Approximate Formulas for the Cell Loss Probability in Finite-Buffer Queues with Correlated Input

Guidance

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Abstract

It is widely recognized that the Internet and video traffic indicate very bursty characteristics. The queueing performance with this kind of traffic is likely to be different from that with the conventional Poisson traffic. From these points, analyzing and quantitatively evaluating the performance of queues with correlated input are of great importance. In particular, the loss probability in finite-buffer queues is the most important performance measure of interest.

In this thesis, we consider discrete-time single-server queues fed by correlated input and study the cell loss probability in finite-buffer queues. We study queues with two different types of input in detail. One is the long-range dependent (LRD) input, which has a subexponentially decaying autocorrelation function. The other is the generalized discrete-time autoregressive input, which is a generalization of the discrete-time autoregressive process of order one (DAR(1)) input and has a geometrically decaying autocorrelation function.

In the LRD input case, we propose an approximate formula for the cell loss probability in finite-buffer queues with LRD input. The approximate formula is constructed by studying an analytically tractable queueing model with LRD input, and it is asymptotically exact in this special case. Through numerical examples, the accuracy and robustness of the approximate formula are discussed throughly, and it is shown that the order of magnitude of the cell loss probability can be well estimated with the approximate formula.

In the generalized discrete-time autoregressive input case, we study queues fed by the generalized discrete-time autoregressive input and construct an approximate formula for the cell loss probability in finite-buffer queues with DAR(1) input, where we propose how to estimate the asymptotic decay rate of the cell loss probability. Through numerical examples, the accuracy of the approximate formula is examined, and it is confirmed that the formula works well when the offered load is high or the correlation of the input is weak.

Contents

1 Introduction

It is widely recognized that the Internet traffic indicates very bursty characteristics so-called long-range dependence or self-similarity, e.g., see [21]. The queueing performance with this kind of traffic is likely to be different from that with the conventional Poisson traffic [15, 19]. Commonly used mathematical models for long-range dependent (LRD) and self-similar traffic are fractional Brownian motion, M/G/∞ input, and on/off source models with subexponential active periods [20], and many research papers have been published for the last decade [20, 22, 23]. Note here that most of those considered infinite-buffer queues and discussed the tail distributions of queue length and waiting time, e.g., [2, 22, 23]. In practice, however, the buffer size is finite and in these circumstances, the loss probability is the most important performance measure of interest.

Some efforts have been made to obtain the loss probability in finite-buffer queues with LRD input. Using the theory of large deviations, Likhanov and Mazumdar studied the asymptotic cell loss probability in a finite-buffer queue when the number of sources and the service rate go to infinity, while their ratio remains constant [13]. Tsybakov and Georganas obtain upper and lower bounds for the cell loss probability in a finite-buffer queue with M/G/∞ input [26]. The M/G/∞ input process is a vital traffic modeling tool because it is very flexible in *time domain*, i.e., the M/G/∞ input process can represent any autocorrelation function with respect to the numbers of arrivals if it is decreasing and convex [11]. However, the distribution of the number of arrivals in unit time is restricted to the family of Poisson distributions, and therefore the variation in *space domain* cannot be incorporated into the model. See also [10]. Zwart studies the asymptotic loss fraction in finite-buffer fluid queues when the buffer size goes to infinity [27]. See [14] also. To the best of our knowledge, however, there exists no handy formula to evaluate the loss probability in finite-buffer queues with LRD input.

This thesis develops a closed-form approximate formula for the cell loss probability in finite-buffer queues with LRD input. The past study on queues with LRD input and queues with heavy-tailed components show that the performance of those queues is determined mainly by the characteristics of LRD and heavy-tailed components, and other stochastic factors play a minor role. Thus we expect that the cell loss probability in queues with LRD input would be determined mainly by the correlation structure in the LRD input process. In other words, a loss probability formula in a specific queue with LRD input would be applicable to other queues, if the input processes of those queues have common characteristics.

Based on this observation, we study a discrete-time queue with a somewhat peculiar LRD input that is characterized by the distribution of the number of arrivals in unit time and two parameters related to the asymptotics of the autocorrelation function with respect to the number of arrivals. Even though this LRD input process is artificial, we can exactly analyze the stationary distribution of buffer contents, both in the finite-buffer and infinite-buffer queues. To the best of our knowledge, this LRD input model is the first one having such a feature. Further, we derive a closed-form asymptotic formula for the cell loss probability, combining the asymptotic tail distribution of buffer contents in the infinite-buffer queue [25] with the relationship between the stationary distributions of buffer contents in a finite-buffer queue and the corresponding infinite-buffer queue [9]. As you will see, the formula is given in terms of up to the second order statistics of the input process that can be readily computed. We then propose using this formula as an approximate formula for the cell loss probability in queues with LRD input.

The accuracy and robustness of the approximate formula are investigated by numerical experiments. Because the approximate formula is derived based on the asymptotic cell loss probability for large buffer, we first examine to what extent the approximate formula is accurate for relatively small buffer systems, comparing with the exact result. Next we apply the approximate formula to queues with LRD on/off sources, $M/G/\infty$ input, and their generalized ones.

It is also interesting to consider the input that has a geometrically decaying autocorrelation function and a queue fed by this kind of input. For a few decades in the past, some input models with a geometrically decaying autocorrelation function were proposed and studied. One of the useful traffic models is the discrete-time autoregressive process of order one (DAR(1)). The DAR(1) processes have been used as a model of video traffic [6] and queues with DAR(1) input were studied [7, 8]. The DAR(1) process has the following futures. In *space domain*, it can represent any distribution of the number of arrivals in unit time. On the other hand, in *time domain*, the autocorrelation function is determined by one parameter and it decays geometrically.

Hwang, Choi, and Kim studied the tail of the waiting time distribution with the theory of the GI/G/1 queue [7]. Hwang and Sohraby studied the queue length distribution and the mean queue length [8]. To the best of our knowledge, they considered and examined the infinite-buffer queues. In practice, however, the buffer size is finite, and in these circumstances, the cell loss probability is the most important measure.

In this thesis, we generalize the DAR(1) process and study finite buffer queues with this input. In what follows, we call this input generalized discrete-time autoregressive input. Note here that the generalized discrete-time autoregressive input includes DAR(1) input as a special case and can represent any marginal distribution, which is similar to DAR(1). Further its autocorrelation function is given by discrete phase-type (PH) distribution whose number of parameters can be greater than one, while the number of parameters of DAR(1) is limited to only one. Thus, it is worth considering and studying the generalized discrete-time autoregressive input.

As in the case with LRD input, we consider finite buffer queues with generalized discrete-time autoregressive input and derive the cell loss probability using geometric asymptotics of Markov chains of M/G/1 type [25] with the relationship between the stationary distributions of buffer contents in a finite-buffer queue and the corresponding infinite-buffer queue [9]. Further, we consider the special case where the number of phases of discrete PH distribution of the input is equal to one, that is, DAR(1). We construct the approximate formula for the cell loss probability for DAR(1) input, where an approximate decay rate of the cell loss probability is proposed. Through numerical examples, the accuracy of the formula is examined.

There are some common points in deriving approximate formulas for queues with LRD input and with the generalized discrete time autoregressive input. Thus we first derive the cell loss probability that is applicable to both cases. Next we study queues with LRD input and construct an approximate formula for the cell loss probability in finite-queues with LRD input. We then consider queues fed by the generalized discrete-time autoregressive input and derive the approximate formula for queues with DAR(1) input.

The rest of this thesis is organized as follows. In section 2, we explain the discrete-time single-server queue and the input process. In section 3, we derive the cell loss probability in general settings. In section 4, we propose the approximate formula for the cell loss probability in queues with LRD input. In section 5, we study the cell loss probability with generalized discrete-time autoregressive input and construct the approximate formula for queues with DAR(1) input. Finally, we conclude this thesis in section 6.

2 Preliminaries

In this section, we explain the discrete-time single-server queue and the terms and mathematical definitions of the input process.

2.1 Discrete-Time Single-Server Queue

Throughout this thesis, we consider a stationary discrete-time single-server queue with a buffer of finite capacity N. We assume that the time axis is divided into slots with equal length. Let B_n $(n = 1, 2, \ldots)$ and $Q_n^{(N)}$ $(n = 0, 1, \ldots)$ denote the numbers of cells arriving to the queue and present in the queue, respectively, at the nth slot. Given $Q_0^{(N)}$, $Q_n^{(N)}$ is assumed to be determined by the following recursion: for $n = 1, 2, \ldots,$

$$Q_n^{(N)} = \min\left(\left(Q_{n-1}^{(N)} - 1\right)^+ + B_n, N\right), \tag{1}$$

where $(x)^+$ stands for $\max(x, 0)$. Let $L_n^{(N)}$ $(n = 1, 2, \ldots)$ denote the number of cells lost in the nth slot, i.e.,

$$L_n^{(N)} = \left(\left(Q_{n-1}^{(N)} - 1\right)^+ + B_n - N\right)^+, \quad n = 1, 2, \ldots.$$

We then define $P_{\text{loss}}^{(N)}$ as the cell loss probability when the buffer size is equal to N:

$$P_{\text{loss}}^{(N)} = \lim_{m\to\infty} \frac{\sum_{n=1}^{m} L_n^{(N)}}{\sum_{n=1}^{m} B_n},$$

where the limit is assumed to exist with probability 1. In this thesis, we mainly study the cell loss probability $P_{\text{loss}}^{(N)}$.

2.2 Autocorrelation Function of Correlated Input

In this subsection, we introduce the important and useful indicators of input processes.

We define B as a generic random variable for B_n's. Further let $\gamma(k)$ $(k = 1, 2, \ldots)$ denote the autocorrelation function of B_n's $(n = 1, 2, \ldots)$:

$$\gamma(k) = \frac{\text{Cov}[B_n B_{n+k}]}{\text{Var}[B]}, \qquad k = 0, 1, \ldots.$$

It is well known that there exist two quite different types of correlation of the input. One is the input with the subexponentially decaying autocorrelation function and the other is the input with the geometrically decaying autocorrelation function.

We say that the autocorrelation function $\gamma(k)$ of the input is subexponentially decaying if the input process $\{B_n; n = 0, 1, \ldots\}$ satisfies

$$\lim_{k\to\infty} \frac{\gamma(k)}{k^{-\theta}} = \alpha,$$

where $\alpha > 0$. In particular, if θ satisfies $0 < \theta < 1$, the input process is called LRD input [3]. We study queues with LRD input in section 4.

On the other hand, we say that the autocorrelation function $\gamma(k)$ of the input is geometrically decaying, if the input process $\{B_n; n = 0, 1, \ldots\}$ satisfies

$$\lim_{k\to\infty} \frac{\gamma(k)}{\sigma^k} = c_g,$$

where $0 < \sigma < 1$ and c_g is some positive number. We call σ asymptotic decay rate. One example of this type of input is the generalized discrete-time autoregressive input. We study queues with this input process in section 5.

The autocorrelation function of the input represents a feature in *time domain*. On the other hand, the marginal distribution of the input expresses a property in *space domain*. Generally speaking, the characteristics of the input process are mainly expressed by these measures.

3 Correlated Input Model and General Analysis

In this section, we consider a discrete-time single server queue with a specific input process and derive the cell loss probability in finite buffer queues with this input. As we will see, we use the relationship between the stationary distributions of buffer contents in a finite-buffer queue and the corresponding infinite-buffer queue [9] and express the cell loss probability approximately in terms of the buffer contents in the corresponding infinite-buffer queue. Under the specific condition, the approximation becomes exact.

3.1 Mathematically Tractable Correlated Input Model

To explain our input model, the origin of time is set to be time $-T+1$ ($T > 0$). The cell arrival process is then defined to be $\{B_n;\ n = -T+1, -T+2, \ldots\}$. We assume that there exists an underlying bivariate stochastic process $\{(Z_\nu, D_\nu);\ \nu = 1, 2, \ldots\}$, where Z_ν and D_ν take nonnegative and positive integer values, respectively, for all $\nu = 1, 2, \ldots$. Associated with this, the cell arrival process $\{B_n;\ n = -T+1, -T+2, \ldots\}$ is determined in the following way. We define T_ν ($\nu = 1, 2, \ldots$) as

$$T_\nu = -T + \sum_{n=1}^{\nu} D_n, \quad \nu = 1, 2, \ldots. \tag{2}$$

B_n ($n = -T+1, -T+2, \ldots$) is then determined by

$$B_{T_{\nu-1}+1} = B_{T_{\nu-1}+2} = \cdots = B_{T_\nu} = Z_\nu, \quad \nu = 1, 2, \ldots,$$

sample path wise, where $T_0 = -T$. Thus the cell arrival process is completely described in terms of the underlying bivariate stochastic process $\{(Z_\nu, D_\nu);\ \nu = 1, 2, \ldots\}$. Fig. 1 shows a sample path of the arrival process. For some ν, $(Z_\nu, D_\nu) = (3, 2)$, so that three cells arrive in two consecutive slots. Next we have $(Z_{\nu+1}, D_{\nu+1}) = (0, 4)$, and therefore no cells arrive in the next four consecutive slots. Further we have $(Z_{\nu+2}, D_{\nu+2}) = (1, 3)$, so that one cell arrives in the next three consecutive slots, in this example.

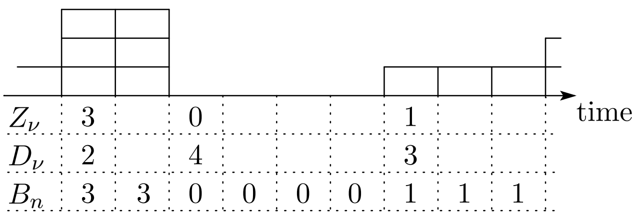

Fig. 1: Input model.

To make things tractable, we assume the followings.

Assumption 1

(i) *Z_ν's are independent and identically distributed (i.i.d.) random variables with finite mean and variance.*

(ii) $\mathrm{E}[D_\nu \mid Z_\nu = m] = (1-\sigma)^{-1}$ *for all* $m = 0, 1, \ldots,$ *where* $0 \le \sigma < 1$.

(iii) *Given $Z_\nu \ge 1$, the conditional D_ν has the probability mass function $d(l)$ ($l = 1, 2, \ldots$), i.e., for all $m = 1, 2, \ldots,$*

$$\Pr[D_\nu = l \mid Z_\nu = m] = d(l), \qquad l = 1, 2, \ldots.$$

(iv) *Given $Z_\nu = 0$, the conditional D_ν follows a discrete phase-type (PH) distribution with irreducible representation $(\boldsymbol{\eta}, \boldsymbol{R})$, i.e.,*

$$\Pr[D_\nu = l \mid Z_\nu = 0] = \boldsymbol{\eta}\boldsymbol{R}^{l-1}\boldsymbol{r}, \quad l = 1, 2, \ldots,$$

where $\boldsymbol{\eta}$ and $\boldsymbol{R}$ denote a $1 \times M$ probability vector and an $M \times M$ substochastic matrix, respectively, and $\boldsymbol{r}$ denotes an $M \times 1$ vector which satisfies $\boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{R})\boldsymbol{e}$.

□

In what follows, we consider a situation such that $T \to \infty$, and the input process $\{B_n\}$ is assumed to be stationary at slot 0. Let Z and B denote generic random variables for Z_n's and B_n's, respectively, in steady state. It then follows from Assumption 1 (ii) that Z and B have the same distribution. Thus we define $b(m)$ $(m = 0, 1, \ldots)$ as

$$b(m) = \Pr[Z = m] = \Pr[B = m], \quad m = 0, 1, \ldots.$$

Note that Assumptions 1 (iii) and 1 (iv) are introduced so as to make the steady-state analysis simple.

3.2 Analysis with Markov Chains

In this subsection, we first consider a finite-buffer single-server queue with an input process satisfying Assumption 1. We then consider the relationship between the queue length distributions in the finite-buffer queue and in the corresponding infinite-buffer queue.

