Variational Inequality Approaches to Generalized Nash Equilibrium Problems

Guidance

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Abstract

We consider the generalized Nash equilibrium problem (GNEP), in which each player's strategy set may depend on the rivals' strategies. The GNEP can be formulated as a quasi-variational inequality (QVI). However, unlike the standard variational inequality (VI), there are only a few methods available for solving a QVI efficiently. A practical approach to find a GNE is to solve a related VI rather than solving the corresponding QVI directly, and it has been receiving much attention recently. From the viewpoint of game theory, it is important to find GNEs as many as possible, or all of them if possible. We propose the price-directed and the resource-directed parametrization methods that construct parametrized families of VIs related to the GNEP. We then show that those VIs yield all GNEs under suitable conditions. Moreover, by means of some numerical examples, we show that GNEs obtained by the proposed VI approaches are widely distributed in the GNEP solution set.

Contents

1 Introduction

The generalized Nash equilibrium problem (GNEP) is a generalization of the standard Nash equilibrium problem (NEP), in which each player's strategy set may depend on the rivals' strategies [1, 23, 8]. Recently, the GNEP has attracted much attention [5, 14] because there are many interesting applications in the fields of economics, mathematics and engineering. For example, Robinson [20, 21] discussed a two-sided game model of combat as an application of GNEP. Wei and Smeers [26] and Hobbs [11] formulated oligopolistic electricity models as GNEPs.

It is well known that NEP where each player solves a convex programming problem can be formulated as a finite-dimensional variational inequality (VI) [9, 6]. The VI has a long history and the state-of-the-art and various computational methods are presented in the recent monograph [6]. On the other hand, GNEP can be formulated as a quasi-variational inequality (QVI) [8, 19]. However, unlike the VI, there are only a few methods available for computing a generalized Nash equilibrium (GNE) or solving a QVI efficiently [17, 19, 18]. A practical approach to GNEP is to find a GNE via a VI rather than solving a QVI directly [8, 26, 18, 19, 3, 4] and it has received much attention recently.

Pang and Fukushima [19] propose a penalty method for GNEP, which solves a sequence of penalized NEPs, and establish its convergence under some assumptions. Pang [18] formulates GNEP as a partitioned VI via the equivalent Karush-Kuhn-Tucker (KKT) system and proposes to apply the Josephy-Newton method that solves a linearized VI at each iteration. Local superlinear convergence is established under convexity and regularity assumptions.

Wei and Smeers [26] formulate an oligopolistic electricity model as a GNEP with "shared constraints", which means the constraint functions that depend on rivals' strategies are identical among all players. They present a VI formulation such that its solution is GNE, and establish the uniqueness of GNE under some restrictive assumptions. Facchinei et al. [3] also consider a GNEP with shared constraints and propose to apply Newton-type methods to solve its VI formulation. This approach seems to be promising because we can find a GNE by solving a single VI. However, the obtained GNE is a particular equilibrium such that the multipliers of the shared constraints are identical [3, 4]. In general, GNEP has multiple, or even infinitely many, solutions [8]. In practical situations, the players usually have different objective functions, and the multipliers of the shared constraints need not be identical. Therefore, the VI formulation considered in [26, 3, 4] may fail to identify some important GNEP solutions.

Another approach for GNEP is the algorithms [24, 13, 10] based on the Nikaido–Isoda function [16]. Uryasev and Krawczyk [24, 13] develop the relaxation method and establish the global convergence of the algorithm under some assumptions, which are, however, rather restrictive. In a recent paper, von Heusinger and Kanzow [10] consider the regularization of the Nikaido–Isoda function and reformulate a GNEP with shared constraints as a smooth optimization problem. However, it is shown that these methods also find only a part of GNEP solutions in general.

From the practical viewpoint of game theory, it is important to find as many GNEs as possible, or all of them if possible [25]. To the best of our knowledge, little attempt has been made to develop a method to achieve this. The goal of the present paper is to construct, for GNEP with shared constraints, a parametrized family of VIs whose solutions include all GNEs. This extends the VI approaches studied in [26, 3, 4] and show that all GNEP solutions are contained in the solution sets of those VIs. Moreover, we clarify the conditions that guarantee a solution of those VIs to be a GNE.

