max} - L. \quad (124)$$

We now consider maximal y such that $\int_y^{\tilde{s}} r(t)dt = L_{\max} - L$. Thus we have $y \leq d$ and

$$S_i(0, d) \geq S_i(0, y). \quad (125)$$

The increase rate of S_i is equal or less than $r(t)$:

$$\begin{aligned}S_i(y, \tilde{s}) &\leq \int_y^{\tilde{s}} r(x)dx = L_{\max} - L.\\ S_i(0, y) &\geq S_i(0, \tilde{s}) + L - L_{\max}.\end{aligned}$$

We then have

$$S_i(0, d) \geq S_i(0, \tilde{s}) + L - L_{\max}. \quad (126)$$

From (121),

$$S_i(0, \tilde{s}) - \tilde{S}_i(0, \tilde{s}) \leq L_{\max}.$$

B Proofs in section 4

B.1 Proof of Lemma 4.1

We first prove that for a session $j \in H_1$,

$$\frac{\phi_j}{\sum_{i \in I(m)} \phi_i} \sum_{i \in I(m)} \rho_i > \rho_j. \quad (127)$$

Note that

$$\frac{\phi_i^m}{\phi_j^m} \leq \frac{\rho_i^m}{\rho_j^m},$$

for any session $i \in I(m)$, by definition of H_1. Summing over $i \in I(m)$, we have (127).

We now prove that there exists $\bar{t}^m < \infty$ such that

$$\bar{t}^m = \inf\{\ t \mid \hat{r}^m(t) > \sum_{i \in I(m)} \rho_i\}.$$

If $\hat{r}^m(t) \leq \sum_{i \in I(m)} \rho_i$ for all times t, the busy period under $\hat{A}^m$ may not end in finite time. This is contradiction to Proposition 3.1. $\hat{r}^m(t)$ is monotone nondecreasing. Thus we have

$$\hat{t}^m \leq \bar{t}^m < \infty.$$

We have the result.

B.2 Proof of Lemma 4.4

Let $\tau_1, \ldots, \tau_{k+1}$ denote the minimizing values of (42). From definition,

$$\tau_{m+1} - \tau_m \leq t_m^B,$$

for each $m = 1, 2, \ldots, k$. For $t = \sum_{m=1}^k t_m^B$, we have $\tau_{m+1} - \tau_m = t_m^B$. Thus we have

$$\hat{S}_i^m(0, \tau_{m+1} - \tau_m) = \hat{S}_i^m(0, t_m^B) = \hat{A}_i(0, t_m^B).$$

The second equality follows from definition of t_m^B. We then have

$$G_i^k(t) = \sum_{m=1}^{k} \hat{S}_i^m(0, t_m^B) \quad (128)$$

$$= \sum_{m=1}^{k} \hat{A}_i^m(0, t_m^B) \quad (129)$$

$$\geq \hat{A}_i^m(0, \sum_{m=1}^{k} t_m^B). \quad (130)$$

B.3 Proof of Lemma 4.5

We prove this lemma by induction. For $m = 1$, (41) states that

$$S_i^1(\tau, t) \geq \min_{V \in [\tau, t]} \{A_i(\tau, t) + \hat{S}_i^1(0, t - V)\}.$$

Choosing V to be the last time in an interval $[\tau, t]$ that session i begins a busy period at node 1, we have

$$S_i^1(\tau, t) \geq A_i(\tau, V) + \hat{S}_i(0, t - V) \quad (131)$$

$$\geq \min_{V \in [\tau, t]} \{A_i(\tau, t) + \hat{S}_i^1(0, t - V)\}. \quad (132)$$

We now assume the result for nodes $1, 2, \ldots, m-1$. Let t_m denote the last time in the interval $[\tau, t]$ that session i is in an idle period at node m. We then have

$$S_i^m(\tau, t) = S_i^{m-1}(\tau, t_m) + S_i^m(t_m, t). \quad (133)$$

From the induction hypothesis,

$$S_i^{m-1}(\tau, t_m) \geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V)\}. \quad (134)$$