Recall that the queue length process $\{Q_n^{(N)};\ n = 0, 1, \ldots\}$ in the finite-buffer queue is governed by (1). Let ρ denote the expected number of cells arriving in a slot, i.e., $\rho = \mathrm{E}[B]$. We define $\rho^{(N)}$ as the time-average probability of the server being busy. Since the service time of a cell is equal to the length of one slot, the cell loss probability $P_{\text{loss}}^{(N)}$ is given by

$$P_{\text{loss}}^{(N)} = \frac{\rho - \rho^{(N)}}{\rho}. \tag{3}$$

To obtain $\rho^{(N)}$, we construct an embedded (bivariate) Markov chain, where all slots T_ν's (see (2)) and all slots in which no arrival happens $(B_n = 0)$ are chosen as embedded Markov points. Let $X_n^{(N)}$ $(n = 1, 2, \ldots)$ and H_n $(n = 1, 2, \ldots)$ denote the number of cells in the system at the nth embedded Markov point and the length between the $(n-1)$st and the nth embedded Markov points, respectively. Further we define G_n $(n = 1, 2, \ldots)$ as the number of cells arriving in each slot during the nth interval H_n. We then have

$$X_n^{(N)} = \min\left((X_{n-1}^{(N)} - 1)^+ + A_n, N\right), \quad n = 1, 2, \ldots,$$

where

$$A_n = H_n(G_n - 1) + 1, \quad n = 1, 2, \ldots. \tag{4}$$

We introduce the phase variable S_n $(n = 1, 2, \ldots)$ at the nth embedded Markov point:

$$S_n = \begin{cases} 0, & \text{if the } n\text{th embedded Markov point corresponds to } T_\nu \text{ for some } \nu, \\ j, & \text{if the } n\text{th embedded Markov point does not correspond to } T_\nu \text{ for any } \nu \\ & \qquad \text{and the corresponding state of the discrete PH distribution of input is } j. \end{cases}$$

It is readily seen that the bivariate process $\{(X_n^{(N)}, S_n);\ n = 1, 2, \ldots\}$ constitutes a Markov chain because of Assumption 1.

Let $\boldsymbol{A}_k$ $(k = 0, 1, \ldots)$ denote an $(M+1)\times(M+1)$ matrix whose $(i+1, j+1)$st $(i, j = 0, 1, \ldots, M)$ element represents $\Pr[A_{n+1} = k, S_{n+1} = j \mid S_n = i]$. The transition probability matrix $\boldsymbol{P}^{(N)}$ of the bivariate Markov chain $\{(X_n^{(N)}, S_n);\ n = 1, 2, \ldots\}$ is then given by

$$\boldsymbol{P}^{(N)} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_{N-1} & \overline{\boldsymbol{A}}_{N-1} \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_{N-1} & \overline{\boldsymbol{A}}_{N-1} \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{N-2} & \overline{\boldsymbol{A}}_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A}_1 & \overline{\boldsymbol{A}}_1 \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A}_0 & \overline{\boldsymbol{A}}_0 \end{pmatrix},$$

where $\overline{\boldsymbol{A}}_k = \sum_{l=k+1}^{\infty} \boldsymbol{A}_l$. It is easy to see that the z-transform of $\boldsymbol{A}_k$'s $(k = 0, 1, \ldots)$ is given by

$$\boldsymbol{A}^*(z) = \begin{pmatrix} A^*_{0,0}(z) & b(0)\boldsymbol{\eta}\boldsymbol{R} \\ \boldsymbol{r} & \boldsymbol{R} \end{pmatrix}, \tag{5}$$

where

$$A^*_{0,0}(z) = b(0)\boldsymbol{\eta}\boldsymbol{r} + \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} b(m)d(l)z^{ml-l+1}.$$

Let $X^{(N)}$ and S denote generic random variables for $X_n^{(N)}$'s and S_n's, respectively, in steady state. We then define $\boldsymbol{x}_k^{(N)}$ $(k = 0, 1, \ldots, N)$ as a $1\times(M+1)$ vector whose $(j+1)$st $(j = 0, 1, \ldots, M)$ element represents $\Pr[X^{(N)} = k, S = j]$. We denote $(\boldsymbol{x}_0^{(N)}, \boldsymbol{x}_1^{(N)}, \ldots, \boldsymbol{x}_N^{(N)})$ by $\boldsymbol{x}^{(N)}$. Note that $\boldsymbol{x}^{(N)}$ can be obtained numerically by solving $\boldsymbol{x}^{(N)} = \boldsymbol{x}^{(N)}\boldsymbol{P}^{(N)}$ and $\boldsymbol{x}^{(N)}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ denotes a column vector with an appropriate dimension, whose elements are all equal to one. Some efficient numerical algorithms to solve those equations are available in the literature, e.g., see [12].

Let $\boldsymbol{\pi}$ denote a $1 \times (M+1)$ vector whose $(j+1)$st $(j = 0, 1, \ldots, M)$ element π_j represents $\Pr[S = j]$. Note that $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}\boldsymbol{A}^*(1) = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$. Thus we have

$$\begin{cases} \pi_0(1 - b(0) + b(0)\boldsymbol{\eta}\boldsymbol{R}) + \boldsymbol{\pi}_+\boldsymbol{r} = \pi_0, \\ \pi_0 b(0)\boldsymbol{\eta}\boldsymbol{R} + \boldsymbol{\pi}_+\boldsymbol{R} = \boldsymbol{\pi}_+, \\ \pi_0 + \boldsymbol{\pi}_+\boldsymbol{e} = 1, \end{cases}$$

where $\boldsymbol{\pi}_+$ denote a $1 \times M$ vector which means $\boldsymbol{\pi}_+ = (\pi_1, \pi_2, \ldots, \pi_M)$. Thus we obtain

$$\pi_0 = \frac{1}{1 + b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}}, \qquad \boldsymbol{\pi}_+ = \frac{b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}}{1 + b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}}. \tag{6}$$

Let H denote a generic random variable for H_n's in steady state. We then have

$$\begin{aligned} \mathrm{E}[H] &= \pi_0\left[b(0)\times 1 + (1-b(0))\mathrm{E}[D^{[+]}]\right] + (1-\pi_0)\times 1 \\ &= \frac{b(0)\boldsymbol{\eta}(\boldsymbol{I}-\boldsymbol{R})(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e} + (1-b(0))\boldsymbol{\eta}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e} + b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}}{1 + b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}} \\ &= \frac{\boldsymbol{\eta}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}}{1 + b(0)\boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}} = \pi_0\mathrm{E}[D], \end{aligned} \tag{7}$$

where we use Assumptions 1 (ii) and 1 (iv) and (6).

Lemma 1 *$\rho^{(N)}$ is given in terms of $\boldsymbol{x}_0^{(N)}$:*

$$\rho^{(N)} = 1 - \frac{\boldsymbol{x}_0^{(N)}\boldsymbol{e}}{\mathrm{E}[H]}. \tag{8}$$

□

The proof of Lemma 1 is given in Appendix A. The following theorem is immediate from (3), (6), (7), and (8).

Theorem 1 *Under Assumption 1, $P_{\text{loss}}^{(N)}$ is given by*

$$P_{\text{loss}}^{(N)} = 1 - \frac{1}{\rho}\left(1 - \frac{1 - b(0) + b(0)\mathrm{E}[D]}{\mathrm{E}[D]} \cdot \boldsymbol{x}_0^{(N)}\boldsymbol{e}\right).$$

□

Next we consider the corresponding infinite-buffer single-server queue with the same input process as in the finite-buffer queue. Let $Q_n^{(\infty)}$ $(n = 0, 1, \dots)$ denote the number of cells at slot n in the corresponding infinite-buffer queue. We then have

$$Q_n^{(\infty)} = \left(Q_{n-1}^{(\infty)} - 1\right)^+ + B_n, \quad n = 1, 2, \dots.$$

We choose the same embedded Markov points as in the finite-buffer queue and construct the bivariate Markov chain $\{(X_n^{(\infty)}, S_n);\ n = 1, 2, \dots\}$, where $X_n^{(\infty)}$ denotes the number of cells in the system at the nth embedded Markov point. Note that $X_n^{(\infty)}$'s satisfy

$$X_n^{(\infty)} = \left(X_{n-1}^{(\infty)} - 1\right)^+ + A_n, \quad n = 1, 2, \dots,$$

where A_n is given in (4).

It is clear that the transition probability matrix $\boldsymbol{P}^{(\infty)}$ of the embedded bivariate Markov chain $\{(X_n^{(\infty)}, S_n);\ n = 1, 2, \dots\}$ is given by

$$\boldsymbol{P}^{(\infty)} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To proceed further, we assume the following.

Assumption 2 $\mathrm{E}[B] < 1$. □

Note that Assumption 2 ensures the existence of the steady state of the bivariate Markov chain $\{(X_n^{(\infty)}, S_n);\ n = 1, 2, \dots\}$ [18]. We then define $\boldsymbol{x}_k^{(\infty)}$ $(k = 0, 1, \dots)$ as a $1 \times (M+1)$ vector whose $(j+1)$st $(j = 0, 1, \dots, M)$ element represents $\Pr[X^{(\infty)} = k, S = j]$, where $X^{(\infty)}$ denotes a generic random variable for $X_n^{(\infty)}$'s in steady state. Because the transition probability matrix $\boldsymbol{P}^{(\infty)}$ is of M/G/1 type, $\boldsymbol{x}_k^{(\infty)}$ $(k = 0, 1, \dots)$ can be obtained numerically by matrix-analytic methods [18].

We then use the following approximation of the stationary probability vectors of the bivariate Markov chain $\{(X_n^{(\infty)}, S_n);\ n = 1, 2, \dots\}$ [9]:

$$\boldsymbol{x}_0^{(N)} \approx \frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}{\sum_{k=0}^{N} \boldsymbol{x}_k^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}} \cdot \boldsymbol{x}_0^{(\infty)}. \tag{9}$$

Remark 1 *Note here that* $\boldsymbol{A}_0$ *is given by*

$$\boldsymbol{A}_0 = \begin{pmatrix} b(0) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1-\sigma & \sigma \end{pmatrix},$$

when $M = 1$*. Thus we can utilize the relationship between the stationary distributions of buffer contents in the finite-buffer queue and the corresponding infinite-buffer queue in [9]. Namely, using Lemmas 1 and 2 in [9], we obtain*

$$\boldsymbol{x}_0^{(N)} = \frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}{\sum_{k=0}^{N}\boldsymbol{x}_k^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}} \cdot \boldsymbol{x}_0^{(\infty)},$$

i.e., the approximation given in (9) becomes exact.

□

Theorem 2 *Under Assumptions 1 and 2,* $P_{\text{loss}}^{(N)}$ *is given in terms of* $\boldsymbol{x}_k^{(\infty)}$*'s:*

$$P_{\text{loss}}^{(N)} \approx \frac{1-\rho}{\rho} \cdot \frac{\overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} - \overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}, \tag{10}$$

where

$$\overline{\boldsymbol{x}}_N^{(\infty)} = \sum_{k=N+1}^{\infty} \boldsymbol{x}_k^{(\infty)}.$$

When $M = 1$*, the approximation given in (10) becomes exact, i.e.,*

$$P_{\text{loss}}^{(N)} = \frac{1-\rho}{\rho} \cdot \frac{\overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} - \overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}. \tag{11}$$

□

Proof: In the same way as in the proof of Lemma 1, we have

$$\rho = 1 - \frac{\boldsymbol{x}_0^{(\infty)}\boldsymbol{e}}{\mathrm{E}[H]}. \tag{12}$$

It then follows from (8), (9), and (12) that

$$\begin{aligned} \rho^{(N)} &= 1 - (1-\rho)\frac{\boldsymbol{x}_0^{(N)}\boldsymbol{e}}{\boldsymbol{x}_0^{(\infty)}\boldsymbol{e}} \\ &\approx 1 - (1-\rho)\frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}{\sum_{k=0}^{N}\boldsymbol{x}_k^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}. \end{aligned} \tag{13}$$

Substituting (13) into (3) yields

$$P_{\text{loss}}^{(N)} \approx \frac{1-\rho}{\rho}\left(\frac{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}{\sum_{k=0}^{N}\boldsymbol{x}_k^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}} - 1\right).$$

Thus noting $\boldsymbol{\pi} = \sum_{k=0}^{\infty}\boldsymbol{x}_k^{(\infty)}$, we obtain (10).

From Remark 1, the approximation in (13) becomes exact when $M = 1$. Thus we obtain (11) when $M = 1$. ■

The results obtained in this section are used in section 4 and 5 to derive the approximate formulas for the cell loss probability.

4 Queues with LRD Input

In this section, we consider the cell loss probability in finite buffer queues with LRD input.

4.1 Approximate Formula for the Cell Loss Probability

This subsection presents an approximate formula for the cell loss probability. We now present an approximate formula for $P_{\text{loss}}^{(N)}$, assuming $\{B_n;\ n = 1, 2, \ldots\}$ is a stationary sequence of random variables. We define B as a generic random variable for B_n's. Further let $\gamma(k)$ $(k = 1, 2, \ldots)$ denote the autocorrelation function of B_n's $(n = 1, 2, \ldots)$:

$$\gamma(k) = \frac{\text{Cov}[B_n B_{n+k}]}{\text{Var}[B]}, \qquad k = 0, 1, \ldots.$$

Proposition 1 *Suppose* $0 < \text{E}[B] < 1$, $0 < \text{Var}[B] < \infty$, *and there exist* α $(\alpha > 0)$ *and* θ $(0 < \theta < 1)$ *such that*

$$\lim_{k\to\infty} \frac{\gamma(k)}{k^{-\theta}} = \alpha. \tag{14}$$

$P_{\text{loss}}^{(N)}$ *is then approximately given by*

$$P_{\text{loss}}^{(N)} \approx c(\theta)\gamma(N), \tag{15}$$

with

$$c(\theta) = \frac{\overline{B}(\theta)}{\text{E}[B]\left[1 - \dfrac{\Pr[B=0]}{\text{C}_\text{V}^2[B]}\right]},$$

where $\text{C}_\text{V}^2[B] = \text{Var}[B]/\text{E}[B]^2$ *denotes the squared coefficient of variation of* B *and*

$$\overline{B}(\theta) = \sum_{m=2}^{\infty} (m-1)^{\theta+1} \Pr[B = m].$$

□

Because $0 < \theta < 1$, (14) implies that the input process $\{B_n;\ n = 1, 2, \ldots\}$ is LRD [3]. The approximate formula in (15) is asymptotically exact for a certain queueing model considered in the next subsection. We observe the followings from this formula:

(i) The cell loss probability $P_{\text{loss}}^{(N)}$ is (asymptotically) proportional to the autocorrelation function $\gamma(N)$, i.e., $\lim_{N\to\infty} P_{\text{loss}}^{(N)}/\gamma(N) = c(\theta)$, where $c(\theta)$ is given in terms of $\text{E}[B]$, $\text{C}_\text{V}^2[B]$, $\Pr[B = 0]$, and $\overline{B}(\theta)$.

(ii) When $\text{Var}[B]$ is finite, the characteristics of the tail distribution $\Pr[B > k]$ has no impact on the cell loss probability. Note that, in telecommunication networks, B is bounded above by channel capacity, so that the assumption of finite variance does not matter at all in application.

(iii) The factor $\overline{B}(\theta)$ can be obtained numerically, whereas it may be hard to obtain explicit expressions for mathematical models.

(iv) Because $\overline{B}(\theta)$ ($0 < \theta < 1$) is an increasing function of θ, the upper and lower bounds of the asymptotic constant $c(\theta)$ are readily obtained to be

$$c(0) < c(\theta) < c(1),$$

where

$$c(0) = \frac{\mathrm{E}[B] - (1 - \Pr[B = 0])}{\mathrm{E}[B]\left[1 - \dfrac{\Pr[B = 0]}{\mathrm{C}_{\mathrm{V}}^2[B]}\right]},$$
$$c(1) = \frac{\mathrm{Var}[B] + (1 - \mathrm{E}[B])^2 - \Pr[B = 0]}{\mathrm{E}[B]\left[1 - \dfrac{\Pr[B = 0]}{\mathrm{C}_{\mathrm{V}}^2[B]}\right]}.$$

(v) Based on the observation in (iv), we obtain a conservative approximate formula for $P_{\mathrm{loss}}^{(N)}$:

$$P_{\mathrm{loss}}^{(N)} \approx c(1)\gamma(N), \tag{16}$$

whose accuracy may be estimated in advance if the ratio $c(1)/c(0)$ is known.

The advantage of the conservative approximate formula (16) is obvious; it provides an explicit formula for a specific mathematical model. For example, consider a single-server queue with M/G/∞ input [11]. In this case, B follows a Poisson distribution with mean ρ, and therefore we have

$$P_{\mathrm{loss}}^{(N)} \approx \frac{\rho + (1-\rho)^2 - \exp(-\rho)}{\rho(1 - \rho\exp(-\rho))} \cdot \gamma(N), \tag{17}$$

if $0 < \rho < 1$ and (14) holds. Note that the ratio $c(1)/c(0)$ in M/G/∞ input is an increasing function of ρ and lies in $(1, e-1)$ for $0 < \rho < 1$. Therefore we expect that (17) can be used to estimate the order of magnitude of the cell loss probability in single-server queues with M/G/∞ input. For more details, see subsection 4.3.

4.2 Derivation of the Cell Loss Probability

In this subsection, we consider a discrete-time single-server queue with a specific LRD input process, which is given by introducing some assumptions to the model described in section 3, and derive the asymptotic cell loss probability given on the right hand side of (15).

4.2.1 LRD Input Model

We consider input process considered in section 3 again. In addition to Assumption 1, we assume the following.