The paper is organized as follows. In the next section, we recall some definitions and basic concepts. In Section 3, we present the parametrized family of VIs and show that those VIs yield all GNEP solutions. We also show that the parameters involved in VIs can be restricted to a bounded set. In Section 4, we present some numerical results with the proposed approach. Finally, Section 5 concludes the paper.

We use the following notations throughout the paper. For a set X, $\mathcal{P}(X)$ denotes the set of all subsets of X. For a nonempty closed convex set $X \subset \Re^n$, $N_X(x) = \{d \in \Re^n \mid d^T(y-x) \le 0 \ \forall y \in X\}$ denotes the normal cone to X at $x \in X$. For a function $f : \Re^n \times \Re^m \to \Re$, $f(x, \cdot) : \Re^m \to \Re$ denotes the function with x being fixed. We denote the nonnegative and positive orthants in $\Re^n$ by $\Re^n_+$ and $\Re^n_{++}$, respectively, that is,

$$\Re^n_+ := \{x \in \Re^n \mid x \ge 0\} \quad \text{and} \quad \Re^n_{++} := \{x \in \Re^n \mid x > 0\}.$$

For vectors $x, y \in \Re^n$, $\langle x, y \rangle$ denotes the inner product defined by $\langle x, y \rangle := x^T y$ and $x \perp y$ means $\langle x, y \rangle = 0$. For a vector x, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{\langle x, x \rangle}$. A mapping $F : \Re^n \to \Re^n$ is said to be monotone on a nonempty closed convex set $X \subset \Re^n$ if

$$\langle F(x) - F(y), x - y \rangle \ge 0 \quad \forall x, y \in X,$$

strictly monotone on X if the above inequality is strict whenever $x \ne y$, and strongly monotone on X if zero on the right-hand side of the above inequality is replaced by $\alpha\|x-y\|^2$ for some constant $\alpha > 0$.

2 Problem Formulation and Assumptions

The generalized Nash game with N players is to find a profile of strategies such that each player's strategy is an optimal response to the rival players' strategies, where each player's strategy set may depend on the rival players' strategies. For $\nu = 1, \dots, N$, let $x_\nu \in \Re^{n_\nu}$ be a player ν's strategy, where n_ν is a positive integer. The vector formed by all these strategies is denoted $x := (x_\nu)_{\nu=1}^N \in \Re^n$, where $n := \sum_{\nu=1}^N n_\nu$, and the vector formed by all the players' strategies except those of player ν is denoted $x_{-\nu} := (x_{\nu'})_{\nu'=1,\ \nu' \ne \nu}^N \in \Re^{n_{-\nu}}$, where $n_{-\nu} := n - n_\nu$. For $\nu = 1, \dots, N$, let K_ν be a given point-to-set mapping from $\Re^{n_{-\nu}}$ to $\Re^{n_\nu}$. Thus, for each fixed $x_{-\nu}$, $K_\nu(x_{-\nu})$ is a subset of $\Re^{n_\nu}$, which is the strategy set of player ν with the other players' strategies given by $x_{-\nu}$.

Each player $\nu = 1, \dots, N$, taking the other players' strategies $x_{-\nu}$ as exogenous variables, solves the minimization problem:

$$P_\nu(x_{-\nu}) : \quad \begin{array}{ll} \text{minimize} & \theta_\nu(x_\nu, x_{-\nu}) \\ \text{subject to} & x_\nu \in K_\nu(x_{-\nu}), \end{array} \tag{1}$$

where $\theta_\nu : \Re^n \to \Re$ is a given cost function of player ν. A vector $x = (x_\nu)_{\nu=1}^N$ is said to be feasible to GNEP if $x_\nu \in K_\nu(x_{-\nu})$ for each $\nu = 1, \dots, N$.