Since $[t_m, t]$ is in a busy period,

$$S_i^m(t_m, t) \geq \hat{S}_i^m(0, t - t_m). \quad (135)$$

Substituting (134) and (135) into (133), we have

$$S_i^m(\tau, t) \geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V)\} + \hat{S}_i^m(0, t - t_m) \quad (136)$$

$$\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V) + \hat{S}_i^m(0, t - t_m)\} \quad (137)$$

$$\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^m(t - V) \quad (138)$$

$$\geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(t - V). \quad (139)$$

□

B.4 Proof of Theorem 4.2

We first show (45). For a given set of arrival processes $A_1, \ldots, A_N$:

$$Q_i(t) = A_i(0,t) - S_i^K(0,t).$$

From Lemma 4.5, we have

$$\begin{aligned} Q_i(t) &\leq A_i(0,t) - \min_{V\in[0,t]}\{A_i(0,V) + G_i^K(t-V)\} &(140)\\ &= A_i(0,t) - A_i(0,V_{\min}) - G_i^K(t-V_{\min}), &(141)\end{aligned}$$

where V_{min} represents the minimizing value of V. We then have

$$\begin{aligned} Q_i(t) &\leq A_i(V_{\min},t) - G_i^K(t-V_{\min}) &(142)\\ &\leq \hat{A}_i(0,t-V_{\min}) - G_i^K(t-V_{\min}) &(143)\\ &\leq \max_{\tau\geq 0}\{\hat{A}_i(0,\tau) - G_i^K(\tau)\}. &(144)\end{aligned}$$

Thus (45) follows.

Next we show (45). For a given set of arrival processes $A_1, \ldots, A_N$ and $t \geq 0$, from Lemma 4.5, we have

$$S_i^K(0,t) \geq \min_{V\in[o,t]}\{A_i(0,V) + G_i^K(t-V)\}.$$

Thus we have

$$\begin{aligned} D_i(\hat{t}) &= \min\{t : S_i^K(0,t) = A_i(0,\hat{t})\} - \hat{t} &(145)\\ &\leq \min\{t : \min_{V\in[0,t]}\{A_i(0,V) + G_i^K(t-V)\} = A_i(0,\hat{t})\} - \hat{t} &(146)\\ &= \min\{t : A_i(0,V_{\min}) + G_i^K(t-V_{\min}) = A_i(0,\hat{t})\} - \hat{t} &(147)\\ &= \min\{t : G_i^K(t-V_{\min}) = A_i(V_{\min},\hat{t})\} - \hat{t} &(148)\\ &\leq \min\{t : G_i^K(t) = A_i(V_{\min},\hat{t})\} + V_{\min} - \hat{t} &(149)\\ &\leq \max_{\tau\geq 0}\{\min\{t : G_i^K(t) = \hat{A}_i(0,\tau)\} - \tau\}, &(150)\end{aligned}$$

for all $\hat{t}$. We have the results. □

B.5 Proof of Theorem 4.3

The following lemma establishes that for a (K,t)-fictitious worst-case regime, backlogs are not built up at node m prior to time T_m.

Lemma B.1 *Suppose a (K,t)-fictitious worst-case regime is given, which characterized by $(0, T_2, \ldots, T_K)$ for $t \leq B_K$ and a node $k \in \{1, 2, \ldots, K\}$. For each $j = 1, 2, \ldots, k-1$, and $\tau \in [T_j, T_{j+1}]$,*

$$A_i^k(0,\tau) = \sum_{m=1}^{j-1} \hat{S}_i^m(0, T_{m+1} - T_m) + \hat{S}_i^j(0, \tau - T_j). \tag{151}$$

and for $\tau > T_k$,

$$S_i^k(0,\tau) = \min\{\hat{A}_i(0,\tau), \sum_{m=1}^{k-1} \hat{S}_i^m(0, T_{m+1} - T_m) + \hat{S}_i^k(0, \tau - T_k)\}. \tag{152}$$

(Proof)

We proceed by induction on k: For $k = 1$ only (152) applies. Since $S_i^1 = \hat{S}_i$ the basis step is shown. Assume the result for nodes $1, 2, \ldots, k-1$. We show that $Q_i^k(\tau) = 0$ for all $\tau \leq T_k$, i.e. that

$$S_i^k(0,\tau) = S_i^{k-1}(0,\tau), \ \forall \tau \leq T_k. \tag{153}$$

The equation (151) follows from (153). Suppose (153) is false. We then have $Q_i^k(\tau) > 0$ for some $\tau \le T_k$. Since independent sessions at k are quiet during the interval $[0, T_k]$, it follows that there is at least one interval before T_k during which S_i^{k-1} has a slope greater than r^k. Since the slope of $\hat{S}_i^k$ is never greater than r_k for $t \in [0, t_k^B]$ and $T_1, T_2, \ldots, T_k$ are derived from the minimization of (42), it follows that $T_{k+1} - T_k = t_k^B$. We have shown in Proposition 3.1 that no node k busy period can be longer than t_k^B time units, so it follows that $Q_i^k(T_{k+1}) = 0$. Thus