Assumption 3 *$M = 1$, i.e., $\boldsymbol{\eta} = 1$, $\boldsymbol{R} = \sigma$, $\boldsymbol{r} = 1 - \sigma$, and*

$$\Pr[D_\nu = l \mid Z_\nu = 0] = (1 - \sigma)\sigma^{l-1}, \quad l = 1, 2, \ldots.$$

□

In what follows, we use the following notation for simplicity in description.

$$f(k) \overset{k}{\sim} g(k) \iff \lim_{k\to\infty} \frac{f(k)}{g(k)} = 1.$$

Let $D^{[+]}$ denote a generic random variable for the conditional D_ν given $Z_\nu \geq 1$.

Assumption 4 *There exist* θ $(0 < \theta < 1)$ *and* β $(0 < \beta < \infty)$ *such that*

$$\Pr[D^{[+]} > k] \stackrel{k}{\sim} \beta k^{-(\theta+1)}.$$

□

Remark 2 *Note that for real* $x \geq 0$

$$\begin{aligned}\Pr[D^{[+]} > x] &= \Pr[D^{[+]} > \lfloor x \rfloor] \\ &\stackrel{\lfloor x \rfloor}{\sim} \beta(\lfloor x \rfloor)^{-(\theta+1)},\end{aligned}$$

where $\lfloor x \rfloor$ *denotes the integer part of* x*. Because*

$$\begin{aligned}x^{-(\theta+1)} &\leq (\lfloor x \rfloor)^{-(\theta+1)} \\ &\leq (x-1)^{-(\theta+1)} \\ &= x^{-(\theta+1)} \left(1 - \frac{1}{x}\right)^{-(\theta+1)},\end{aligned}$$

we obtain

$$\lim_{x \to \infty} \frac{\Pr\left[D^{[+]} > x\right]}{\beta x^{-(\theta+1)}} = 1,$$

from which it follows that

$$\lim_{x \to \infty} \frac{\Pr[D^{[+]} > tx]}{\Pr[D^{[+]} > x]} = t^{-(\theta+1)}.$$

Thus Assumption 4 implies that $\Pr[D^{[+]} > x]$ *is regularly varying at* ∞ *with index* $-(\theta+1)$ *[1].* □

Theorem 3 *Under Assumptions 1, 3, and 4, the autocorrelation function* $\gamma(k)$ *of the* B_n *satisfies*

$$\gamma(k) \stackrel{k}{\sim} \alpha k^{-\theta}, \tag{18}$$

where

$$\alpha = \frac{\beta}{\theta \mathrm{E}[D]} \left(1 - \frac{b(0)}{\mathrm{C}_{\mathrm{V}}^2[B]}\right). \tag{19}$$

□

Remark 3 *Because* $0 < \theta < 1$*, Theorem 3 implies that the stationary input process* $\{B_n;\ n = 0, 1, \ldots\}$ *is LRD under Assumptions 1, 3, and 4.* □

Proof: We define B_n^* $(n = 1, 2, \ldots)$ as the number of cells arriving in slot n given that $T_\nu = 0$ for some ν. Let $\widetilde{D}^{[+]}$ denote a random variable representing the forward recurrence time of $D^{[+]}$, i.e.,

$$\Pr[\widetilde{D}^{[+]} = n] = \frac{\Pr[D^{[+]} > n]}{\mathrm{E}[D^{[+]}]}, \quad n = 0, 1, \ldots.$$

We then have for $k = 1, 2, \ldots,$

$$
\begin{aligned}
&\mathrm{E}[B_n B_{n+k}] - \mathrm{E}[B]^2 \\
&= \sum_{j=0}^{k-1} \sum_{m=1}^{\infty} m \Pr[B = m, \widetilde{D}^{[+]} = j] \mathrm{E}[B^*_{k-j}] + \sum_{m=1}^{\infty} m^2 \Pr[B = m, \widetilde{D}^{[+]} \geq k] - \mathrm{E}[B]^2 \\
&= \mathrm{E}[B] \sum_{j=0}^{k-1} \Pr[\widetilde{D}^{[+]} = j] \mathrm{E}[B^*_{k-j}] + \mathrm{Var}[B] \Pr[\widetilde{D}^{[+]} \geq k] - \mathrm{E}[B]^2 \Pr(\widetilde{D}^{[+]} < k) \\
&= \mathrm{Var}[B] \Pr[\widetilde{D}^{[+]} \geq k] - \mathrm{E}[B] \sum_{j=0}^{k-1} (\mathrm{E}[B] - \mathrm{E}[B^*_{k-j}]) \Pr[\widetilde{D}^{[+]} = j].
\end{aligned}
$$

Thus we obtain

$$\gamma(k) = \Pr[\widetilde{D}^{[+]} \geq k] - \frac{1}{\mathrm{C}^2_{\mathrm{V}}[B]} \cdot g(k), \tag{20}$$

where

$$g(k) = \frac{1}{\mathrm{E}[B]} \sum_{j=0}^{k-1} (\mathrm{E}[B] - \mathrm{E}[B^*_{k-j}]) \Pr[\widetilde{D}^{[+]} = j]. \tag{21}$$

Lemma 2 (Bingham et al. [1]) *Under Assumption 4, we have (see the proof of Corollary 8.10.4 in [1])*

$$\Pr[\widetilde{D}^{[+]} \geq k] \overset{k}{\sim} \frac{1}{\theta \mathrm{E}[D]} \cdot \beta k^{-\theta}. \tag{22}$$

□

Lemma 3 *Under Assumptions 1, 3, and 4, we have*

$$g(k) \overset{k}{\sim} \frac{b(0)}{\theta \mathrm{E}[D]} \cdot \beta k^{-\theta}. \tag{23}$$

□

The proof of Lemma 3 is given in Appendix B. (18) now follows from (20), (22), and (23). ■

4.2.2 Asymptotic Results

In this subsection, we derive the asymptotic cell loss probability when $N \to \infty$ under Assumption 4 (i.e., assuming the LRD input process).

Theorem 4 *Under Assumptions 1, 2, 3, and 4,* $\overline{\boldsymbol{x}}_N^{(\infty)}$ *satisfies*

$$\overline{\boldsymbol{x}}_N^{(\infty)} \overset{N}{\sim} \frac{\pi_0 \overline{B}(\theta) \beta N^{-\theta}}{\theta (1 - \rho) \mathrm{E}[H]} \cdot \boldsymbol{\pi}. \tag{24}$$

□

Proof: (5) implies that $[\boldsymbol{A}_k]_{i,j} = 0$ for all $k = 1, 2, \ldots$ unless $i = j = 0$. Also for $k \geq 1$,

$$\begin{aligned}
\left[\overline{\boldsymbol{A}}_k\right]_{0,0} &= \sum_{l=k+1}^{\infty} [\boldsymbol{A}_l]_{0,0} \\
&= \Pr[H_{n+1}(G_{n+1}-1)+1 \geq k+1, S_{n+1}=0 \mid S_n = 0] \\
&= \Pr[H_{n+1}(G_{n+1}-1) > k-1 \mid G_{n+1} \geq 1, S_{n+1}=0, S_n=0] \\
&\qquad\qquad\qquad \cdot \Pr[G_{n+1} \geq 1, S_{n+1}=1 \mid S_n=1] \\
&= \Pr[D^{[+]}(Z-1) > k-1 \mid Z \geq 1] \Pr[Z \geq 1] \\
&= \sum_{m=2}^{\infty} b(m) \Pr\left[D^{[+]} > \frac{k-1}{m-1}\right].
\end{aligned} \tag{25}$$

Lemma 4 *Under Assumption 4,*

$$\sum_{m=2}^{\infty} b(m) \Pr\left[D^{[+]} > \frac{k-1}{m-1}\right] \stackrel{k}{\sim} \overline{B}(\theta)\beta(k-1)^{-(\theta+1)}, \tag{26}$$

holds. □

The proof of Lemma 4 is given in Appendix C.

Using (25) and (26), we obtain

$$\left[\overline{\boldsymbol{A}}_k\right]_{0,0} \stackrel{k}{\sim} \overline{B}(\theta)\beta k^{-(\theta+1)},$$

and therefore we have

$$\overline{\boldsymbol{A}}_k \stackrel{k}{\sim} k^{-(\theta+1)}\boldsymbol{C},$$

where

$$\boldsymbol{C} = \begin{pmatrix} \overline{B}(\theta)\beta & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus our model satisfies Assumption 4 in [25]. Using Remark 12 in [25], we obtain

$$\overline{\boldsymbol{x}}_N^{(\infty)} \stackrel{N}{\sim} \frac{\boldsymbol{\pi}\boldsymbol{C}\boldsymbol{e}}{\boldsymbol{x}_0^{(\infty)}\boldsymbol{e}} \sum_{k=N+1}^{\infty} k^{-(\theta+1)} \cdot \boldsymbol{\pi}.$$

Further, noting (12) and

$$\boldsymbol{\pi}\boldsymbol{C}\boldsymbol{e} = \pi_0 \overline{B}(\theta)\beta,$$

we have

$$\overline{\boldsymbol{x}}_N^{(\infty)} \stackrel{N}{\sim} \frac{\pi_0 \overline{B}(\theta)\beta}{(1-\rho)\mathrm{E}[H]} \sum_{k=N}^{\infty} k^{-(\theta+1)} \cdot \boldsymbol{\pi}. \tag{27}$$

Because

$$\int_N^{\infty} x^{-(\theta+1)} dx < \sum_{k=N}^{\infty} k^{-(\theta+1)} < \int_N^{\infty} (x-1)^{-(\theta+1)} dx,$$

we obtain

$$\sum_{k=N}^{\infty} k^{-(\theta+1)} \stackrel{N}{\sim} \frac{N^{-\theta}}{\theta}. \tag{28}$$

Applying (28) to (27) yields (24). ■

Theorem 5 *Under Assumptions 1, 2, 3 and 4, $P_{\text{loss}}^{(N)}$ satisfies*

$$P_{\text{loss}}^{(N)} \overset{N}{\sim} \frac{\overline{B}(\theta)}{\mathrm{E}[B]\left[1 - \dfrac{b(0)}{\mathrm{C}_{\mathrm{V}}^2[B]}\right]} \cdot \alpha N^{-\theta}, \tag{29}$$

where α is given in (19). □

Proof: Note that (11) holds when $M = 1$. Applying (24) to (11), we obtain

$$\begin{aligned} P_{\text{loss}}^{(N)} &\overset{N}{\sim} \frac{1-\rho}{\rho} \cdot \frac{\dfrac{\pi_0 \overline{B}(\theta)\beta N^{-\theta}}{\theta(1-\rho)\mathrm{E}[H]} \cdot \boldsymbol{\pi A}_0 \boldsymbol{e}}{\boldsymbol{\pi A}_0 \boldsymbol{e} - \dfrac{\pi_0 \overline{B}(\theta)\beta N^{-\theta}}{\theta(1-\rho)\mathrm{E}[H]} \cdot \boldsymbol{\pi A}_0 \boldsymbol{e}} \\ &= \frac{1-\mathrm{E}[B]}{\mathrm{E}[B]} \cdot \frac{\pi_0 \overline{B}(\theta)\beta N^{-\theta}}{\theta(1-\mathrm{E}[B])\mathrm{E}[H] - \pi_0 \overline{B}(\theta)\beta N^{-\theta}} \\ &\overset{N}{\sim} \frac{\pi_0 \overline{B}(\theta)\beta N^{-\theta}}{\theta \mathrm{E}[B]\mathrm{E}[H]}, \end{aligned} \tag{30}$$

where we use $\rho = \mathrm{E}[B]$. Substituting (7) into (30), we obtain

$$P_{\text{loss}}^{(N)} \overset{N}{\sim} \frac{\overline{B}(\theta)}{\mathrm{E}[B]} \cdot \frac{\beta}{\theta \mathrm{E}[D]} \cdot N^{-\theta}. \tag{31}$$

(29) now follows from (19) and (31). ■

4.3 Numerical Results

The approximate formula (15) stems from the asymptotic cell loss probability for large buffer. Thus it is not clear to what extent parameters in LRD traffic affect the accuracy of the formula in the small buffer case, even for the model studied in subsection 4.2.1. Thus we first discuss this aspect. Next we investigate the accuracy and robustness of the approximate formula by comparing it with simulation results for queues fed by a superposition of independent on/off LRD sources, M/G/∞ input process, and their generalized ones.

4.3.1 Accuracy in Small Buffer Case

This subsection discusses the accuracy of the approximate formula (15) when the buffer size is small. To do so, we compare the approximate results with those obtained by exact analysis in subsection 3.2. For a while, we assume that the number of cells arriving in a slot in steady state follows a geometric distribution, i.e.,

$$b(m) = \frac{1}{1+\rho}\left(\frac{\rho}{1+\rho}\right)^m, \qquad m = 0, 1, \ldots. \tag{32}$$

Further we assume that

$$\Pr[D^{[+]} \geq l] = \left(\frac{1}{l}\right)^{\theta+1}, \qquad l = 1, 2, \ldots.$$

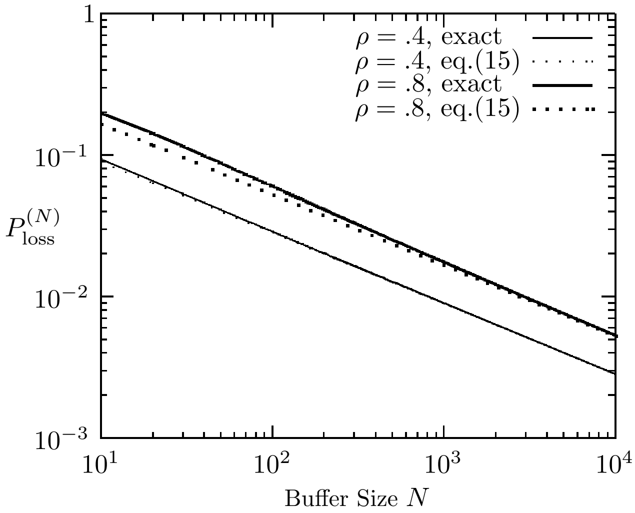

Fig. 2: Exact and asymptotic cell loss probabilities ($\theta = .5$).

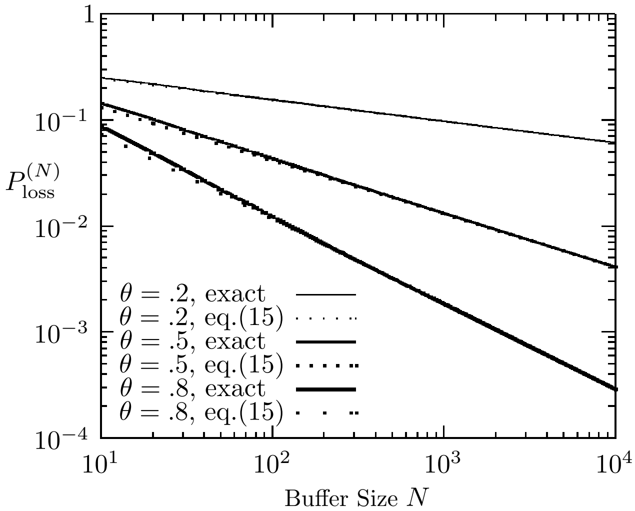

Fig. 3: Exact and asymptotic cell loss probabilities ($\rho = .6$).

Fig. 2 shows the exact and approximate (i.e., asymptotic) cell loss probabilities as a function of the buffer size, where $\theta = .5$. We observe that the approximation is accurate even for a queue with small buffer when the load is light. In a heavily loaded situation, however, the approximate formula slightly underestimates the cell loss probability in queues with small buffer.

Next we examine the impact of θ on the accuracy of the approximation. For this purpose, we set $\rho = .6$. Fig. 3 shows the exact and approximate cell loss probabilities as a function of the buffer size. We observe that the difference between the exact and approximate results becomes small with θ. Thus we expect that the approximate formula is accurate when correlation in arrivals is strong.