The GNEP is to find a vector $x^* = (x^*_\nu)_{\nu=1}^N \in \Re^n$ such that

$$x^*_\nu \text{ is an optimal solution of } P_\nu(x^*_{-\nu}) \quad \text{for all } \nu = 1, \dots, N. \tag{2}$$

A vector x^* satisfying (2) is called a generalized Nash equilibrium (GNE). The set of GNEs is denoted by SOL^*.

In many practical applications, the strategy set $K_\nu(x_{-\nu})$ of player ν is represented by finitely many inequality constraints. In particular, we assume that the feasible strategy set $K_\nu(x_{-\nu})$ of player ν has the form

$$K_\nu(x_{-\nu}) = \{x_\nu \in X_\nu \mid g_\nu(x_\nu, x_{-\nu}) \le 0\}, \tag{3}$$

where $g_\nu = (g_{\nu,i})_{i=1}^{m_\nu} : \Re^n \to \Re^{m_\nu}$ and $X_\nu \subset \Re^{n_\nu}$, with m_ν a nonnegative integer. Thus, player ν's strategy is constrained in two ways; joint constraints that depend also on the other players' strategies, i.e., $g_\nu(x) \leq 0$, and individual constraints that depend only on player ν's strategy, i.e., $x_\nu \in X_\nu$. In what follows, we let

$$X_{-\nu} := \prod_{\substack{\nu'=1 \\ \nu' \neq \nu}}^{N} X_{\nu'}.$$

We distinguish these two types of constraints since our parametrization will involve only the joint constraints. Throughout this paper, we make the following blanket assumption on the smoothness and convexity of functions involved in the GNEP.

Assumption A. *For $\nu = 1, \dots, N$, the set X_ν is nonempty, closed, convex, and, for each fixed $x_{-\nu} \in X_{-\nu}$, the functions $\theta_\nu(\cdot, x_{-\nu})$ and $g_{\nu,i}(\cdot, x_{-\nu})$, $i = 1, \dots, m_\nu$, are differentiable and convex.*

By Assumption A, problem (1) is a differentiable convex programming problem. Thus a necessary and sufficient condition for $x_\nu^* \in K_\nu(x_{-\nu}^*)$ to be optimal for (1) is that the inequalities

$$\langle \nabla_{x_\nu} \theta_\nu(x_\nu^*, x_{-\nu}^*), x_\nu - x_\nu^* \rangle \geq 0 \quad \forall x_\nu \in K_\nu(x_{-\nu}^*)$$

hold. Thus, by defining

$$\begin{aligned} F(x) &:= \left(\nabla_{x_\nu} \theta_\nu(x_\nu, x_{-\nu})\right)_{\nu=1}^{N}, \\ K(x) &:= \prod_{\nu=1}^{N} K_\nu(x_{-\nu}), \end{aligned} \tag{4}$$

it follows that x^* is a GNE if and only if $x^* \in K(x^*)$ and

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in K(x^*).$$

The latter problem is a QVI, which we denote by QVI (F, K).

Suppose that x^* is a solution of GNEP. Then, for each $\nu = 1, \dots, N$, x_ν^* is an optimal solution of the convex programming problem:

$$\begin{aligned} P_\nu(x_{-\nu}^*): \quad & \text{minimize} \quad \theta_\nu(x_\nu, x_{-\nu}^*) \\ & \text{subject to} \quad g_\nu(x_\nu, x_{-\nu}^*) \leq 0, \quad x_\nu \in X_\nu. \end{aligned}$$

Under a suitable CQ at x^* (see, e.g., [2, Section 5.4], [22]), there exists for each $\nu = 1, \dots, N$ a vector $\lambda_\nu^* \in \Re^{m_\nu}$ satisfying the Karush-Kuhn-Tucker (KKT) condition:

$$\begin{aligned} & 0 \in \nabla_{x_\nu} L_\nu(x_\nu, x_{-\nu}^*, \lambda_\nu) + N_{X_\nu}(x_\nu), \\ & 0 \leq \lambda_\nu \perp g_\nu(x_\nu, x_{-\nu}^*) \leq 0, \quad x_\nu \in X_\nu, \end{aligned} \tag{5}$$

where the Lagrangian function L_ν is defined by

$$L_\nu(x, \lambda_\nu) := \theta_\nu(x) + \langle g_\nu(x), \lambda_\nu \rangle.$$