$$S_i^k(0, T_{k+1}) = \hat{A}_i(0, T_{k+1}) = G_i^k(T_{k+1}) - Q_i^k(T_k),$$

where the first equality is from the induction hypothesis and the second equality follows directly from definition of G_i^k. Thus we have $G_i^k(T_{k+1}) \ge \hat{A}_i(0, T_{k+1})$ and $T_{k+1} = B_k$. Let $[a, a+\Delta]$, such that $\Delta > 0$ and $a + \Delta \le T_k$, denote an interval during which $S_i^{(k-1)}$ has the largest slope, and such that this slope belong to a single node $j < k$. As we have already argued, a slope of S_i^{k-1} during this interval must be greater than r_k, since $Q_i^k(\tau) > 0$ for some $\tau < T_k$. Thus the staggered greedy regime characterized by

$$\hat{T} = (0, T_2, \ldots, T_j - \Delta, \ldots, T_k - \Delta).$$

$\hat{T}$ is a $(k, T_{k+1} - \Delta)$-staggered greedy regime. For example,

$$\sum_{m=1}^{k} \hat{S}_i^m(0, \hat{T}_{m+1} - \hat{T}_m) = G_i^k(t - \Delta).$$

where $\hat{T}_{k+1} = T_{k+1} - \Delta$. Since $\hat{T}_{k+1} - \hat{T}_k = t_k$, it follows from similar reasoning as above that under $\hat{T}$ session i is not backlogged at k at time $\hat{T}_{k+1}$. We then have

$$\hat{T}_{k+1} = B_k.$$

Thus we have

$$\hat{T}_{k+1} = T_{k+1} - \Delta = B_k.$$

This implies that $\Delta = 0$, which is a contradiction and so (153) holds.

We are now left to show (152). Since $Q_i^k(T_k) = 0$, it is sufficient to establish that $\hat{S}_i^{k-1}(T_k, \tau) \le \hat{S}_i^k(0, \tau - T_k)$ for all $\tau \in [T_k, T_{k+1}]$. It is straightforward to argue that this must be true from the minimization of (42).

To show Theorem 4, pick $\tau = t > T_k$ in Lemma B.1.

B.6 Proof of Lemma 4.6

For $m = 1$, (50) states that

$$S_i^1(\tau, t) \ge \min_{V \in [\tau, t]} \{A_i(\tau, V) + \hat{S}_i^1(0, t - V)\} - L_i.$$

Choosing V to be the last time in an interval $[\tau, t]$ that session i begins a busy period at node 1,

$$\begin{aligned} S_i^1(\tau, t) &\ge A_i^1(\tau, V) + \hat{S}_i^1(0, t - V) && (154) \\ &\ge A_i(\tau, V) - L_i + \hat{S}_i^1(0, t - V) && (155) \\ &\ge \min_{V \in [\tau, t]} \{A_i(\tau, V) - L_i + \hat{S}_i^1(0, t - V)\}. && (156) \end{aligned}$$

Now assume the result for nodes $1, 2, \ldots, m-1$. Let t_m denote the last time in the interval $[\tau, t]$ that session i begins a busy period at node m, i.e.,

$$S_i^m(\tau, t) = A_i^m(\tau, t_m) + S_i^m(t_m, t).$$

From (49), we have

$$S_i^m(\tau, t) \ge S_i^{m-1}(\tau, t_m) - L_i + S_i^m(t_m, t). \quad (157)$$

By induction hypothesis,

$$S_i^{m-1}(\tau, t_m) \ge \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V) - (m-1)L_i\}. \quad (158)$$

Since session i is busy in the interval $[t_m, t]$ at m,

$$S_i^m(t_m, t) \geq \hat{S}_i^m(0, t - t_m). \tag{159}$$

Substituting (158) and (159) into (157), we have

$$\begin{aligned} S_i^m(\tau, t) + mL_i &\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V)\} + \hat{S}_i^m(0, t - t_m) && (160) \\ &\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^m(t - V)\} && (161) \\ &\geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(t - V)\}, && (162) \end{aligned}$$

by definition of G_i^m.

B.7 Proof of Theorem 4.5

For a given set of arrival processes $A_1, \ldots, A_N$,

$$Q_i(t) = A_i(0, t) - S_i^K(0, t).$$

From Lemma 4.6, we have

$$\begin{aligned} Q_i(t) - KL_i &\leq A_i(0, t) - \min_{V \in [0, t]} \{A_i(0, V) + G_i^K(t - V)\} && (163) \\ &\leq A_i(0, t) - A_i(0, V_{\min}) + G_i^K(t - V_{\min}), && (164) \end{aligned}$$

where $V_{\min}$ represents the minimizing value of V. Thus we have

$$\begin{aligned} Q_i(t) - KL_i &\leq A_i(V_{\min}, t) - G_i^K(t - V_{\min}) && (165) \\ &\leq \hat{A}_i(0, t - V_{\min}) - G_i^K(t - V_{\min}) && (166) \\ &\leq \max_{\tau \geq 0} \{\hat{A}_i(0, \tau) - G_i^K(\tau)\}. && (167) \end{aligned}$$

We have (51).