Finally, we examine the impact of the variance of B on the accuracy of the approximation. For this purpose, we set $\rho = .6$ and $\theta = .5$. As a distribution with modest variance, we choose a geometric distribution given in (32). Note that $\mathrm{E}[B] = .6$, $\mathrm{Var}[B] = 0.96$, and $b(0) = 5/8$ in this case. Besides, we prepare two multi-point distributions, which are given by the solutions of the

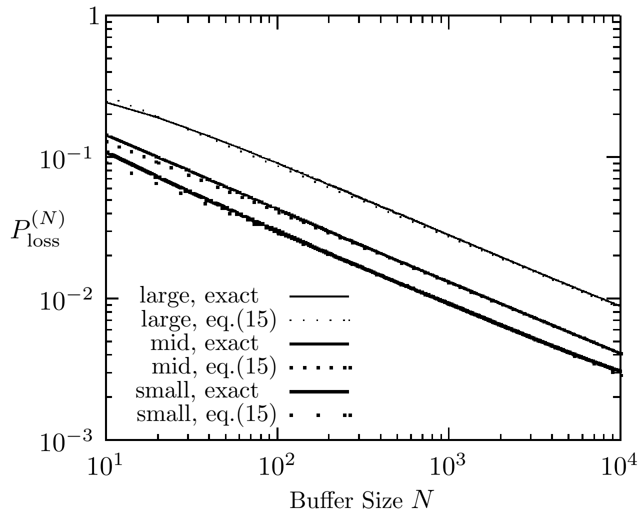

Fig. 4: Exact and asymptotic cell loss probabilities ($\rho = .6$, $\theta = .5$).

following linear programming problems:

$$\begin{aligned}
\text{(P-small)} \quad & \text{minimize} \ \sum_{k=1}^{10} k^2 b_k \\
& \text{subject to} \ b_k \geq 0 \quad (k = 1, 2, \ldots, 10), \\
& \qquad \sum_{k=1}^{10} b_k = 1 - b(0), \qquad \sum_{k=1}^{10} k b_k = \rho.
\end{aligned}$$

$$\begin{aligned}
\text{(P-large)} \quad & \text{maximize} \ \sum_{k=1}^{10} k^2 b_k \\
& \text{subject to} \ b_k \geq 0 \quad (k = 1, 2, \ldots, 10), \\
& \qquad \sum_{k=1}^{10} b_k = 1 - b(0), \qquad \sum_{k=1}^{10} k b_k = \rho.
\end{aligned}$$

As a result, for a distribution with small variance, we have 3-point distribution with $b(1) = 2(1 - b(0)) - \rho = 3/20$ and $b(2) = \rho - (1 - b(0)) = 9/40$ ($\mathrm{Var}[B] = .69$), and for a distribution with large variance, we have 3-point distribution with $b(1) = [10(1 - b(0)) - \rho]/9 = 7/20$ and $b(10) = [\rho - (1 - b(0))]/9 = 1/40$ ($\mathrm{Var}[B] = 2.49$). Fig. 4 shows the exact and approximate cell loss probabilities as a function of the buffer size. We observe that the approximation is accurate when the variance of B is large.

In summary, the approximate formula is fairly accurate even in the small buffer case, and at worst it seems to be used to estimate the order of magnitude of the cell loss probability. Besides, the approximate formula (15) is likely to underestimate the cell loss probability, and therefore the conservative approximate formula (16) might be more suitable for an engineering purpose. This will be discussed further in the following subsection.

4.3.2 Accuracy and Robustness of Approximate Formula

In this subsection, we apply the approximate formula to queues fed by a superposition of independent on/off LRD sources and M/G/∞ input, and compare the approximation with simulation results. Generally speaking, it is rather hard to conduct simulation experiments for rare events in stochastic models with LRD and/or heavy-tailed components. To overcome this difficulties, we adopt a pseudo-random number generator called the Mersenne Twister [16], which has a period of $2^{19937} - 1$ and 623-dimensional equidistribution property[1]. The Mersenne Twister enables us to generate very long sample paths.

4.3.2.A Queues with On/Off LRD Sources

We consider a finite-buffer queue with K independent and homogeneous on/off sources. Each source becomes on and off alternately, and on- and off-periods form a discrete-time alternating renewal process. While being on, each source generates exactly one cell in each slot, and no cells are generated in off-periods.

We define F_{on} and F_{off} as generic random variables for the lengths of on- and off-periods, respectively. Let $\mu_{\text{on}} = \mathrm{E}[F_{\text{on}}]$ and $\mu_{\text{off}} = \mathrm{E}[F_{\text{off}}]$. We assume that F_{on} has a discrete Pareto distribution with shape parameter $\theta + 1$, i.e.,

$$\Pr[F_{\text{on}} > l] = \left(\frac{1}{l+1}\right)^{\theta+1}, \qquad l = 0, 1, \dots,$$

where $0 < \theta < 1$. Note that $\mu_{\text{on}} = \sum_{l=1}^{\infty} (1/l)^{\theta+1}$. Let ρ denote the overall traffic intensity. We then have $\rho = K\mu_{\text{on}}/(\mu_{\text{on}} + \mu_{\text{off}})$, from which it follows that $\mu_{\text{off}} = (K - \rho)\mu_{\text{on}}/\rho$. We assume that F_{off} has a geometric distribution, i.e.,

$$\Pr[F_{\text{off}} > l] = \left(1 - \frac{1}{\mu_{\text{off}}}\right)^{l}, \qquad l = 0, 1, \dots.$$

Because sources are homogeneous, the correlation coefficient $\gamma(k)$ ($k = 1, 2, \dots$) of the numbers of cells arriving at lag k is identical with that of an individual source. Further, because the variance of the number of cells generated by a source is given by $\mu_{\text{on}}\mu_{\text{off}}/(\mu_{\text{on}} + \mu_{\text{off}})^2$, it follows from Theorem 4.3 in [5] that

$$\gamma(k) \overset{k}{\sim} \frac{\dfrac{\mu_{\text{off}}^2}{\theta(\mu_{\text{on}} + \mu_{\text{off}})^3} \cdot k^{-\theta}}{\dfrac{\mu_{\text{on}}\mu_{\text{off}}}{(\mu_{\text{on}} + \mu_{\text{off}})^2}} = \frac{K - \rho}{\theta K \mu_{\text{on}}} \cdot k^{-\theta}.$$

On the other hand, the number B of cells arriving in a slot follows a binomial distribution, i.e., for $m = 0, 1, \dots, K$,

$$\Pr[B = m] = \binom{K}{m} \left(\frac{\rho}{K}\right)^{m} \left(1 - \frac{\rho}{K}\right)^{K-m}.$$

We conduct regenerative simulation experiments for queues with on/off sources described above. Because off-periods of each source are geometrically distributed, time instants n such that $(X_n, B_n) = (0, 0)$ are regenerative points. We then define a regenerative cycle as an interval between regenerative points, during which at least one cell arrives (i.e., an idle period and the following busy period). All simulation results given below are obtained from 10 billion regenerative cycles. Further 95% confidence intervals are shown. Note that in computing the 95% confident intervals, we regard 1,000 successive regenerative cycles as a unit cycle, due to lack of memory capacity.

Fig. 5 shows the cell loss probability $P_{\text{loss}}^{(N)}$ for $\rho = .4$ and $.8$ as a function of the buffer size N, where $K = 5$ and $\theta = .5$. We observe that when ρ is small, the approximate formula (15) bounds

[1]A pseudo-random number generated by the Mersenne Twister is composed of a 53-bit fixed point part and a 11-bit exponent including a sign digit. A C program (zmtrand.h and zmtrand.c) is downloadable from [17].

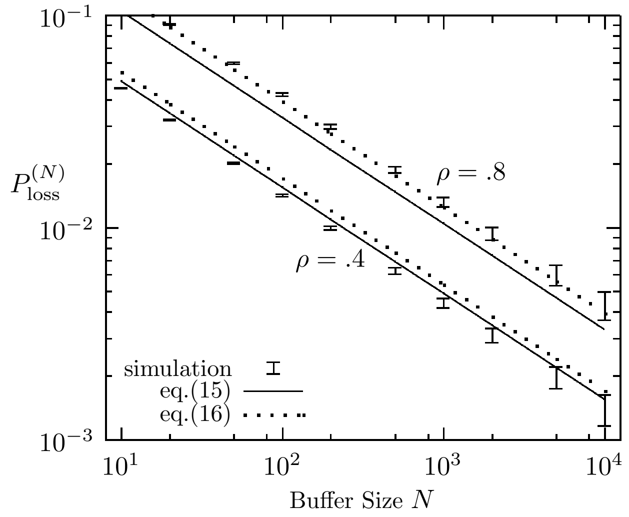

Fig. 5: Queues with on/off LRD sources ($K = 5$, $\theta = .5$).

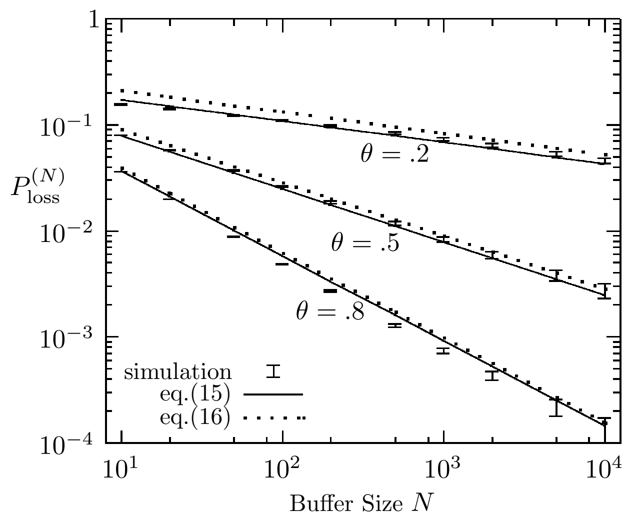

Fig. 6: Queues with on/off LRD sources ($K = 5$, $\rho = .6$).

$P_{\rm loss}^{(N)}$ from the above, whereas it slightly underestimates $P_{\rm loss}^{(N)}$ when ρ is large. On the other hand, the conservative formula (16) works well in both cases. Fig. 6 shows $P_{\rm loss}^{(N)}$ for $\theta = .2, .5$ and $.8$ as a function of the buffer size N, where $K = 5$ and $\rho = .6$. We observe that when θ is large (i.e., correlation is weak), the approximate formula (15) provides an upper bound of $P_{\rm loss}^{(N)}$. However, it slightly underestimates $P_{\rm loss}^{(N)}$ when θ is small. The conservative formula (16) works well in all cases.

4.3.2.B Queues with M/G/∞ Input

Next we consider queues with M/G/∞ input. Note that the M/G/∞ input process is obtained as the limit $K \to \infty$ in the model described above, while ρ remains constant and the distribution of $F_{\rm on}$ is fixed. Thus the number B of cells arriving in a slot follows a Poisson distribution with mean ρ, and the correlation coefficient $\gamma(k)$ ($k = 1, 2, \ldots$) of the numbers of cells arriving at lag

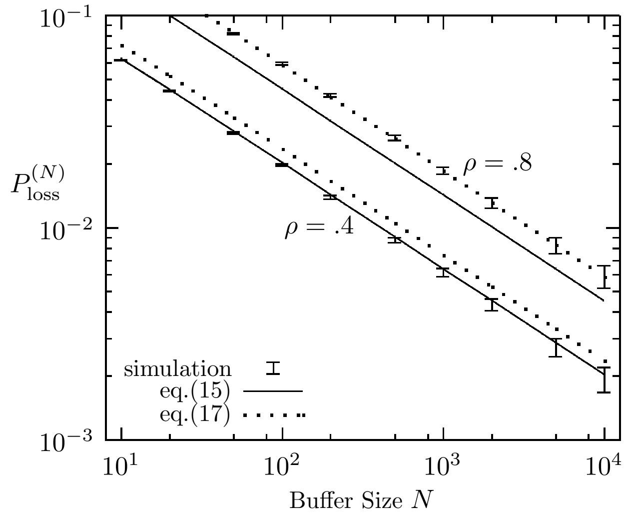

Fig. 7: Queues with M/G/∞ input ($\theta = .5$).

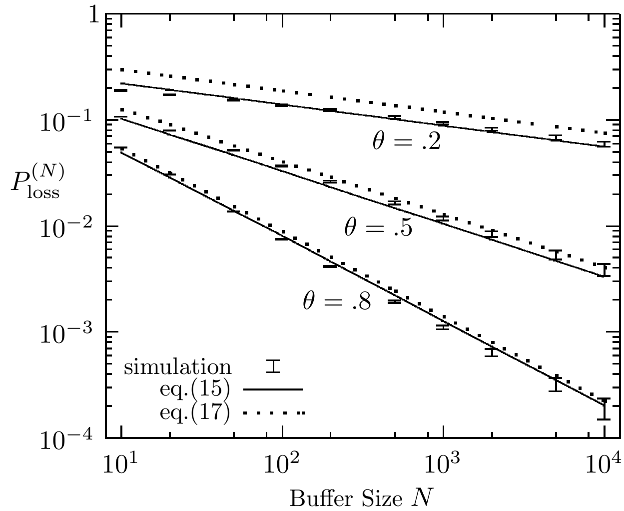

Fig. 8: Queues with M/G/∞ input ($\rho = .6$).

k is given by [11]

$$\gamma(k) = \frac{1}{\mu_{\rm on}} \sum_{l=k}^{\infty} \left(\frac{1}{l}\right)^{\theta+1} \overset{k}{\sim} \frac{1}{\theta\mu_{\rm on}} \cdot k^{-\theta}. \tag{33}$$

We conduct simulation experiments in exactly the same way as in the case of queues with on/off LRD sources. Fig. 7 shows the cell loss probability $P^{(N)}_{\rm loss}$ for $\rho = .4$ and .8 as a function of the buffer size N, where $\theta = .5$. Also, Fig. 8 shows $P^{(N)}_{\rm loss}$ for $\theta = .2, .5$ and .8 as a function of the buffer size N, where $\rho = .6$. Form these two figures, we observe that the general tendency of the accuracy of approximation for M/G/∞ input is very similar to that for a superposition of on/off LRD sources given in Figs. 5 and 6. It is worth noting that such a simple formula (17) works very well.

4.3.2.C Queues with Generalized Input

In queues with on/off LRD sources (resp. M/G/∞ input), the distribution of B is restricted to a family of binomial (resp. Poisson) distributions. Thus we discuss the accuracy and robustness of the approximate formulas for more general B, i.e., queues with a superposition of generalized on/off LRD sources. We use the term "generalized" in the following sense: Each source being on can generate multiple cells in each slot. We assume that sources are homogeneous. Further we assume the numbers of cells generated from each source being on are i.i.d., and let G denote a generic random variable for them. In what follows, we consider the following three cases, i.e.,

$$\begin{aligned} &\text{(G-V0)} \quad \Pr[G = n] = \begin{cases} 1, & n = 2, \\ 0, & \text{otherwise}, \end{cases} \\ &\text{(G-V2)} \quad \Pr[G = n] = (1/2)^n, \quad n = 1, 2, \ldots, \\ &\text{(G-V4)} \quad \Pr[G = n] = \begin{cases} 4/5, & n = 1, \\ 1/5, & n = 6, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

It is easy to see that $\mathrm{E}[G] = 2$ in all three cases. Further $\mathrm{Var}[G] = 0, 2$, and 4 in cases (G-V0), (G-V2), and (G-V4), respectively.

Let G_k $(k = 1, 2, \ldots)$ denote a sequence of i.i.d. random variables with the same distribution as G. In the steady state, the total number B of cells arriving in a slot is given by

$$B = \sum_{k=1}^{B^*} G_k, \tag{34}$$

where B^* follows a binomial distribution, i.e., for $m = 0, 1, \ldots, K$,

$$\Pr[B^* = m] = \binom{K}{m} \left(\frac{\rho}{\mathrm{E}[G]K} \right)^m \left(1 - \frac{\rho}{\mathrm{E}[G]K} \right)^{K-m}.$$

Thus $\mathrm{E}[B] = \rho$, $\mathrm{Var}[B] = \rho \mathrm{E}[G^2]/\mathrm{E}[G] - \rho^2/K$, and $\Pr[B = 0] = \{1 - \rho/(\mathrm{E}[G]K)\}^K$. Further $\overline{B}(\theta)$ can be numerically computed.

The autocorrelation function of the generalized on/off process can be derived in the same way as in 4.3.2.A, i.e.,

$$\gamma(k) \overset{k}{\sim} \frac{K - \rho/\mathrm{E}[G]}{\theta K \mu_{\mathrm{on}}} \cdot k^{-\theta}.$$

Note that $\mu_{\mathrm{on}} = \sum_{l=1}^{\infty} (1/l)^{\theta+1}$, Thus we have $\mu_{\mathrm{on}}/(\mu_{\mathrm{on}} + \mu_{\mathrm{off}}) = \rho/(\mathrm{E}[G]K)$, from which it follows that $\mu_{\mathrm{off}} = (K - \rho/\mathrm{E}[G])\mu_{\mathrm{on}}/(\rho/\mathrm{E}[G])$.

We conduct simulation experiments in exactly the same way as in 4.3.2.A. Fig. 9 shows the cell loss probability $P_{\mathrm{loss}}^{(N)}$ as a function of the buffer size N, where $K = 5$, $\rho = .6$, and $\theta = .5$). We observe that the approximate formula (15) tends to underestimate the cell loss probability in all cases (G-V0), (G-V2), and (G-V4), even though the difference is not so large. On the other hand, the conservative formula (16) does not work well, especially for large variance cases, because the ratio $c(1)/c(0)$ is somewhat large in these cases (i.e., 1.66, 3.46, and 5.25 in cases (G-V0), (G-V2), and (G-V4), respectively).