The vector $\lambda^* = (\lambda_\nu^*)_{\nu=1}^N$ is called a *Lagrange multiplier* vector. Under Assumption A, if (x^*, λ^*) satisfies (5), then x^* is a GNE. A well-known CQ at x is the Mangasarian-Fromovitz CQ (MFCQ) [22, page 198]:

$$\left\{ \begin{array}{c} 0 \in \nabla_{x_\nu} g_\nu(x) \lambda_\nu + N_{X_\nu}(x_\nu), \\ 0 \leq \lambda_\nu \perp g_\nu(x) \leq 0, \quad x_\nu \in X_\nu \end{array} \right\} \quad \Longrightarrow \quad \lambda_\nu = 0, \quad \nu = 1, \dots, N,$$

where

$$\nabla_{x_\nu} g_\nu(x) := \left(\nabla_{x_\nu} g_{\nu,1}(x) \quad \cdots \quad \nabla_{x_\nu} g_{\nu,m_\nu}(x)\right).$$

Another useful CQ at x is the Linear Independence CQ (LICQ):

$$\left\{ \begin{array}{c} 0 \in \nabla_{x_\nu} g_\nu(x)\lambda_\nu + N_{X_\nu}(x_\nu) + (-N_{X_\nu}(x_\nu)), \\ \lambda_\nu \perp g_\nu(x) \le 0, \quad x_\nu \in X_\nu \end{array} \right\} \implies \lambda_\nu = 0, \quad \nu = 1, \dots, N. \tag{6}$$

This CQ implies uniqueness of the multiplier vector λ^* for each x^*.

3 Parameterized VI Approaches to GNE with Shared Constraints

In this section, we consider the case of "shared constraints", that is, all players share common constraints that depend on all the players' strategies. Specifically, we make the following blanket assumption.

Assumption B. *For some $\bar{m}$ and $g = (g_i)_{i=1}^{\bar{m}} : \Re^n \to \Re^{\bar{m}}$, we have $m_1 = \cdots = m_N = \bar{m}$ and $g^1 = \cdots = g^N = g$. Moreover, $g_i : \Re^n \to \Re$ is differentiable and convex for all $i = 1, \dots, \overline{m}$.*

Then, (x^*, λ^*) satisfies the KKT condition (5) if and only if (x^*, λ^*) satisfies

$$\begin{aligned} 0 \in \nabla_{x_\nu}\theta_\nu(x) + \nabla_{x_\nu} g(x)\lambda_\nu + N_{X_\nu}(x_\nu), \quad \nu = 1, \dots, N, \\ 0 \le \lambda_\nu \perp g(x) \le 0, \quad x_\nu \in X_\nu, \quad \nu = 1, \dots, N. \end{aligned} \tag{7}$$

Note from the complementarity condition that $g_i(x) < 0$ implies that $\lambda_{\nu,i} = 0$ for all $\nu = 1, \dots, N$ and $\lambda_{\nu,i} > 0$ for some ν implies that $g_i(x) = 0$.

The VI approach to finding a GNE [26, 3, 4] is to define

$$X := \{x \in X_1 \times \cdots \times X_N \mid g(x) \le 0\},$$

which is a closed convex (possibly empty) set, and solve the following VI (F, X):

$$\text{Find } x^* \in X \quad \text{such that} \quad \langle F(x^*), x - x^*\rangle \ge 0 \quad \forall x \in X,$$

where $F : \Re^n \to \Re^n$ is defined by (4). The solution set of VI (F, X) is denoted by SOL (F, X) [1]. A vector x^* belongs to SOL (F, X) if and only if it is an optimal solution of the convex programming problem:

$$\begin{aligned} &\text{minimize} \quad \langle F(x^*), x\rangle \\ &\text{subject to} \quad x \in X, \end{aligned} \tag{8}$$

whose KKT condition is a special case of (7) with $\lambda_1 = \cdots = \lambda_N$. Specifically, the following result is known [3, Theorem 3.6] (also see [26, Theorem 2] and [4, Theorem 2.1]).