Next, we show (52). We restrict our value of t to be such that a session i packet, p_m, departs node m at time t. Given such a time t, let the corresponding packet arrive at time a_m, i.e.,

$$A_i(0, a_m) = S_i^m(0, t).$$

Note that a_m is also the time that packet arrives at node 1, i.e.,

$$A_i^1(0, a_m) = A_i(0, a_m). \tag{168}$$

We will show that for each node m, $1 \leq m \leq K$:

$$S_i^m(\tau, t) \geq \min_{V \in [\tau, a_m]} \{A_i(\tau, V) + G_i^m(0, t - V)\} - (m - 1)L_i. \tag{169}$$

Note that the last session i busy period at node m before t cannot start at time strictly greater than the time that p_m arrives at node m. We now use induction on m. For $m = 1$, we are done by (154) and (168). (Session i is busy at node 1 in $[a_1, t]$.)

In the inductive step, proceed as in the proof of Lemma 4.6. From the induction hypothesis:

$$S_i^{m-1}(\tau, t_m) \geq \min_{V \in [\tau, a_{m-1}]} \{A_i(\tau, V) + G_i^{m-1}(0, t - V)\} - (m - 2)L_i, \tag{170}$$

where t_m represents the beginning of the last session i busy period at node m in the interval $[\tau, t]$. a_{m-1} represents the arrival time of the packet (call it p_{m-1}) that departs node $m - 1$ at time t_m. Note that such a packet must exist for a busy period to begin at node m at time t_m. Now since p_{m-1} cannot arrive at node m after p_m does. Since the relative order o the packets is preserved over all links, it follows that

$$a_{m-1} \leq a_m. \tag{171}$$

Now proceeding analogously to the proof of Lemma 4.6:

$$
\begin{aligned}
S_i^m(\tau,t)+(m-1)L_i &\geq \min_{V\in[\tau,a_{m-1}]}\{A_i(\tau,V)+G_i^{m-1}(t_m-V)\}+\hat{S}_i^m(0,t-t_m) && (172)\\
&\geq \min_{V\in[\tau,a_{m-1}]}\{A_i(\tau,V)+G_i^{m-1}(t_m-V)+\hat{S}_i^m(0,t-t_m)\} && (173)\\
&\geq \min_{V\in[\tau,a_{m-1}]}\{A_i(\tau,V)+G_i^{m}(t_m-V)\} && (174)\\
&\geq \min_{V\in[\tau,a_{m}]}\{A_i(\tau,V)+G_i^{m}(t_m-V)\}, && (175)
\end{aligned}
$$

by definition of G_i^m.

For a given set of arrival and service processes, $A_1,\ \ldots,\ A_N,\ r^1(t),\ \ldots,\ r^K(t)$ and t such that a packet departs node K at time $t \geq 0$, we have :

$$S_i^{(K)} \geq \min_{V\in[0,\hat{t}]}\{A_i(0,V)+G_i^K(t-V)\}-(K-1)L_i,$$

where the packet departing at time t arrived at time $\hat{t}$. Thus for all packet arrival times $\hat{t}\geq 0$:

$$
\begin{aligned}
D_i(\hat{t}) &= \min\{t: S_i^{(K)}(0,t)=A_i(0,\hat{t})\}-\hat{t} && (176)\\
&\leq \min\{t: \min_{V\in[0,\hat{t}]}\{A_i(0,V)+G_i^K(t-V)-(K-1)L_i\}=A_i(0,\hat{t})\}-\hat{t} && (177)\\
&\leq \min\{t: A_i(0,V_{\min})+G_i^K(t-V_{\min})=A_i(0,\hat{t})+(K-1)L_i\}-\hat{t} && (178)\\
&\leq \min\{t: G_i^K(t)=A_i(V_{\min},\hat{t})+(K-1)L_i\}+V_{\min}-\hat{t} && (179)\\
&\leq \min\{t: G_i^K(t)=\hat{A}_i(0,\hat{t}-V_{\min})+(K-1)L_i\}-(\hat{t}-V_{\min}) && (180)\\
&\leq \max_{\tau\geq 0}\{\min\{t: G_i^K(t)=\hat{A}_i(0,\tau)+(K-1)L_i\}-\tau\}. && (181)
\end{aligned}
$$

B.8 Proof of Theorem 4.6

From Theorem 3.3, the completion of a packet arrival time under PGPS is delayed by at most $\inf\{\ t \mid \hat{R}^m(0,t)\geq L_i\}$. We now define $\hat{Y}^m=\inf\{\ t\mid \hat{R}^m(0,t)\geq L_i\}$. We then have the following lemma.