Finally we apply the formula to queues with a generalized M/G/∞ input. Note that the generalized M/G/∞ input process is obtained as the limit $K \to \infty$ in a superposition of independent and homogeneous generalized on/off LRD sources described above. Thus the number B of cells arriving in a slot is expressed by (34), where B^* follows a Poisson distribution with mean $\rho/\mathrm{E}[G]$ and the autocorrelation function $\gamma(k)$ is given by (33). As a result, we have $\mathrm{E}[B] = \rho$, $\mathrm{Var}[B] = \rho \mathrm{E}[G^2]/\mathrm{E}[G]$, and $\Pr[B = 0] = e^{-\rho/\mathrm{E}[G]}$. Further $\overline{B}(\theta)$ can be numerically computed.

We conduct simulation experiments in the same way as in 4.3.2.B. Fig. 10 shows the cell loss probability $P_{\mathrm{loss}}^{(N)}$ as a function of the buffer size N, where $\theta = .5$ and $\rho = .6$. We observe that the approximate formula (15) works very well in all cases (G-V0), (G-V2), and (G-V4). We also observe that the general tendency of the accuracy of the conservative formula (16) is similar to that for a superposition of generalized on/off LRD sources.

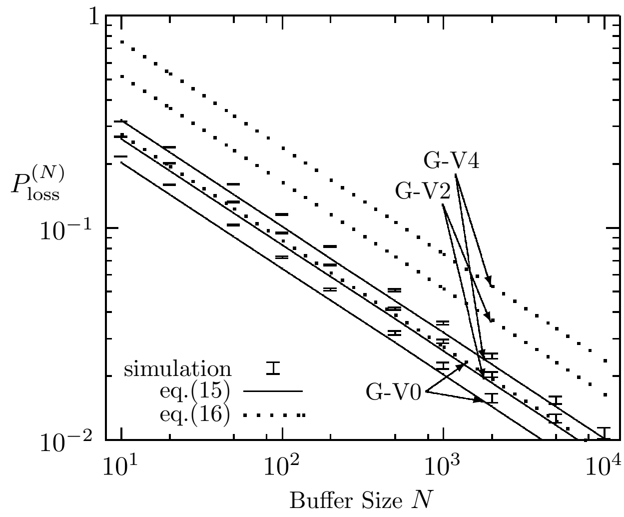

Fig. 9: Queues with generalized on/off LRD sources ($K = 5$, $\rho = .6$, $\theta = .5$).

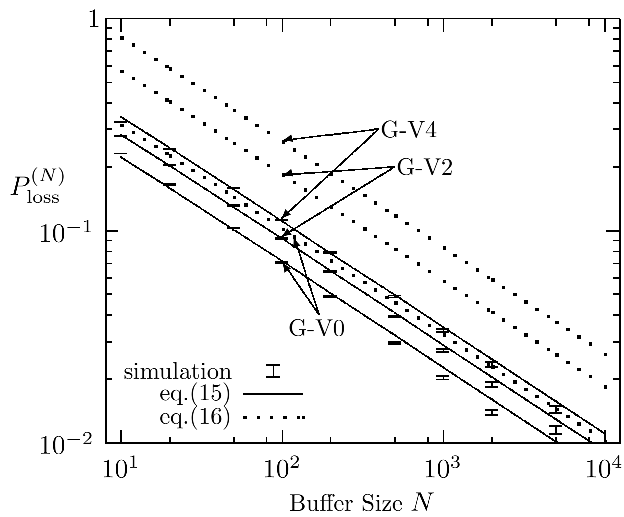

Fig. 10: Queues with M/G/∞ input ($\rho = .6, \theta = .5$).

4.4 Conclusion

In this section, we proposed a closed-form approximate formula for the cell loss probability in discrete-time single-server queues with LRD input. The formula is constructed by studying an analytically tractable queueing model with LRD input, and it is given in terms of up to the second order statistics of the input process. We conducted simulation experiments and apply the approximate formula to discrete-time single-server queues fed by LRD input processes, i.e., superposition of independent on/off sources, M/G/∞ input and their generalized ones. Through numerical experiments, we showed that the approximate formula (15) works well for these LRD input processes, especially when the ratio $c(1)/c(0)$ is large, and the conservative formula (16) works well when the ratio $c(1)/c(0)$ is small. In summary, the order of magnitude of the cell loss probability in queues with LRD input can be estimated with our approximate formulas.

5 Queues with Generalized Discrete-Time Autoregressive Input

In this section, we consider the queues with generalized discrete-time autoregressive input. This input is given by introducing an assumption to the model described in section 3.

5.1 Generalized Discrete-Time Autoregressive Input

In this section, we consider the input process described in section 3 again. In addition to Assumption 1, we assume the following.

Assumption 5 *The conditional distribution of D_ν given $Z_\nu \geq 1$ is identical to that of D_ν given $Z_\nu = 0$, i.e., $d(l) = \boldsymbol{\eta}\boldsymbol{R}^{l-1}\boldsymbol{r}$ $(l = 1, 2, \ldots)$. In other words, D_ν satisfies*

$$\Pr[D_\nu = l \mid Z_\nu = m] = \boldsymbol{\eta}\boldsymbol{R}^{l-1}\boldsymbol{r}, \quad l = 1, 2, \ldots,$$

for all $m = 0, 1, \ldots$. □

We call this input process generalized discrete-time autoregressive input. The generalized discrete-time autoregressive input has the following futures. Note first that under Assumptions 1 and 5, the marginal distribution of the B_n is identical to that of Z_ν. Thus for n such that $T_{\nu-1} < n \leq T_\nu$,

$$\begin{aligned} \mathrm{E}[B_n B_{n+k}] &= \Pr[\tilde{D}_\nu \geq k]\mathrm{E}[B_n B_{n+k} \mid \tilde{D}_\nu \geq k] + \Pr[\tilde{D}_\nu < k]\mathrm{E}[B_n B_{n+k} \mid \tilde{D}_\nu < k] \\ &= \Pr[\tilde{D}_\nu \geq k]\mathrm{E}[Z^2] + (1 - \Pr[\tilde{D}_\nu \geq k])\mathrm{E}[Z]^2 \\ &= \Pr[\tilde{D}_\nu \geq k]\mathrm{Var}[Z] + \mathrm{E}[Z]^2, \end{aligned}$$

where Z denotes a generic random variable for Z_ν and $\tilde{D}_\nu$ denotes a random variable that follows the equilibrium distribution of D_ν, i.e.,

$$\Pr[\tilde{D}_\nu = k] = \frac{\Pr[D_\nu > k]}{\mathrm{E}[D_\nu]}, \quad k = 1, 2, \ldots,$$

from which it follows that

$$\frac{\mathrm{Cov}[B_n B_{n+k}]}{\mathrm{Var}[B_n]} = \frac{\mathrm{E}[B_n B_{n+k}] - \mathrm{E}[Z]^2}{\mathrm{Var}[Z]} = \Pr[\tilde{D}_\nu \geq k].$$

Therefore the autocorrelation function $\gamma(k)$ of the B_n at lag k is given by

$$\gamma(k) = \Pr[\tilde{D}_\nu \geq k] = \boldsymbol{w}\boldsymbol{R}^k\boldsymbol{e}, \quad k = 1, 2, \ldots, \tag{35}$$

where $\boldsymbol{w}$ denotes a $1 \times M$ vector which satisfies

$$\boldsymbol{\omega}(\boldsymbol{R} + \boldsymbol{r}\boldsymbol{\eta}) = \boldsymbol{\omega}, \quad \boldsymbol{\omega}\boldsymbol{e} = 1.$$

(35) implies that the autocorrelation function $\gamma(k)$ is completely determined by the distribution of D_ν, that is, the marginal distribution has no impact to the autocorrelation function.

5.2 Asymptotic Analysis

In this subsection, we consider queues with generalized discrete-time autoregressive input and derive the asymptotic cell loss probability with this input.

Let $\delta(z)$ denote the Perron-Frobenius eigenvalue of $\boldsymbol{A}^*(z)$, and $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ denote the left and right eigenvectors associated with $\delta(z)$, respectively, with normalizing condition $\boldsymbol{u}(z)\boldsymbol{v}(z) = \boldsymbol{u}(z)\boldsymbol{e} = 1$. Further we define ρ_A as

$$\rho_A = \boldsymbol{\pi} \sum_{k=1}^{\infty} k\boldsymbol{A}_k\boldsymbol{e}.$$

Assumption 6 *The convergence radius r_A of $\boldsymbol{A}^*(z)$ is greater than one and there exists λ such that $1 < \lambda < r_A$ and $\lambda = \delta(\lambda)$.* □

Remark 4 (Neuts [18]) *Suppose $r_A > 1$. Then there exists at most one λ such that $1 < \lambda < r_A$ and $\lambda = \delta(\lambda)$, because $\delta(z)$ is a convex function of z and $\frac{d}{dz}\delta(z)|_{z=1} = \rho_A < 1$.* □

Let $\boldsymbol{G}$ denote an $(M+1) \times (M+1)$ stochastic matrix which satisfies

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{G}^k,$$

and we denote the invariant probability vector of $\boldsymbol{G}$ by $\boldsymbol{g}$:

$$\boldsymbol{g} = \boldsymbol{g}\boldsymbol{G}, \qquad \boldsymbol{g}\boldsymbol{e} = 1.$$

The cell loss probability in finite buffer queues with generalized discrete-time autoregressive input is approximately given as follows.

Theorem 6 *The cell loss probability $P_{\text{loss}}^{(N)}$ is approximately given by*

$$P_{\text{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \cdot \frac{\boldsymbol{g}\boldsymbol{v}(\lambda)}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} - 1} \cdot \frac{u_0(\lambda)b(0) + 1 - u_0(\lambda)}{b(0)} \cdot \lambda^{-N}, \tag{36}$$

where u_0 denotes the first element of $\boldsymbol{u}$. □

Proof: When the buffer size N is large enough, $\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} >> \overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}$, we then can rewrite $P_{\text{loss}}^{(N)}$ in (10) as

$$\begin{aligned} P_{\text{loss}}^{(N)} &\approx \frac{1-\rho}{\rho}\left(\frac{\overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} - \overline{\boldsymbol{x}}_N^{(\infty)}\boldsymbol{A}_0\boldsymbol{e}}\right) \\ &\approx \frac{1-\rho}{\rho} \cdot \overline{\boldsymbol{x}}_N^{(\infty)} \cdot \frac{\boldsymbol{A}_0\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}. \end{aligned}$$

Further from Remark 8 in [25], we obtain

$$P_{\text{loss}}^{(N)} \approx \frac{1-\rho}{\rho} \cdot \frac{(1-\rho_A)\boldsymbol{g}\boldsymbol{v}(\lambda)}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} - 1}\boldsymbol{u}(\lambda)\lambda^{-N} \cdot \frac{\boldsymbol{A}_0\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}}. \tag{37}$$

From $1 - \rho_A = \boldsymbol{x}_0^{(\infty)}\boldsymbol{e}$ [18] and (12), we have $\boldsymbol{x}_0^{\infty}\boldsymbol{e} = (1-\rho)\mathrm{E}[H]$. Thus (37) can be rewritten to be

$$P_{\text{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \frac{\boldsymbol{g}\boldsymbol{v}(\lambda)}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} - 1}\boldsymbol{u}(\lambda)\boldsymbol{A}_0\boldsymbol{e} \cdot \frac{\mathrm{E}[H]}{\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}} \cdot \lambda^{-N}. \tag{38}$$

On the other hand, $\mathrm{E}[H]$ and $\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}$ are given by

$$\begin{aligned} \mathrm{E}[H] &= \pi_0 \mathrm{E}[D] \\ &= \frac{\mathrm{E}[D]}{1 + b(0)\mathrm{E}[D-1]}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e} &= \begin{pmatrix} \pi_0 & \boldsymbol{\pi}_+ \end{pmatrix} \begin{pmatrix} b(0)\boldsymbol{\eta}\boldsymbol{r} & b(0)\boldsymbol{\eta}\boldsymbol{R} \\ \boldsymbol{r} & \boldsymbol{R} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{e} \end{pmatrix} \\ &= \pi_0 b(0) + (1 - \pi_0) \\ &= \frac{b(0)\mathrm{E}[D]}{1 + b(0)\mathrm{E}[D-1]}, \end{aligned} \tag{40}$$

respectively, where we use

$$\mathrm{E}[D] = \boldsymbol{\eta}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e}, \quad \mathrm{E}[D-1] = \boldsymbol{\eta}\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1}\boldsymbol{e},$$

(6), and (7). Note here that $\boldsymbol{A}_0\boldsymbol{e}$ is given by

$$\boldsymbol{A}_0\boldsymbol{e} = \begin{pmatrix} b(0) \\ \boldsymbol{e} \end{pmatrix}.$$

Substituting (39) and (40) into (38), we obtain

$$P_{\mathrm{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \frac{\boldsymbol{g}\boldsymbol{v}(\lambda)}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} -1} \frac{u_0(\lambda)b(0)+\sum_{i=1}^{M}u_i(\lambda)}{b(0)} \cdot \lambda^{-N},$$

where $u_i(\lambda)$ $(i=0,1,\ldots,M)$ denote the $(i+1)$st element of $\boldsymbol{u}(\lambda)$. ■

From (36), it is easy to see that the cell loss probability decays geometrically when the input process is generalized discrete-time autoregressive process. We call λ asymptotic decay rate of the cell loss probability $P_{\mathrm{loss}}^{(N)}$.

5.3 Special Case: DAR(1) Input

In this subsection, we consider the case of $M=1$, that is, we assume the following:

Assumption 7 *$\boldsymbol{\eta}=1$, $\boldsymbol{R}=\sigma$, and $\boldsymbol{r}=1-\sigma$, i.e.,*

$$\Pr[D_\nu = l \mid Z_\nu = m] = (1-\sigma)\sigma^{l-1}, \qquad l=1,2,\ldots,$$

for all $m=0,1,\ldots$. □

Remark 5 *Under Assumptions 1 and 7, this input process is called DAR(1) [6, 8].* □

Under Assumptions 1 and 7, it is easy to see from (35) that the autocorrelation function $\gamma(k)$ of the B_n at lag k is given by

$$\gamma(k) = \sigma^k, \qquad k=1,2,\ldots.$$

Hereafter we study the queues with DAR(1) input.

5.3.1 Derivation of the Cell Loss Probability

Note first that under Assumptions 1 and 7, the approximation in Theorem 2 becomes exact. In other words, the approximation in (10) becomes exact and the equation (11) holds in this case (see Remark 1).

We now consider the components given in (11). Because the number M of phases of the input is equal to one, we obtain the expression of $\boldsymbol{A}^*(z)$. Further we have the explicit expressions for $\boldsymbol{\pi}$, $\boldsymbol{G}$, and $\boldsymbol{g}$. They are given by following theorem:

Theorem 7 *Under Assumptions 1 and 7, $\boldsymbol{A}^*(z)$ is given by*

$$\boldsymbol{A}^*(z) = \begin{pmatrix} a(z) & b(0)\sigma \\ 1-\sigma & \sigma \end{pmatrix},$$

where $a(z)$ denotes $A_{0,0}^*(z)$, i.e.,

$$a(z) = b(0)(1-\sigma) + \sum_{m=1}^{\infty} b(m)\frac{1-\sigma}{1-\sigma z^{m-1}} z^m.$$

Further $\boldsymbol{\pi}$, $\boldsymbol{G}$, and $\boldsymbol{g}$ are given by

$$\boldsymbol{\pi} = \left(\begin{array}{cc} \frac{1-\sigma}{1-\sigma+b(0)\sigma} & \frac{b(0)\sigma}{1-\sigma+b(0)\sigma} \end{array} \right), \tag{41}$$

$$\boldsymbol{G} = \left(\begin{array}{cc} 1-\sigma & \sigma \\ 1-\sigma & \sigma \end{array} \right),$$

$$\boldsymbol{g} = \left(\begin{array}{cc} 1-\sigma & \sigma \end{array} \right), \tag{42}$$

respectively. □

Proof: It then follows from $\boldsymbol{\eta} = 1$, $\boldsymbol{R} = \sigma$, $\boldsymbol{r} = 1-\sigma$, and (5) that

$$a(z) = b(0)(1-\sigma) + \sum_{m=1}^{\infty} b(m) \frac{1-\sigma}{1-\sigma z^{m-1}} z^m.$$

Substituting $\boldsymbol{\eta} = 1$, $\boldsymbol{R} = \sigma$, and $\boldsymbol{r} = 1-\sigma$ into (6), we obtain

$$\pi_0 = \frac{1}{1+b(0)\sigma(1-\sigma)^{-1}}, \quad \pi_1 = \frac{b(0)\sigma(1-\sigma)^{-1}}{1+b(0)\sigma(1-\sigma)^{-1}},$$

respectively. Thus we obtain (41).