Theorem 3.1. *Every $x^* \in$ SOL (F, X) is a GNE. Furthermore, if x^* together with some Lagrange multiplier vector satisfies the KKT condition for (8), then x^* and some $\lambda^* = (\lambda^*_\nu)_{\nu=1}^N$ with $\lambda^*_1 = \cdots = \lambda^*_N$ satisfy (7).*

[1] A solution of VI (F, X) is called a variational equilibrium in [5] and is shown to be a particular instance of a normalized equilibrium introduced by Rosen [23]. In Section 3.3, we will show the relation between normalized equilibrium and the solutions of VI formulations.

In many practical situations, since the players have different objective functions as well as their own constraints, the multipliers of the shared constraints in GNEP may not be identical. Therefore, in general, there would be many GNEs that are not normalized equilibria. This is illustrated in the following example.

Example 1. Consider the two-person game, where the problems of player 1 and player 2 are defined by

$$\begin{array}{lll} P_1(x_2): & \text{minimize} & x_1^2 - x_1x_2 - x_1 \\ & \text{subject to} & x_1 \geq 0 \\ & & x_1 + x_2 \leq 1, \end{array}$$

and

$$\begin{array}{lll} P_2(x_1): & \text{minimize} & x_2^2 - \frac{1}{2}x_1x_2 - 2x_2 \\ & \text{subject to} & x_2 \geq 0 \\ & & x_1 + x_2 \leq 1, \end{array}$$

respectively. The set of GNEs consists of infinitely many vectors

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 1-t \end{pmatrix}, \quad 0 \leq t \leq \tfrac{2}{3}.$$

On the other hand, the corresponding $F : \Re^2 \to \Re^2$ and $X \subseteq \Re^2$ are given by

$$F(x) = \begin{pmatrix} 2x_1 - x_2 - 1 \\ -\frac{1}{2}x_1 + 2x_2 - 2 \end{pmatrix}, \quad X = \{x \in \Re^2_+ \mid x_1 + x_2 \leq 1\},$$

and the solution of VI (F, X) is uniquely given by $x = \left(\frac{4}{11}, \frac{7}{11}\right)^T$.

In this example, the mapping F is strongly monotone and hence SOL (F, X) is a singleton [6], but there are infinitely many GNEs. Uniqueness of GNE requires restrictive assumptions [8, 26] which cannot be expected to hold in most applications. In general, the above VI approach can find only a part of the GNEs.

3.1 Price-Directed Parametrization

Now, we construct a family of VIs that contains VI (F, X) as a particular instance. Let $\Delta = (\Delta_\nu)_{\nu=1}^N \in \Re^m_+$ with $m = N\bar{m}$ and $\Delta_\nu \in \Re^{\bar{m}}_+$, $\nu = 1, \ldots, N$, be a vector of parameters. Define the function $F^\Delta : \Re^n \to \Re^n$ by

$$F^\Delta(x) := \left(\nabla_{x_\nu}\theta_\nu(x) + \nabla_{x_\nu}g(x)\Delta_\nu\right)_{\nu=1}^N. \tag{9}$$

Note that VI (F^Δ, X) reduces to VI (F, X) if the parameter Δ is set to be zero. The following proposition shows that, under appropriate assumptions, the monotonicity of F implies the monotonicity of F^Δ for any $\Delta \in \Re^m_+$.