Lemma B.2 *Given the same set of arrival processes at a node, m*

$$\tilde{S}_i^m(0,t)\geq S_i^m(0,t-\hat{Y}_i^m). \qquad (182)$$

Lemma B.3 *Suppose arrival processes $A_1,\ldots,A_N$ at a single GPS server are given, such that for a particular session i, the k^{th} session i packet arrives at time a_k, and has length $l_k < L_i$. Replace A_i with $\bar{A}_i$, such that for all k:*

$$\bar{l}_k = l_k,$$

and

$$\bar{a}_k \geq a_k,$$

where $\bar{l}_k$ and $\bar{a}_k$ represent the length and arrival time respectively, of the k^{th} session packet under the new regime. If f_k represents the time that the k^{th} session i packet is served under the old regime and $\bar{f}_k$ represents the time it is served under the new regime, we have:

$$\bar{f}_k \geq f_k,$$

for all k.

(Proof)

This proof is the same as in the constant rate. See [3].

□

Now suppose a PGPS network with arrival processes $A_1,\ldots,A_N$ is given. We would like to bound delay for a particular session i. Without loss of generality, assume that $P(i)=\{1,2,\ldots,K\}$. The arrival functions A_j^m, for each m, $1\leq m\leq K$, and $j\in I(m)$ are completely determined and are assumed to be known.

Construct a GPS network consisting of nodes $1, 2, \ldots, K$ connected in a line, i.e., the links are given by $\{(e, e+1) := 1, 2, \ldots, K-1\}$. The rates of the links are the same as the corresponding links in the PGPS network, but the link leaving node m has a fixed propagation delay of $\hat{Y}_i^m$. The GPS network supports a session i, with route $1, 2, \ldots, K$ and the arrival function A_i, i.e., the route and arrival processes are identical to those in the PGPS network. The other sessions on the GPS network have a route of exactly one hop and are defined as follows. At each node m, for every session j in the PGPS network define a session j' such that

$$A_{j'}^m(0,t) = A_j^m(0,t).$$

Now for each node, m, let $\tilde{S}_i^m$ describe the session i departures from node m in the PGPS network and S_i^m describe the session i departures from node m in the corresponding GPS network. We them have the following result.

Lemma B.4 *For* $k = 1, 2, \ldots, K$

$$S_i^k(0, t - \hat{Y}_i^k) \leq \tilde{S}_i^k(0,t),$$

for all t.

(Proof)

We use induction on k. For $k = 1$ we are done from Lemma B.2. Assume the result at nodes $1, 2, \ldots, k-1$ and show it at node $k \leq K$: We first consider the service function $\bar{S}_i^k$ that results at node m, if session i arrives at node k in the GPS network are identical to session i arrivals at node k in the PGPS network. We then have

$$\bar{S}_i^k(0, t - \hat{Y}_i^k) \leq \tilde{S}_i^k(0,t). \tag{183}$$

By the induction hypothesis, we have

$$S_i^{k-1}(0, t - \hat{Y}_i^{k-1}) \leq \tilde{S}_i^{k-1}(0,t), \tag{184}$$

for all t. The LHS of (184) describes the traffic that has traversed link $(k-1, k)$ in the GPS network in the interval $[0, t]$. Thus every session i packet arrives at node k earlier in the PGPS network, than it does in the GPS network. From Lemma B.3, we have

$$S_i^k(0, t - \hat{Y}_i^k) \leq \bar{S}_i^k(0, t - \hat{Y}_i^k), \tag{185}$$

for all t. From (185) and (183),

$$S_i^k(0, t - \hat{Y}_i^k) \leq \tilde{S}_i^k(0,t).$$

We have the result. □

We now assume a fixed network topology with no propagation delay. Also assume a fixed internal characterization for all the sessions. Let D_i^* denote the worst-case session i delay when the nodes have GPS servers and let $\tilde{D}_i^*$ denote the worst-case session i delay when the nodes have PGPS servers. We then have

$$\tilde{D}_i^* \leq D_i^* + \sum_{m=1}^{K} \hat{Y}_i^m.$$

Note that the GPS network being considered here has internal characterization identical to the PGPS network. Now using bounds in Theorem 4.2, we have the result.