Note here that $\boldsymbol{A}_0$ is given by

$$\boldsymbol{A}_0 = \left(\begin{array}{c} b(0) \\ 1 \end{array} \right) \cdot \left(\begin{array}{cc} 1-\sigma & \sigma \end{array} \right),$$

and $\boldsymbol{A}_k$'s $(k = 1, 2, \ldots)$ are given by

$$\boldsymbol{A}_k = \left(\begin{array}{cc} [A(k)]_{0,0} & 0 \\ 0 & 0 \end{array} \right).$$

Thus we can easily confirm that

$$\begin{aligned} \sum_{k=0}^{\infty} \boldsymbol{A}_k \boldsymbol{G}^k &= \left(\begin{array}{c} b(0) \\ 1 \end{array} \right) \cdot \left(\begin{array}{cc} 1-\sigma & \sigma \end{array} \right) + \sum_{k=1}^{\infty} \left(\begin{array}{cc} [A(k)]_{0,0} & 0 \\ 0 & 0 \end{array} \right) \left(\begin{array}{cc} 1-\sigma & \sigma \\ 1-\sigma & \sigma \end{array} \right)^k \\ &= \left(\begin{array}{cc} b(0)(1-\sigma) & b(0)\sigma \\ 1-\sigma & \sigma \end{array} \right) + \left(\begin{array}{cc} (1-b(0)(1-\sigma) & (1-b(0))\sigma \\ 0 & 0 \end{array} \right) \\ &= \boldsymbol{G}, \end{aligned}$$

and

$$\boldsymbol{g}\boldsymbol{G} = \boldsymbol{g}.$$

■

To express the cell loss probability, we consider $\boldsymbol{u}$ and $\boldsymbol{v}$ and express them in terms of $b(0)$, λ, and σ. They are given as follows:

Lemma 5 *Under Assumptions 1 and 7, $\boldsymbol{u}(\lambda)$ and $\boldsymbol{v}(\lambda)$ are given by*

$$\boldsymbol{u}(\lambda) = \left(\begin{array}{cc} \frac{\lambda-\sigma}{\lambda+b(0)\sigma-\sigma}, & \frac{b(0)\sigma}{\lambda+b(0)\sigma-\sigma} \end{array} \right), \tag{43}$$

$$\boldsymbol{v}(\lambda) = \left(\begin{array}{c} \frac{(\lambda-\sigma)(\lambda-\sigma+b(0)\sigma)}{b(0)\sigma(1-\sigma)+(\lambda-\sigma)^2} \\ \frac{(1-\sigma)(\lambda-\sigma+b(0)\sigma)}{b(0)\sigma(1-\sigma)+(\lambda-\sigma)^2} \end{array} \right), \tag{44}$$

respectively. □

The proof of Lemma 5 is given in Appendix D.

Now we can express the cell loss probability in terms of ρ, σ, $b(0)$, and λ.

Theorem 8 *If $M = 1$, i.e., the input process is DAR(1) input, the cell loss probability $P_{\rm loss}^{(N)}$ is approximately given by*

$$P_{\rm loss}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \cdot \frac{1}{\frac{d}{dz}(z)|_{z=\lambda} - 1} \cdot \frac{1-\sigma}{b(0)\sigma(1-\sigma) + (\lambda-\sigma)^2} \cdot \lambda^{-N+2}, \tag{45}$$

where λ ($\lambda > 1$) is the solution of $\delta(z) = z$. □

Proof: From (42) and (44), we obtain

$$\begin{aligned} \boldsymbol{gv}(\lambda) &= \begin{pmatrix} 1-\sigma & \sigma \end{pmatrix} \begin{pmatrix} v_0(\lambda) \\ v_1(\lambda) \end{pmatrix} \\ &= (1-\sigma)v_0(\lambda) + \sigma v_1(\lambda) \\ &= \frac{\lambda(1-\sigma)(\lambda + b(0)\sigma - \sigma)}{b(0)\sigma(1-\sigma) + (\lambda-\sigma)^2}. \end{aligned} \tag{46}$$

Substituting (43) and (46) into (36), we obtain

$$\begin{aligned} P_{\rm loss}^{(N)} &= \frac{(1-\rho)^2}{\rho} \frac{1}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} -1} \boldsymbol{gv}(\lambda) \cdot \frac{u_0(\lambda)b(0) + 1 - u_0(\lambda)}{b(0)} \cdot \lambda^{-N} \\ &= \frac{(1-\rho)^2}{\rho} \frac{1}{\frac{d}{dz}\delta(z)\mid_{z=\lambda} -1} \cdot \frac{\lambda(1-\sigma)(\lambda+b(0)\sigma-\sigma)}{b(0)\sigma(1-\sigma)+(\lambda-\sigma)^2} \cdot \frac{1}{b(0)} \cdot \frac{(\lambda-\sigma)b(0)+b(0)\sigma}{\lambda+b(0)\sigma-\sigma} \cdot \lambda^{-N}, \end{aligned}$$

from which (45) follows. ■

5.3.2 Approximation of the Asymptotic Decay Rate

We should calculate λ and $\frac{d}{dz}\delta(z)|_{z=\lambda}$ in (45), to estimate the cell loss probability numerically with Theorem 8. In this subsection, we show how to approximate them easily.

Hereafter we use the following conventions. For any function $f(z)$ of z, let $f^{(n)}$ denote the nth derivative of $f(z)$ evaluated at $z = 1-$, i.e.,

$$f^{(n)} = \lim_{z \to 1-} \frac{d^n}{dz^n} f(z).$$

Now we consider the Taylor expansion of $\delta(z)$ around $z = 1$. For simplicity, let ϕ denote $\phi = \lambda - 1$. $\delta(1+\phi)$ then satisfies

$$\delta(1+\phi) = \sum_{l=0}^{\infty} \frac{\phi^l}{l!} \delta^{(l)}. \tag{47}$$

To estimate ϕ, we truncate infinite series on the right hand side of (47) at n^* and we define δ_{n^*} as

$$\delta_{n^*} = \sum_{l=0}^{n^*} \frac{\phi^l}{l!} \delta^{(l)}. \tag{48}$$

We then consider (48) as an alternative of (47).

ϕ is approximately given by the solution of the polynomial equation as follows:

Theorem 9 *Suppose that there exists one* ϕ *such that*

$$\sum_{l=1}^{n^*} \frac{\phi^{l-1}}{l!} \delta^{(l)} = 1, \tag{49}$$

and $\phi > 0$*. The solution of* $\delta(1+\phi) = 1+\phi$ *is then approximately given by the solution of (49) with* $\phi > 0$*.* □

Proof: Note here that $\delta(1+\phi) = 1+\phi$. Thus we have

$$\begin{aligned} 1+\phi &= \delta(1+\phi) \\ &\approx \delta_{n^*} \\ &= \delta^{(0)} + \sum_{l=1}^{n^*} \frac{\phi^l}{l!} \delta^{(l)}. \end{aligned}$$

Thus noting $\delta^{(0)} = \delta(1) = 1$ and $\phi \neq 0$, we have (49). ■

Next we approximate $\frac{d}{dz}\delta(z)|_{z=\lambda}$. To do so, we consider the Taylor series expansion:

$$\frac{d}{dz}\delta(z)|_{z=1+\phi} = \sum_{l=1}^{\infty} \frac{\phi^{l-1}}{(l-1)!} \delta^{(l)}.$$

In the same way as we consider (48), we truncate the infinite series on the right hand side of (47) at n^* and define δ'_{n^*} as

$$\delta'_{n^*} = \sum_{l=1}^{n^*} \frac{\phi^{l-1}}{(l-1)!} \delta^{(l)}.$$

We then express the cell loss probability $P_{\text{loss}}^{(N)}$ approximately without $\frac{d}{dz}\delta(z)|_{z=\lambda}$.

Theorem 10 *If* $M = 1$*, i.e., the input process is DAR(1) input, the cell loss probability* $P_{\text{loss}}^{(N)}$ *is approximately given by*

$$P_{\text{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \cdot \frac{1}{\sum_{l=2}^{n^*} \frac{(l-1)}{l!} \phi^{l-1} \delta^{(l)}} \cdot \frac{1-\sigma}{b(0)\sigma(1-\sigma) + (\lambda-\sigma)^2} \cdot \lambda^{-N+2}, \tag{50}$$

where $\lambda = 1+\phi$ *and* ϕ *is obtained with Theorem 9.* □

Proof: Note here that (49) and

$$\begin{aligned} \frac{d}{dz}\delta(z) \mid_{z=\lambda} -1 &\approx \delta'_{n^*} - 1 \\ &= \delta^{(1)} - 1 + \sum_{l=2}^{n^*} \frac{\phi^{l-1}}{(l-1)!} \delta^{(l)}. \end{aligned}$$

Thus we have

$$\frac{d}{dz}\delta(z) \mid_{z=\lambda} -1 = \sum_{l=2}^{n^*} \frac{\phi^{l-1}(l-1)}{l!} \delta^{(l)}. \tag{51}$$

Substituting (51) into (45), we have (50). ■

Next we consider $\delta^{(n)}$ because $\delta^{(n)}$ is needed to solve the equation (49) and apply the approximate formula (50). Note here that, when n^* is large, it is difficult to solve equation (49). Further it is also difficult to obtain the explicit expression for $\delta^{(n)}$ for large n. Therefore we mainly consider $\delta^{(n)}$ for $n = 1, 2, 3$, and 4.

To obtain $\delta^{(n)}$, we consider the relationship between $a(z)$ and $\delta(z)$. Note that $\det(\boldsymbol{A}^*(z) - \delta(z)\boldsymbol{I}) = 0$. Thus we have

$$\det \begin{vmatrix} a(z) - \delta(z) & b(0)\sigma \\ 1 - \sigma & \sigma - \delta(z) \end{vmatrix} = 0,$$

from which it follows

$$\delta(z)^2 - \Big(a(z) + \sigma\Big)\delta(z) + a(z)\sigma - b(0)\sigma(1 - \sigma) = 0. \tag{52}$$

Considering the differentiations of the both sides of (52), we obtain $\delta^{(n)}$.

Lemma 6 *$\delta^{(n)}$ $(n = 1, 2, \ldots)$ is recursively given by*

$$\delta^{(n)} = \frac{(1-\sigma)a^{(n)} + \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)} (a^{(n-l)} - \delta^{(n-l)})}{1 - \sigma + b(0)\sigma}. \tag{53}$$

In particular, $\delta^{(1)}$, $\delta^{(2)}$, $\delta^{(3)}$, and $\delta^{(4)}$ are given by

$$\begin{aligned}
\delta^{(1)} &= \frac{(1-\sigma)a^{(1)}}{1 - \sigma + b(0)\sigma}, \\
\delta^{(2)} &= \frac{(1-\sigma)a^{(2)} + 2\delta^{(1)}a^{(1)} - 2(\delta^{(1)})^2}{1 - \sigma + b(0)\sigma}, \\
\delta^{(3)} &= \frac{(1-\sigma)a^{(3)} + 3\delta^{(1)}a^{(2)} + 3\delta^{(2)}a^{(1)} - 6\delta^{(1)}\delta^{(2)}}{1 - \sigma + b(0)\sigma}, \\
\delta^{(4)} &= \frac{(1-\sigma)a^{(4)} + 4\delta^{(1)}a^{(3)} + 6\delta^{(2)}a^{(2)} + 4\delta^{(3)}a^{(1)} - 8\delta^{(1)}\delta^{(3)} - 6(\delta^{(2)})^2}{1 - \sigma + b(0)\sigma},
\end{aligned}$$

respectively. □

The poof of Lemma 6 is given in Appendix E.

To calculate $\delta^{(n)}$, we must use $a^{(n)}$'s as we can see in (53).

Lemma 7 *$a^{(n)}$ $(n = 1, 2, 3, 4)$ are given by*

$$\begin{aligned}
a^{(1)} &= \frac{\mathrm{E}[B] - \sigma(1 - b(0))}{1 - \sigma}, \\
a^{(2)} &= \frac{(1+\sigma)\mathrm{E}[B(B-1)] - 2\sigma\mathrm{E}[B] + 2\sigma(1 - b(0))}{(1-\sigma)^2}, \\
a^{(3)} &= \frac{(\sigma^2 + 4\sigma + 1)\mathrm{E}[B(B-1)(B-2)] + 6\sigma\mathrm{E}[B] - 6\sigma(1 - b(0))}{(1-\sigma)^3}, \\
a^{(4)} &= \frac{(\sigma^3 + 11\sigma^2 + 11\sigma + 1)\mathrm{E}[B(B-1)(B-2)(B-3)]}{(1-\sigma)^4} \\
&\qquad + \frac{4(\sigma^3 + 7\sigma^2 + 4\sigma)\mathrm{E}[B(B-1)(B-2)]}{(1-\sigma)^4} \\
&\qquad\qquad + \frac{12(\sigma^2 + 1)\mathrm{E}[B(B-1)] - 24\sigma\mathrm{E}[B] + 24\sigma(1 - b(0))}{(1-\sigma)^4},
\end{aligned} \tag{54}$$

respectively. □

The poof of lemma 7 is given in Appendix F.

To calculate the cell loss probability, we consider two cases, $n^* = 4$ and $n^* = 3$. When we set $n^* = 4$, we can estimate ϕ and $P_{\text{loss}}^{(N)}$ by the following theorem.

Theorem 11 *ϕ is approximately given by the solution of the cubic equation:*

$$\frac{\delta^{(4)}}{24}\phi^3 + \frac{\delta^{(3)}}{6}\phi^2 + \frac{\delta^{(2)}}{2}\phi - (1 - \delta^{(1)}) = 0, \tag{55}$$

with $\phi > 0$. Further the cell loss probability $P_{\text{loss}}^{(N)}$ is approximately given by

$$P_{\text{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \frac{1}{\dfrac{\delta^{(2)}\phi}{2} + \dfrac{\delta^{(3)}\phi^2}{3} + \dfrac{\delta^{(4)}\phi^3}{8}} \cdot \frac{1-\sigma}{b(0)\sigma(1-\sigma) + (\lambda - \sigma)^2} \cdot \lambda^{-N+2}, \tag{56}$$

where $\lambda = 1 + \phi$. □

Proof: From (49), (55) is obtained immediately. Further (56) is immediately obtained from (50). ■

When we set $n^* = 3$, we obtain ϕ and $P_{\text{loss}}^{(N)}$ as the following corollary.

Corollary 1 *Suppose $\delta^{(2)} > 0$. ϕ is approximately given by*

$$\phi = \frac{-3\delta^{(2)} + \sqrt{9(\delta^{(2)})^2 + 24(1 - \delta^{(1)}\delta^{(3)})}}{2\delta^{(3)}}. \tag{57}$$

Further the cell loss probability $P_{\text{loss}}^{(N)}$ is approximately given by

$$P_{\text{loss}}^{(N)} \approx \frac{(1-\rho)^2}{\rho} \frac{1}{\dfrac{\delta^{(2)}\phi}{2} + \dfrac{\delta^{(3)}\phi^2}{3}} \cdot \frac{1-\sigma}{b(0)\sigma(1-\sigma) + (\lambda - \sigma)^2} \cdot \lambda^{-N+2}, \tag{58}$$

where $\lambda = 1 + \phi$. □

Proof: Substituting $n^* = 3$ into (49), we have

$$\delta^{(3)}\phi^2 + 3\delta^{(2)}\phi - 6(1 - \delta^{(1)}) = 0.$$

It is easy to see that the two solutions of this quadratic equation are given by

$$\phi = \frac{-3\delta^{(2)} \pm \sqrt{9(\delta^{(2)})^2 + 24(1 - \delta^{(1)}\delta^{(3)})}}{2\delta^{(3)}}.$$

Thus the solution which satisfies $\phi > 0$ is given by (57). Further (58) is immediately obtained from (50). ■

Finally, we explain how to apply the approximate formula (56) (resp. (58)). We first obtain $a^{(n)}$'s from input parameters with Lemma 7. Next we calculate $\delta^{(n)}$'s with Lemma 6. After that we calculate ϕ, λ, where λ is given by $\lambda = \phi + 1$, and $P_{\text{loss}}^{(N)}$ approximately with Theorem 11 (resp. Corollary 1).

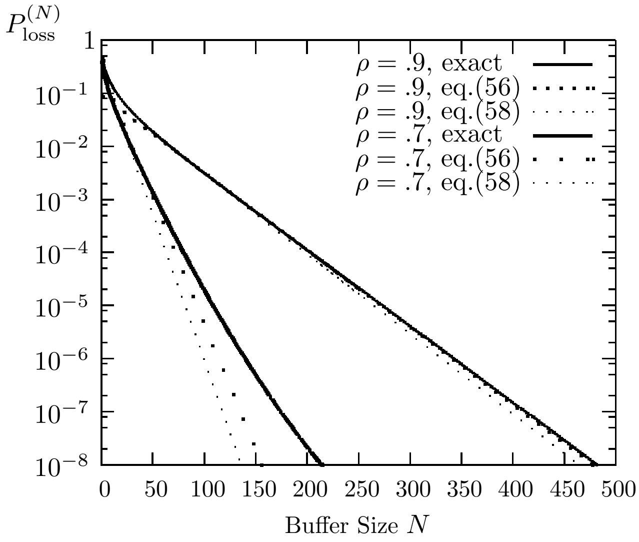

Fig. 11: Queues with DAR(1) input ($\sigma = .5$).