Proposition 3.1. *Assume that F is monotone (strictly monotone, strongly monotone) on X. Assume further that the shared constraint function g is separable, that is, $g(x)$ can be written as*

$$g(x) = \sum_{\nu=1}^N \hat{g}_\nu(x_\nu), \tag{10}$$

where $\hat{g}_\nu = (\hat{g}_{\nu,i})_{i=1}^{\bar{m}} : \Re^{n_\nu} \to \Re^{\bar{m}}$, $\nu = 1, \ldots, N$, are differentiable convex functions. Then, for any $\Delta \in \Re^m_+$, F^Δ is also monotone (strictly monotone, strongly monotone) on X.

Proof. Since g is separable, we have

$$F^{\Delta}(x) = F(x) + (\nabla \hat{g}_{\nu}(x_{\nu})\Delta_{\nu})_{\nu=1}^{N}.$$

For each $\nu = 1,\dots,N$ and $i = 1,\dots,\bar{m}$, $\nabla \hat{g}_{\nu,i}$ is monotone on X_{ν} since $\hat{g}_{\nu,i}$ is a differentiable convex function. Therefore F^{Δ} is monotone (strictly monotone, strongly monotone) on X for any $\Delta \in \Re^m_+$. □

Under the assumptions of this proposition, we can apply, for example, Newton-type or projection-type methods to solve VI(F^{Δ}, X). A sufficient condition for F to be strictly monotone in the setting of spatial oligopolistic electricity models is given in [26, Theorems 5 and 6].

We now investigate the relationship between VI(F^{Δ}, X) and GNEP. The KKT condition for VI(F^{Δ}, X) can be written as

$$\begin{aligned} 0 \in (\nabla_{x_\nu}\theta_\nu(x) + \nabla_{x_\nu} g(x)\Delta_\nu) + \nabla_{x_\nu} g(x)\pi + N_{X_\nu}(x_\nu), \quad \nu = 1,\dots,N, \\ 0 \le \pi \perp g(x) \le 0, \quad x_\nu \in X_\nu, \quad \nu = 1,\dots,N. \end{aligned} \tag{11}$$

For each GNE x, let

$$\mathcal{M}(x) := \{\lambda \in \Re^m \mid (x,\lambda) \text{ satisfies the KKT condition (7)}\}.$$

By comparing this KKT condition (11) with (7), we have the following result.

Theorem 3.2. *For any GNE x^*, if $\lambda^* \in \mathcal{M}(x^*)$, then $x^* \in \mathrm{SOL}(F^{\lambda^*}, X)$.*

Proof. Fix any GNE x^*, and assume that $\lambda^* \in \mathcal{M}(x^*)$. Then (x^*, λ^*) satisfies the KKT condition (7). This in turn shows that $(x^*, 0)$ satisfies the KKT condition for VI(F^{λ^*}, X) and hence x^* is a solution of VI(F^{λ^*}, X). □

Corollary 3.1. *If $\mathcal{M}(x^*) \neq \emptyset$ for every GNE x^*, then*

$$\bigcup_{\Delta \in \Re^m_+} \mathrm{SOL}(F^{\Delta}, X) \supset \mathrm{SOL}^*.$$

For an arbitrary GNE x^*, let $\lambda^* \in \mathcal{M}(x^*)$ and define

$$\begin{aligned} \delta_i &:= \min_{\nu=1,\dots,N} \lambda^*_{\nu,i} && i = 1,\dots,\bar{m} \\ \bar{\lambda}_{\nu,i} &:= \lambda^*_{\nu,i} - \delta_i && i = 1,\dots,\bar{m},\ \nu = 1,\dots,N. \end{aligned}$$

Then x^* along with δ satisfies the KKT condition for VI$(F^{\bar{\lambda}}, X)$, and hence x^* also belongs to SOL$(F^{\bar{\lambda}}, X)$. This implies that for any GNE x^* satisfying $\mathcal{M}(x^*) \neq \emptyset$, there always exists a $\Delta \in \Re^m_+$ such that $x^* \in \mathrm{SOL}(F^{\Delta}, X)$ and, for each i, $\Delta_{\nu,i} = 0$ for some ν. This observation yields the following result that sharpens Corollary 3.1.