C Proofs in section 5

C.1 Proof of Lemma 5.1

(65) is equivalent to $\sum_{j=1}^{i} \eta_j(t) \leq 0$. We prove this inequality by induction on i. When $i = 1$, let τ denote the beginning of a session 1 busy period that contains t, then $Q_1(\tau) = 0$ and $\eta_1(\tau) = -\delta_1(\tau) \leq 0$. From definition of GPS and the fact that $r_1 < \phi_1 / \sum_{j=1}^{N} \phi_j$, we have

$$S_1(\tau, t) \geq \frac{\phi_1}{\sum_{j=1}^{N} \phi_j} \int_{\tau}^{t} r(y)dy. \tag{186}$$

From (59),

$$\int_\tau^t r_1(y)dy = \frac{r_1}{r}\int_\tau^t r(y)dy \tag{187}$$

$$< \frac{\phi_1}{\sum_{j=1}^N \phi_j}\int_\tau^t r(y)dy. \tag{188}$$

From (64),

$$\eta_1(t) \leq \eta_1(\tau) + \int_\tau^t r_1(y)dy - S_1(\tau,t) \tag{189}$$

$$\leq \eta_1(\tau) + (\frac{r_1}{r} - \frac{\phi_1}{\sum_{j=1}^N \phi_j})\int_\tau^t r(y)dy \leq 0. \tag{190}$$

We now assume that the lemma is true for $1, 2, \ldots, i-1$. We show that it is also true for i. First, if $\eta_i(t) \leq 0$, then by the induction hypothesis, the claim follows easily. The case where $\eta_i(t) > 0$ is a bit harder. Note that $\eta_i(t) > 0$ implies that $Q_i(t) > \delta_i(t) \geq 0$. Let τ denote the beginning of a session i busy period that contains t, thus $Q_i(\tau) = 0$, and $\eta_i(\tau) = -\delta_i(\tau) \leq 0$. From (63),

$$S_i(\tau,t) \leq \int_\tau^t r_i(y)dy - \eta_i(t) < \int_\tau^t r_i(y)dy. \tag{191}$$

Let $x > 0$ be such that

$$S_i(\tau,t) = \int_\tau^t r_i(y)dy - x. \tag{192}$$

From (59) and (192), we have

$$S_i(\tau,t) = \int_\tau^t r_i(y)dy - x \tag{193}$$

$$< \frac{r_i}{r}\int_\tau^t r(y)dy - x \tag{194}$$

$$< \frac{\phi_i}{\sum_{j=i}^N \phi_j}(1 - \frac{\sum_{j=1}^{i-1} r_j}{r})\int_\tau^t r(y)dy - x. \tag{195}$$

$$\tag{196}$$

Moreover, by definition of GPS, for any j,

$$S_i(\tau,t) \geq \frac{\phi_i}{\phi_j} S_j(\tau,t). \tag{197}$$

$$\Rightarrow \sum_{j=i}^N S_j(\tau,t) \leq \frac{\sum_{j=i}^N \phi_j}{\phi_i} S_i(\tau,t). \tag{198}$$

Using (195), we have

$$\frac{\phi_i}{\sum_{j=i}^N \phi_j}\sum_{j=i}^N S_j(\tau,t) \leq \frac{\phi_i}{\sum_{j=i}^N \phi_j}(1 - \frac{\sum_{j=1}^{i-1} r_j}{r})\int_\tau^t r(y)dy - x. \tag{199}$$

$$\Rightarrow \sum_{j=i}^N S_j(\tau,t) \leq (1 - \frac{\sum_{j=1}^{i-1} r_j}{r})\int_\tau^t r(y)dy - x \tag{200}$$

$$\leq (1 - \frac{\sum_{j=1}^{i-1} r_j}{r})\int_\tau^t r(y)dy - x. \tag{201}$$

Since $\sum_{j=1}^{i-1} S_j(\tau,t) + \sum_{j=i}^{N} S_j(\tau,t) = \int_\tau^t r(y)dy$,

$$\int_\tau^t r(y)dy - \sum_{j=1}^{i-1} S_j(\tau,t) \leq (1 - \frac{\sum_{j=1}^{i-1} r_j}{r}) \int_\tau^t r(y)dy - x. \tag{202}$$

$$\Rightarrow \sum_{j=1}^{i-1} S_j(\tau,t) \geq \frac{\sum_{j=1}^{i-1} r_j}{r} \int_\tau^t r(y)dy + x. \tag{203}$$

From (192),

$$\sum_{j=1}^{i} S_j(\tau,t) \geq \frac{\sum_{j=1}^{i} r_j}{r} \int_\tau^t r(y)dy. \tag{204}$$

We have

$$\eta_j(t) \leq \eta_j(\tau) + \int_\tau^t r_j(y)dy - S_j(\tau,t).$$

for $1 \leq j \leq i$. Summing over j,

$$\sum_{j=1}^{i} \eta_j(t) \leq \sum_{j=1}^{i} \eta_j(\tau) + \sum_{j=1}^{i} \int_\tau^t r_j(y)dy - \sum_{j=1}^{i} S_j(\tau,t) \leq \sum_{j=1}^{i} \eta_j(\tau) = \sum_{j=1}^{i-1} \eta_j(\tau) + \eta_i(\tau) \leq 0.$$

We have the result.