5.4 Numerical Results

In this subsection, we provide some numerical results to show the accuracy of the approximate formula for the cell loss probability with DAR(1) input. To do so, we compare the approximate results in subsection 5.3 with those obtained by exact analysis in section 3. For a while, we assume that the number of cells arriving in a slot in steady state follows a geometric distribution (32), i.e.,

$$b(m) = \frac{1}{1+\rho}\left(\frac{\rho}{1+\rho}\right)^m, \qquad m = 0, 1, \ldots,$$

as in subsection 4.3.1. Note here that the lighter the load becomes, the larger the asymptotic decay rate λ tends to become. Thus when ρ is small, ϕ tends to be quite large and the truncation in (47) yields large errors. Hence the approximate formula does not work well when the load is low, that is, we expect that the approximate formula works fairly when the load is moderate or high. Therefore we mainly conduct numerical experiment and apply the formula to queues with moderate or high load.

We first examine the difference between the accuracy of (56) (i.e., we set $n^* = 4$.) and that of (58) (i.e., we set $n^* = 3$.). Fig. 12 shows the exact and approximate cell loss probabilities for $\rho = .9$ and $\rho = .7$ as a function of the buffer size, where $\sigma = .5$. We observe that the difference among the exact cell loss probability, (58), and (56) is not so large when the input is strong. It is clear that the advantage of (58) is that there exists no need to solve a cubic equation. Thus when the input is very strong, we may use (58) to evaluate the cell loss probability more easily. As a whole, however, we observe that (56) is more accurate than (58) in both cases. This is because (56) is derived by considering higher order of the Taylor series expansion. Thus we recommend using (56) to evaluate the cell loss probability.

In what follows, we use (56) to examine the accuracy of the approximate formula for the cell loss probability. In other words, we set $n^* = 4$ and apply the formula with Theorem 11 to calculate the decay rate λ.

We now examine when the offered load is very heavy, i.e., $\rho = .9$. Fig. 12 shows the exact and approximate cell loss probabilities as a function of the buffer size, where $\rho = .9$. We observe that the approximation is very accurate for all cases. As we can see in Fig. 12, the difference between the exact cell loss probability and the approximated cell loss probability is very small in all cases,

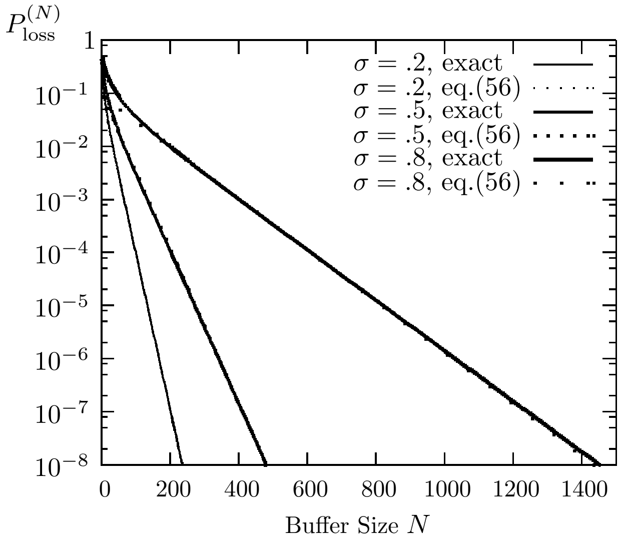

Fig. 12: Queues with DAR(1) input ($\rho = .9$).

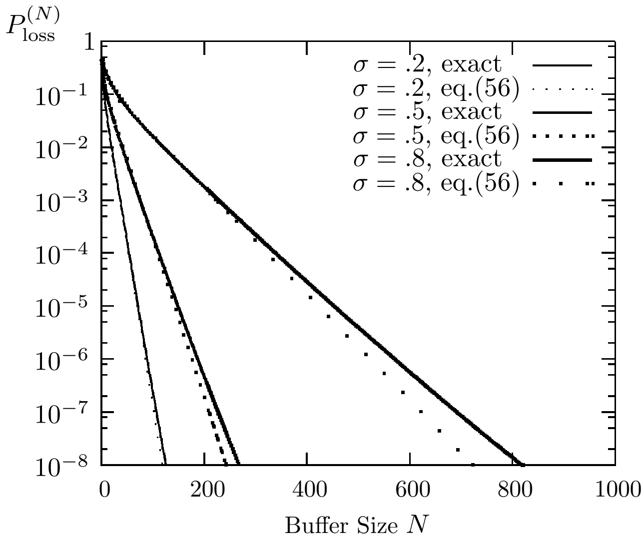

Fig. 13: Queues with DAR(1) input ($\rho = .8$).

i.e., correlation in arrivals.

Next we examine when the load ρ is lighter than $\rho = .9$. Fig. 13 shows the exact and approximate cell loss probabilities as a function of the buffer size, where $\rho = .8$. We observe that the approximation is accurate when correlation in arrival is weak ($\sigma = .2$). In a modest or strong correlated situation ($\sigma = .5$ or $\sigma = .8$), however, the approximate formula (56) underestimates the cell loss probability, even though the difference is not so large.

Further, we examine how accurate when the offered load ρ is $\rho = .7$. Fig. 14 shows the exact and approximate cell loss probabilities as a function of the buffer size. We observe that the approximate formula is fairly accurate when the correlation in arrivals is weak ($\sigma = .2$). On the other hand, the approximate formula (56) does not work well for the queue with strong correlation input.

It seems that the approximate formula does not work well because the approximation for the asymptotic decay rate λ is not so accurate. When offered load is not so heavy, λ tends to be apart from 1, that is, ϕ tends to be apart from 0. Thus it is difficult to estimate λ because we truncate the right hand side of (47). That is the reason why the approximate formula does not work well

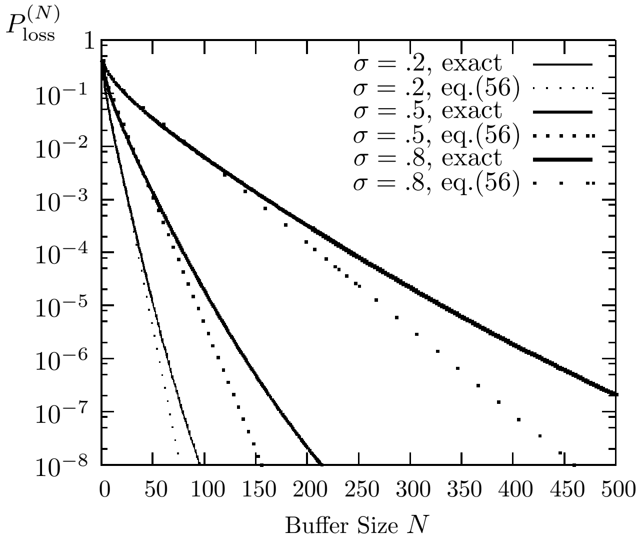

Fig. 14: Queues with DAR(1) input ($\rho = .7$).

when the offered load is light.

From Fig. 13 and Fig. 14, we also observe that the approximate formula works well when the correlation of the input is weak ($\sigma = .2$). It is also observed that the heavier the correlation becomes, the worse the approximation becomes. When the correlation of the input is strong, σ approaches 1. Thus $a^{(n)}$ and $\delta^{(n)}$ for $n = 1, 2, 3$, and 4 tend to be large. This fact is easily checked from the denominator on the right hand side of (54), for example. Thus it may be expected that $a^{(n)}$ and $\delta^{(n)}$ for $n \geq 5$ also become quite large. In such a condition, the component $\delta^{(l)}\phi^l$ ($l = 5, 6, \ldots$) in (47) is not small enough, and therefore, it is difficult to estimate λ because of the truncation in (47).

Finally, we examine the impact of the marginal distribution B on the accuracy of the approximation. For this purpose, we set $\rho = .7$ and $\sigma = .5$ and apply the formula to queues with different marginal distributions. In the same way as in subsection 4.3.1, we study for three marginal distributions. One is the geometric distribution given by (32), as a distribution with modest variance. Note that $\mathrm{E}[B] = .7$, $\mathrm{Var}[B] = 119/100 = 1.19$, and $b(0) = 10/17$ in this case. Besides, we prepare two multi-point distributions as in subsection 4.3.1. Note here that $b(0) = 10/17$ and $\rho = .7$. As a result, for a distribution with small variance, we have 3-point distribution with $b(1) = 21/170$ and $b(2) = 49/170$ ($\mathrm{Var}[B] = 1337/1700 = 0.786\ldots$). On the other hand, for a distribution with large variance, we have 3-point distribution with $b(1) = 581/1530$ and $b(10) = 49/1530$ ($\mathrm{Var}[B] = 47313/15300 = 3.092\ldots$).

Fig. 15 shows the exact and approximate cell loss probabilities as a function of the buffer size. We observe that the approximation is quite accurate when the variance is small. On the other hand, we observe that the approximation is not so bad when the variance is large. Therefore we may use the approximate formula regardless of the marginal distribution of the input.

In summary, the heavier the load is or the weaker the correlation of input is, the better the approximate formula works. If the load ρ is larger than .7 or σ is small, the approximate formula proposed here works well regardless of the marginal distribution. In the light load case, the formula may not be suitable to estimate the cell loss probability.

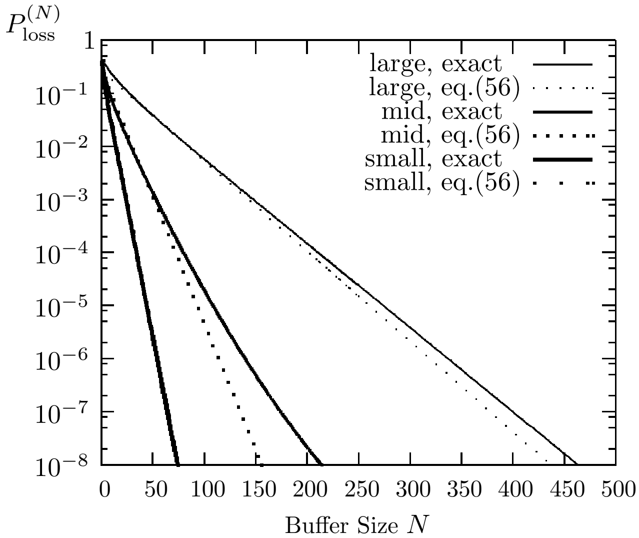

Fig. 15: Queues with DAR(1) input ($\rho = .7$, $\sigma = .5$).

5.5 Conclusion

In this section, we studied a single server queue with generalized discrete-time autoregressive input. We derived the cell loss probability in finite-buffer queues with this input process. We also derived the approximate formula for the cell loss probability in finite-buffer queues with DAR(1) input. To do so, we proposed the approximation for the asymptotic decay rate of the cell loss probability. We conducted numerical experiments and apply the approximate formula to queues fed by DAR(1) input. Through numerical examples, we showed that the formula works well when the offered load is strong or the correlation in arrivals is weak.

6 Conclusion

This thesis studied discrete-time single-server queues with correlated input. We studied analytically tractable queueing model with correlated input. This input can represent wide range of autocorrelation function in arrival process, that is, both geometrically decaying and subexponentially decaying autocorrelation functions.

Based on this model, we developed a closed-form approximate formula for the cell loss probability with long-range dependent (LRD) input. We conducted simulation experiments to study the accuracy and robustness of the formulas. Through numerical examples, we showed the order of magnitude of the cell loss probability in queues with LRD input can be estimated with our approximate formulas.

We also studied queues with generalized discrete-time autoregressive process and developed the approximate formula for the cell loss probability in finite buffer queues with DAR(1) input. Through numerical experiments, we showed that the formula works well when the offered load is strong or the correlation of input is weak.

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Appendix A Proof of Lemma 1

Let G and $Q^{(N)}$ denote generic random variables for G_n's and $Q_n^{(N)}$'s, respectively, in steady state. Further, let $\widehat{H}$ denote a random variable representing the backward recurrence time of H, i.e.,

$$\Pr[\widehat{H} = k] = \frac{\Pr[H > k]}{\mathrm{E}[H]}, \quad k = 0, 1, \ldots.$$

We then have

$$Q^{(N)} = \begin{cases} X^{(N)}, & \text{if } \widehat{H} = 0, \\ \min\Big((X^{(N)} - 1)^+ + \widehat{H}(G - 1) + 1, N\Big), \\ & \text{if } \widehat{H} \geq 1. \end{cases}$$

Note here that $Q^{(N)} \geq 1$ if $\widehat{H} \geq 1$, because $\widehat{H} \geq 1$ implies $G \geq 1$. Thus, we obtain

$$\begin{aligned} 1 - \rho^{(N)} &= \Pr[Q^{(N)} = 0] = \Pr[\widehat{H} = 0, X^{(N)} = 0] \\ &= \Pr[\widehat{H} = 0] \Pr[X^{(N)} = 0] = \frac{1}{\mathrm{E}[H]} \cdot \boldsymbol{x}_0^{(N)} \boldsymbol{e}, \end{aligned}$$

which completes the proof. ■

Appendix B Proof of Lemma 3

By definition, $\mathrm{E}[B_n^*]$'s ($n = 1, 2, \ldots$) satisfy the following renewal equation: for $n = 1, 2, \ldots$,

$$\begin{aligned} \mathrm{E}[B_n^*] &= \sum_{m=1}^{\infty} m \Pr[Z = m, D \geq n] + \sum_{l=1}^{n-1} \Pr[D = l] \mathrm{E}[B_{n-l}^*] \\ &= \mathrm{E}[B] \Pr[D^{[+]} \geq n] + \sum_{l=1}^{n-1} \Pr[D = l] \mathrm{E}[B_{n-l}^*]. \end{aligned} \tag{59}$$

We then define $r^{(k)}(m)$ ($k, m = 0, 1, \ldots$) as

$$r^{(0)}(m) = \begin{cases} 1, & m = 0, \\ 0, & m = 1, 2, \ldots, \end{cases}$$

and for $k = 1, 2, \ldots$,

$$r^{(k)}(m) = \begin{cases} 0, & m = 0, \\ \displaystyle\sum_{l=1}^{m} \Pr[D = l] r^{(k-1)}(m - l), & m = 1, 2, \ldots. \end{cases}$$

Further, we define $u(m)$ ($m = 0, 1, ...$) as

$$u(m) = \sum_{k=0}^{\infty} r^{(k)}(m), \quad m = 0, 1, \ldots.$$

(59) then implies

$$\begin{aligned} \mathrm{E}[B_n^*] &= \mathrm{E}[B] \sum_{l=1}^{n} u(n - l) \Pr[D^{[+]} \geq l] \\ &= \mathrm{E}[B] \mathrm{E}[D^{[+]}] \sum_{l=0}^{n-1} u(n - 1 - l) \Pr[\widetilde{D}^{[+]} = l]. \end{aligned} \tag{60}$$

Substituting (60) into (21) and noting $\mathrm{E}[D^{[+]}] = \mathrm{E}[D]$, we obtain

$$\begin{aligned} g(k) &= \Pr[\widetilde{D}^{[+]} \le k-1] - \mathrm{E}[D^{[+]}] \sum_{j=0}^{k-1} \Pr[\widetilde{D}^{[+]} = j] \sum_{l=0}^{k-1-j} u(k-1-j-l) \Pr[\widetilde{D}^{[+]} = l] \\ &= \Pr[\widetilde{D}^{[+]} \le k-1] - \mathrm{E}[D] \sum_{l=0}^{k-1} u(l) f(k-1-l) \\ &= \mathrm{E}[D] \left(\frac{1}{\mathrm{E}[D]} - \sum_{l=0}^{k-1} u(l) f(k-1-l) \right) \Pr[\widetilde{D}^{[+]} \ge k], \end{aligned} \tag{61}$$

where

$$f(k) = \sum_{j=0}^{k} \Pr[\widetilde{D}^{[+]} = j] \Pr[\widetilde{D}^{[+]} = k-j], \quad k = 0, 1, \ldots.$$

In what follows, we consider the asymptotic property of the first term on the right hand side of (61).

Note first that $f(k)$ is the probability mass function of the sum of two i.i.d. regularly varying random variables. See Remark 2 also. Note that regularly varying distributions belong to the subclass $\mathcal{S}^*$ of subexponential distributions [4]. Thus from the statement after Definition A.1 in [10], we obtain

$$\begin{aligned} f(k) &\overset{k}{\sim} 2 \Pr[\widetilde{D}^{[+]} = k] \\ &\overset{k}{\sim} \frac{2\beta}{\mathrm{E}[D]} k^{-(\theta+1)}. \end{aligned} \tag{62}$$

On the other hand,

$$\begin{aligned} \Pr[D > k] &= \Pr[Z = 0] \Pr[D > k \mid Z = 0] + \Pr[Z \ge 1] \Pr[D^{[+]} > k] \\ &= b(0)\sigma^k + \Pr[Z \ge 1] \beta k^{-(\theta+1)} \\ &\overset{k}{\sim} \Pr[Z \ge 1] \beta k^{-(\theta+1)}. \end{aligned} \tag{63}$$

Thus, from (62) and (63), we have

$$f(k) \overset{k}{\sim} \frac{2}{\Pr[Z \ge 1]} \cdot \Pr[D > k].$$

Because $\theta + 1 > 1$, we can use the result for the convergence rate of the key renewal theorem in [5]. Namely, Theorem 3.1 (ii) in [5] implies

$$\begin{aligned} \frac{1}{\mathrm{E}[D]} - \sum_{l=0}^{k-1} f(k-1-l) u(l) &\overset{k}{\sim} \frac{1}{\theta \mathrm{E}[D]} \left(\frac{2}{\Pr[Z \ge 1] \mathrm{E}[D]} - \frac{1}{\mathrm{E}[D]} \right) \cdot k^{-\theta} \Pr[Z \ge 1] \beta \\ &\overset{k}{\sim} \frac{2 - \Pr[Z \ge 1]}{\theta \mathrm{E}[D]^2} \cdot \beta k^{-\theta}, \end{aligned} \tag{64}$$

because $\sum_{k=0}^{\infty} f(k) = 1$. Applying (22) and (64) to (61) and noting $\Pr[Z \ge 1] = 1 - b(0)$ yield (23). ■

Appendix C Proof of Lemma 4

It follows from Assumption 4 and Remark 2 that for an arbitrarily fixed $\varepsilon > 0$, there exists $x^* > 0$ such that for all real $x > x^*$,

$$\frac{\Pr[D^{[+]} > x]}{x^{-(\theta+1)}} < \beta(1 + \varepsilon).$$

We define k^* as $\lfloor x^* \rfloor + 1$. It then follows that

$$\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{\left(\dfrac{k-1}{m-1}\right)^{-(\theta+1)}} < \beta(1+\varepsilon),$$

for every integer k such that $(k-1)/(m-1) \geq k^* > x^*$. In other words, for all $k = (m-1)k^* + 1, (m-1)k^* + 2, \ldots,$

$$\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{(k-1)^{-(\theta+1)}} < \beta(1+\varepsilon)(m-1)^{\theta+1}. \tag{65}$$

On the other hand, for all $k = 2, 3, \ldots, (m-1)k^*$,

$$\begin{aligned}\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{(k-1)^{-(\theta+1)}} &\leq \frac{1}{(k-1)^{-(\theta+1)}} \\ &< k^{\theta+1} \\ &\leq (m-1)^{\theta+1}(k^*)^{\theta+1}.\end{aligned} \tag{66}$$

It then follows from (65) and (66) that

$$\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{(k-1)^{-(\theta+1)}} < \zeta(m-1)^{\theta+1},$$

where $\zeta = \max(\beta(1+\varepsilon), (k^*)^{\theta+1}) < \infty$. Note here that

$$\sum_{m=2}^{\infty} b(m)\zeta(m-1)^{\theta+1} = \overline{B}(\theta)\zeta < \infty,$$

because $0 < \theta < 1$ and $\mathrm{Var}(B) < \infty$. Thus the dominated convergence theorem yields

$$\begin{aligned}\lim_{k\to\infty}\sum_{m=2}^{\infty} b(m)\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{(k-1)^{-(\theta+1)}} &= \sum_{m=2}^{\infty} b(m)\lim_{k\to\infty}\frac{\Pr\left[D^{[+]} > \dfrac{k-1}{m-1}\right]}{(k-1)^{-(\theta+1)}} \\ &= \sum_{m=2}^{\infty} b(m)\beta(m-1)^{\theta+1} \\ &= \overline{B}(\theta)\beta,\end{aligned}$$

which completes the proof. ■

Appendix D Proof of Lemma 5

By definition,

$$\begin{pmatrix} u_0(\lambda) & u_1(\lambda) \end{pmatrix}\begin{pmatrix} a(\lambda) & b(0)\sigma \\ 1-\sigma & \sigma \end{pmatrix} = \delta(\lambda)\begin{pmatrix} u_0(\lambda) & u_1(\lambda) \end{pmatrix},$$

from which it follows that

$$b(0)\sigma u_0(\lambda) + \sigma u_1(\lambda) = \lambda u_1(\lambda),$$

where we use $\delta(\lambda) = \lambda$. Note here that $u_0(\lambda) + u_1(\lambda) = 1$, we obtain,

$$u_0(\lambda) = \frac{\lambda - \sigma}{\lambda + b(0)\sigma - \sigma}, \qquad u_1(\lambda) = \frac{b(0)\sigma}{\lambda + b(0)\sigma - \sigma}. \tag{67}$$

Also, by definition, we have

$$\begin{pmatrix} a(\lambda) & b(0)\sigma \\ 1-\sigma & \sigma \end{pmatrix} \begin{pmatrix} v_0(\lambda) \\ v_1(\lambda) \end{pmatrix} = \delta(\lambda) \begin{pmatrix} v_0(\lambda) \\ v_1(\lambda) \end{pmatrix},$$

from which it follows that

$$(1-\sigma)v_0(\lambda) + \sigma v_1(\lambda) = \lambda v_1(\lambda).$$

Note here that $u_0(\lambda)v_0(\lambda) + u_1(\lambda)v_1(\lambda) = 1$. Thus we obtain

$$v_0(\lambda) = \frac{1 - u_1(\lambda)v_1(\lambda)}{u_0(\lambda)},$$

from which it follows that

$$\begin{cases} v_0(\lambda) = \dfrac{\lambda - \sigma}{(1-\sigma)u_1(\lambda) + (\lambda - \sigma)u_0(\lambda)}, \\ v_1(\lambda) = \dfrac{1 - \sigma}{(1-\sigma)u_1(\lambda) + (\lambda - \sigma)u_0(\lambda)}. \end{cases} \tag{68}$$

Substituting (67) into (68), we have (44). ■

Appendix E Proof of Lemma 6

For simplicity, we first define $h_1(z)$ and $h_0(z)$ as $h_1(z) = a(z) + \sigma$ and $h_0 = \sigma(a(z) - b(0)(1-\sigma))$, respectively.

Differentiating (52) with respect to z, we obtain

$$\sum_{l=0}^{n} \binom{n}{l} \delta^{(l)}(z)\delta^{(n-l)}(z) - \sum_{l=0}^{n} \binom{n}{l} \delta^{(l)}(z) h_1^{(n-l)}(z) + h_0^{(n)}(z) = 0.$$

It then follows from $h_1^{(n)}(z) = a^{(n)}(z)$ $(n = 1, 2, \ldots)$ and $h_0^{(n)} = \sigma a^{(n)}(z)$ $(n = 1, 2, \ldots)$ that

$$\begin{aligned} &2\delta^{(n)}(z)\delta(z) + \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z)\delta^{(n-l)}(z) \\ &\qquad -\delta^{(n)}(z)(a(z) + \sigma) - \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z) a^{(n-l)}(z) - \delta(z)a^{(n)}(z) + \sigma a^{(n)}(z) = 0. \end{aligned}$$

Thus we obtain

$$\delta^{(n)}(z)(2\delta(z) - a(z) - \sigma) = \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z) a^{(n-l)}(z) - \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z)\delta^{(n-l)}(z) + (\delta(z) - \sigma)a^{(n)}(z),$$

from which it follows that

$$\delta^{(n)}(z) = \frac{\sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z) a^{(n-l)}(z) - \sum_{l=1}^{n-1} \binom{n}{l} \delta^{(l)}(z)\delta^{(n-l)}(z) + (\delta(z) - \sigma)a^{(n)}(z)}{2\delta(z) - a(z) - \sigma}.$$

Taking the limit $z \to 1-$, we obtain

$$\begin{aligned}\delta^{(n)} &= \frac{\sum_{l=1}^{n-1}\binom{n}{l}\delta^{(l)}a^{(n-l)} - \sum_{l=1}^{n-1}\binom{n}{l}\delta^{(l)}\delta^{(n-l)} + (1-\sigma)a^{(n)}(z)}{1-\sigma+b(0)\sigma} \\ &= \pi_0 a^{(n)} + \frac{\sum_{l=1}^{n-1}\binom{n}{l}\delta^{(l)}a^{(n-l)} - \sum_{l=1}^{n-1}\binom{n}{l}\delta^{(l)}\delta^{(n-l)}}{1-\sigma+b(0)\sigma} \\ &= \pi_0 a^{(n)} + \frac{\sum_{l=1}^{n-1}\binom{n}{l}\delta^{(l)}(a^{(n-l)} - \delta^{(n-l)})}{1-\sigma+b(0)\sigma}. \end{aligned} \tag{69}$$

Substituting (41) into (69), we have (53). ■

Appendix F Proof of Lemma 7

Note here that the kth factorial moment of D's ($k = 1, 2, \dots$) are given by

$$\mathrm{E}[D(D-1)(D-2)\cdots(D-k+1)] = \frac{k!\sigma^{k-1}}{(1-\sigma)^k},$$

from which it follows that

$$\mathrm{E}[D] = \frac{1}{1-\sigma}, \tag{70}$$

$$\begin{aligned}\mathrm{E}[D^2] &= \mathrm{E}[D(D-1)] + \mathrm{E}[D] \\ &= \frac{2\sigma + (1-\sigma)}{(1-\sigma)^2} \\ &= \frac{1+\sigma}{(1-\sigma)^2},\end{aligned} \tag{71}$$

$$\begin{aligned}\mathrm{E}[D^3] &= \mathrm{E}[D(D-1)(D-2)] + 3\mathrm{E}[D(D-1)] + \mathrm{E}[D] \\ &= \frac{6\sigma^2 + 3\cdot 2\sigma(1-\sigma) + (1-\sigma)^2}{(1-\sigma)^3} \\ &= \frac{\sigma^2+4\sigma+1}{(1-\sigma)^3},\end{aligned} \tag{72}$$

$$\begin{aligned}\mathrm{E}[D^4] &= \mathrm{E}[D(D-1)(D-2)(D-3)] + 6\mathrm{E}[D(D-1)(D-2)] + 7\mathrm{E}[D(D-1)] + \mathrm{E}[D] \\ &= \frac{24\sigma^3 + 6\cdot 6\sigma^2(1-\sigma) + 7\cdot 2\sigma(1-\sigma)^2 + (1-\sigma)^3}{(1-\sigma)^4} \\ &= \frac{\sigma^3+11\sigma^2+11\sigma+1}{(1-\sigma)^4},\end{aligned} \tag{73}$$

respectively.

Now $a^{(1)}(z)$ is given by

$$a^{(1)}(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(ml-l+1)b(m)d(l)z^{ml-l+1}.$$

Thus we obtain

$$\begin{aligned}a^{(1)} &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(ml-l+1)b(m)d(l) \\ &= \mathrm{E}[B]\mathrm{E}[D] - \mathrm{E}[D](1-b(0)) + 1 - b(0) \\ &= \frac{\mathrm{E}[B] - (1-b(0)) + (1-b(0))(1-\sigma)}{1-\sigma} \\ &= \frac{\mathrm{E}[B] - \sigma(1-b(0))}{1-\sigma},\end{aligned}$$

where we use (70). $a^{(2)}(z)$ is given by

$$a^{(2)}(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(ml-l+1)(ml-l)b(m)d(l)z^{ml-l+1},$$

from which it follows that

$$\begin{aligned}
a^{(2)} &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(l^2(m^2-2m+1)+l(m-1))b(m)d(l)\\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Big(l^2(m(m-1)-m+1)+l(m-1)\Big)b(m)d(l)\\
&= \mathrm{E}[D^2]\Big(\mathrm{E}[B(B-1)]-\mathrm{E}[B]+(1-b(0)\Big)+\mathrm{E}[D]\Big(\mathrm{E}[B]-(1-b(0))\Big)\\
&= \frac{(1+\sigma)\mathrm{E}[B(B-1)]+(-(1+\sigma)+(1-\sigma))(\mathrm{E}[B]-(1-b(0))}{(1-\sigma)^2}\\
&= \frac{\mathrm{E}[B(B-1)](1+\sigma)-2\sigma\mathrm{E}[B]+2\sigma(1-b(0))}{(1-\sigma)^2},
\end{aligned}$$

where we use (70) and (71). Further $a^{(3)}(z)$ is given by

$$a^{(3)}(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(ml-l+1)(ml-l)(ml-l-1)b(m)d(l)z^{ml-l+1}.$$

Thus we obtain

$$\begin{aligned}
a^{(3)} &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(l^3(m-1)^3-l(m-1))b(m)d(l)\\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Big(l^3(m(m-1)(m-2)+m-1)-l(m-1)\Big)b(m)d(l)\\
&= \mathrm{E}[D^3]\Big(\mathrm{E}[B(B-1)(B-2)]+\mathrm{E}[B]-(1-b(0))\Big)+\mathrm{E}[D]\Big(-\mathrm{E}[B]+(1-b(0))\Big)\\
&= \frac{(\sigma^2+4\sigma+1)\mathrm{E}[B(B-1)(B-2)]+((\sigma^2+4\sigma+1)-(1-\sigma)^2)(\mathrm{E}[B]-(1-b(0)))}{(1-\sigma)^3}\\
&= \frac{(\sigma^2+4\sigma+1)\mathrm{E}[B(B-1)(B-2)]+6\sigma\mathrm{E}[B]-6\sigma(1-b(0))}{(1-\sigma)^3},
\end{aligned}$$

where we use (70), (71), and (72). $a^{(4)}(z)$ is given by

$$a^{(4)}(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(ml-l+1)(ml-l)(ml-l-1)(ml-l-2)b(m)d(l)z^{ml-l+1},$$

from which it follows that

$$\begin{aligned}
a^{(4)} &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(l(m-1)-1)(l(m-1)+1)l(m-1)(l(m-1)-2)b(m)d(l)\\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(l^2(m-1)^2-1)l(m-1)(l(m-1)-2)b(m)d(l)\\
&= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Big(l^4(m^4-4m^3+6m^2-4m+1)-2l^3(m^3-3m^2+3m-1)\\
&\qquad\qquad -l^2(m^2-2m+1)+2l(m-1)\Big)b(m)d(l)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \Big(l^4((m^4 - 6m^3 + 11m^2 - 6m) + 2(m^3 - 3m^2 + 2m) + m(m-1) - m + 1) \\
&\qquad -2l^3((m^3 - 3m^2 + 2m) + m - 1) - l^2(m(m-1) - m + 1) + 2l(m-1)\Big) b(m)d(l) \\
&= \mathrm{E}[D^4]\Big(\mathrm{E}[B(B-1)(B-2)(B-3)] + 2\mathrm{E}[B(B-1)(B-2)] \\
&\qquad + \mathrm{E}[B(B-1)] - \mathrm{E}[B] + (1 - b(0))\Big) \\
&\qquad -2\mathrm{E}[D^3]\Big(\mathrm{E}[B(B-1)(B-2)] + \mathrm{E}[B] - (1 - b(0))\Big) \\
&\qquad -\mathrm{E}[D^2]\Big(\mathrm{E}[B(B-1)] - \mathrm{E}[B] + (1 - b(0))\Big) + 2\mathrm{E}[D]\Big(\mathrm{E}[B] - (1 - b(0))\Big) \\
&= \frac{(\sigma^3 + 11\sigma^2 + 11\sigma + 1)}{(1-\sigma)^4}\mathrm{E}[B(B-1)(B-2)(B-3)] \\
&\quad + \frac{2(\sigma^3 + 11\sigma^2 + 11\sigma + 1) - 2(\sigma^2 + 4\sigma + 1)(1-\sigma)}{(1-\sigma)^4}\mathrm{E}[B(B-1)(B-2)] \\
&\quad + \frac{(\sigma^3 + 11\sigma^2 + 11\sigma + 1) - (\sigma + 1)(1-\sigma)^2}{(1-\sigma)^4}\mathrm{E}[B(B-1)] \\
&\quad + \frac{-(\sigma^3 + 11\sigma^2 + 11\sigma + 1) - 2(\sigma^2 + 4\sigma + 1)(1-\sigma) + (\sigma + 1)(1-\sigma)^2 + 2(1-\sigma)^3}{(1-\sigma)^4} \\
&\qquad \cdot(\mathrm{E}[B] - (1 - b(0))) \\
&= \frac{(\sigma^3 + 11\sigma^2 + 11\sigma + 1)\mathrm{E}[B(B-1)(B-2)(B-3)] + 4(\sigma^3 + 7\sigma^2 + 4\sigma)\mathrm{E}[B(B-1)(B-2)]}{(1-\sigma)^4} \\
&\quad + \frac{12(\sigma^2 + 1)\mathrm{E}[B(B-1)] - 24\sigma\mathrm{E}[B] + 24\sigma(1 - b(0)))}{(1-\sigma)^4},
\end{aligned}
$$

where we use (70), (71), (72), and (73). ■