C.2 Proof of Lemma 5.2

As session i is busy in $[\tau,t]$, from definition of GPS, we can easily derive that

$$S_i(\tau,t) \geq \frac{\phi_i}{\sum_{j=i}^{N} \phi_j} (\int_\tau^t r(x)dx - \sum_{j=1}^{i-1} S_j(\tau,t)). \tag{205}$$

From (64) and Lemma 5.1, we have

$$\sum_{j=1}^{i-1} S_j(\tau,t) \leq \sum_{j=1}^{i-1} \int_\tau^t r_j(y)dy + \sum_{j=1}^{i-1} \delta_j(t).$$

Substituting the above inequality into (205),

$$S_i(\tau,t) \geq \frac{\phi_i}{\sum_{j=i}^{N} \phi_j} (\int_\tau^t r(x)dx - \sum_{j=1}^{i-1} \int_\tau^t r_j(y)dy - \sum_{j=1}^{i-1} \delta_j(t)) \tag{206}$$

$$= \frac{\phi_i}{\sum_{j=i}^{N} \phi_j} \frac{r - \sum_{j=1}^{i-1} r_j}{r} \int_\tau^t r(y)dy - \frac{\phi_i}{\sum_{j=i}^{N} \phi_j} \sum_{j=1}^{i-1} \delta_j(t). \tag{207}$$

(208)

From (59), we have the result. □

C.3 Proof of Lemma 5.3

Let τ denote the beginning of a session i busy period that contains t. As $Q_i(t) = A_i(\tau,t) - S_i(\tau,t)$, from (62) and Lemma 5.2, we obtain the result. □

C.4 Proof of Lemma 5.5

It is well known that $\delta_i(t)$ can be expressed in the form,

$$\delta_i(t) = \max_{\tau \le t}\{A_i(\tau,t) - R_i(\tau,t)\}.$$

Let τ denote the beginning of a session i busy period that contains time t. We have $\delta_i(t) = A_i(\tau,t) - R_i(\tau,t)$. We now use the similar technique as in [11] based on busy period analysis and union bound approximation. We introduce a small discritization parameter $\xi > 0$. Note that there exists some k which satisfy $t - k\xi < \tau \le t - (k-1)\xi$.

From union-bound approximation, for any $p(0 < p < 1)$,

$$\begin{aligned} Pr(A_i(\tau,t) - R_i(\tau,t) \ge x) &\le \sum_{k=1}^{\infty} Pr(A_i(t-k\xi,t) - R_i(t-(k-1)\xi,t) \ge x) \\ &\le \sum_{k=1}^{\infty} Pr(A_i(t-k\xi,t) \ge \rho_i k\xi + pY) \\ &\quad + \sum_{k=1}^{\infty} Pr(R_i(t-(k-1)\xi,t) \le r_i(k-1)\xi - (1-p)Y), \end{aligned}$$

where $Y = x + \epsilon_i(k-1)\xi - \rho_i\xi$. We have

$$Pr(\delta_i(t) \ge x) \le \sum_{k=1}^{\infty} \Lambda_i e^{-\alpha_i pY} + M_i e^{-\beta_i(1-p)Y}.$$

Equating $\alpha_i p = \beta_i(1-p) = \gamma_i$ gives:

$$\begin{aligned} Pr(\delta_i(t) \ge x) &\le \sum_{k=1}^{\infty}(\Lambda_i + M_i)e^{-\gamma_i Y} \\ &\le \sum_{k=1}^{\infty}(\Lambda_i + M_i)e^{-\gamma_i(x+\epsilon_i(k-1)\xi-\rho_i\xi)} \\ &\le \frac{(\Lambda_i + M_i)e^{\gamma_i\rho_i\xi}}{1 - e^{-\gamma_i\epsilon_i\xi}} e^{-\gamma_i x}. \end{aligned}$$

We have the result.

C.5 Proof of Lemma 5.6

Let τ denote the beginning of a session i busy period that contains time t. Note that $\delta_i(t) = A_i(\tau,t) - \int_\tau^t r_i(u)du$.

We derive the moment generating function of $\delta_i(t)$, i.e., $E[e^{\theta\delta_i(t)}]$ for some θ.

Define $X_i = A_i(\tau,t) - \rho_i(t-\tau)$. We have

$$\begin{aligned} E[e^{\theta X_i}] &= E[e^{\theta X_i}|X > 0] + E[e^{\theta X_i}|X \le 0] \\ &\le E[e^{\theta X_i}|X > 0] + 1. \end{aligned}$$

Note that

$$\begin{aligned} &Pr(X_i \ge x) \le \Lambda_i e^{-\alpha_i x},\ x > 0. \\ \Rightarrow\ &Pr(e^{\theta X_i} \ge e^{\theta x}) \le \Lambda_i e^{-\alpha_i x},\ x > 0. \end{aligned}$$

Denote $e^{\theta x}$ as y.

$$Pr(e^{\theta X_i} \ge y) \le \Lambda_i y^{-\frac{\alpha_i}{\theta}}, y > 1.$$

We then have

$$E[e^{\theta X_i}|X > 0] = \int_1^{\infty} Pr(e^{\theta X_i} \ge y)dy. \tag{209}$$

Thus we have

$$E[e^{\theta A_i(\tau,t)}] \le e^{\theta(\rho_i(t-\tau)+\hat{\sigma}_i(\theta))},$$

where $\hat{\sigma}_i(\theta) = \frac{1}{\theta}\log(1+\frac{\theta\Lambda_i}{\alpha_i-\theta})$. We now consider output process.

$$Pr(R_i(\tau,t) \le r_i(t-\tau)-x) \le M_i e^{-\beta_i x}. \tag{210}$$
$$\Rightarrow Pr(r_i(t-\tau)-R_i(\tau,t) \ge x) \le M_i e^{-\beta_i x}. \tag{211}$$

Similarly, we denote $r_i(t-\tau)-R_i(\tau,t)$ as X. We have

$$E[e^{-\theta R_i(\tau,t)}] \le e^{-r_i\theta(t-\tau)+\eta_i(\theta)},$$

where $\eta_i(\theta) = \frac{1}{\theta}\log(1+\frac{M_i\theta}{\beta_i-\theta})$. We then have

$$E[e^{\theta\delta_i}] \le \sum_{k=1}^{\infty} E[e^{\theta(A_i(t-k,t)-R_i(t-(k-1),t))}] \tag{212}$$
$$\le \sum_{k=1}^{\infty} e^{\theta(\rho_i k+\hat{\sigma}(\theta)-r_i(k-1)+\hat{\eta(\theta)})} \tag{213}$$
$$= \frac{e^{\theta(\hat{\sigma}(\theta)+\hat{\eta}(\theta)+\rho_i)}}{1-e^{\theta\xi(\epsilon_i)}}. \tag{214}$$

□

C.6 Proof of Theorem 5.4

From Lemma 5.3 and 5.5, (81) holds. We show that (82) holds.

From Lemma 5.2, for any $t > 0$

$$S_i(\tau,t) \ge R_i(\tau,t), \tag{215}$$

where let τ denote the beginning of a session i busy period that contains t. We can obtain the distribution of the session i delay. Note that

$$Pr\,(D_i(t) \ge d) = Pr(A_i(\tau,t)-R_i(\tau,t+d) \ge 0).$$

From union-bound approximation, for any $p(0<p<1)$,

$$\begin{aligned} & Pr(A_i(\tau,t)-R_i(\tau,t+d)\ge 0) \\ \le & \sum_{k=1}^{\infty} Pr(A_i(t-k,t)-R_i(t-(k-1),t+d)\ge 0) \\ \le & \sum_{k=1}^{\infty} Pr(A_i(t-k,t)\ge \rho_i k+pY) + Pr(R_i(t-(k-1),t)\le r_i(d+k-1)-(1-p)Y), \end{aligned}$$

where $Y = r_i d - \rho_i\epsilon_i(k-1)$. We have

$$Pr\,(D_i(t)\ge d) \le \sum_{k=1}^{\infty} \Lambda_i e^{-\alpha_i pY} + M_i e^{-\beta_i(1-p)Y}.$$

Equating $\alpha_i p = \beta_i(1-p) = \gamma_i$ gives:

$$\begin{aligned} Pr\,(D_i(t)\ge d) &\le \sum_{k=1}^{\infty}(\Lambda_i+M_i)e^{-\gamma_i Y} \\ &\le \sum_{k=1}^{\infty}(\Lambda_i+M_i)e^{-\gamma_i(r_i d-\rho_i\epsilon_i(k-1))} \\ &\le \frac{(\Lambda_i+M_i)e^{\gamma_i\rho_i}}{1-e^{-\gamma_i\epsilon_i}}e^{-\gamma_i r_i d}. \end{aligned}$$

We have the result.