Corollary 3.2. *If $\mathcal{M}(x^*) \neq \emptyset$ for every GNE x^*, then*

$$\bigcup_{\Delta \in \mathcal{P}} \mathrm{SOL}(F^{\Delta}, X) \supset \mathrm{SOL}^*,$$

where the set $\mathcal{P}$ is defined by

$$\mathcal{P} := \prod_{i=1}^{\bar{m}} \left(\bigcup_{\nu=1}^{N} \{ \Delta \in \Re^N_+ \mid \Delta_\nu = 0 \} \right) \subset \Re^m_+.$$

This corollary suggests that, for finding GNEs, it is enough to restrict ourselves to the parameter set $\mathcal{P}$ instead of $\Re^m_+$.

In general, a solution of VI (F^Δ, X) for some $\Delta \in \Re^m_+$ need not be a GNE. To see this, let us consider the VI (F^Δ, X) in Example 1 with $\Delta = (1,1)^T$. Then

$$F^\Delta(x) = \begin{pmatrix} 2x_1 - x_2 \\ -\frac{1}{2}x_1 + 2x_2 - 1 \end{pmatrix},$$

and VI (F^Δ, X) has the unique solution $x = \left(\frac{2}{7}, \frac{4}{7}\right)^T$ which is not a GNE.

The next result gives a sufficient condition for a solution of VI (F^Δ, X) to be a GNE.

Theorem 3.3. *For any $\Delta \in \Re^m_+$ and any (x^*, π^*) satisfying* (11)*, a sufficient condition for x^* to be a GNE is that*

$$\langle g(x^*), \Delta_\nu \rangle = 0, \quad \nu = 1, \dots, N. \tag{12}$$

If in addition the LICQ (6) *holds at x^*, then* (12) *is also a necessary condition for x^* to be a GNE.*

Proof. Let

$$\lambda^*_\nu := \pi^* + \Delta_\nu \geq 0, \quad \nu = 1, \dots, N.$$

By (11) and (12), we have

$$\langle g(x^*), \lambda^*_\nu \rangle = \langle g(x^*), \pi^* + \Delta_\nu \rangle = 0.$$

Hence (x^*, λ^*) satisfies the KKT condition (7). This shows that x^* is a GNE.

Conversely, suppose that x^* is a GNE and the LICQ (6) holds at x^*. By LICQ, the Lagrange multiplier vector λ^* satisfying (7) with $x = x^*$ and the Lagrange multiplier vector π^* satisfying (11) with $x = x^*$ are both unique. Then we must have

$$\lambda^*_\nu = \pi^* + \Delta_\nu, \ \nu = 1, \dots, N.$$

Moreover, $\langle g(x^*), \lambda^*_\nu \rangle = 0, \ \nu = 1, \dots, N$ and $\langle g(x^*), \pi^* \rangle = 0$, which together yield

$$\langle g(x^*), \Delta_\nu \rangle = \langle g(x^*), \lambda^*_\nu - \pi^* \rangle = 0, \quad \nu = 1, \dots, N.$$

□

Remark 3.1. When $\Delta = 0$, we have $F^\Delta \equiv F$. Moreover, (12) clearly holds. Therefore, Theorem 3.3 contains the result of Theorem 3.1.

Corollary 3.1 shows that SOL^* is contained in the union of SOL (F^Δ, X) over all $\Delta \in \Re^m_+$. We show below that, under the following sequentially bounded CQ (SBCQ), the range of parameter Δ can be restricted to a bounded set $\Lambda \subseteq \Re^m_+$ and the union of SOL (F^Δ, X) still contains an "arbitrarily large" subset of SOL^*.

Definition 3.1 (SBCQ). *For every bounded sequence $\{x^k\} \subset \mathrm{SOL}^*$, there exists a bounded sequence $\{\lambda^k\}$ satisfying $\lambda^k \in \mathcal{M}(x^k)$ for all k.*

The SBCQ was introduced in the study of the mathematical program with equilibrium constraints (MPEC) [15]. It is a unification of well-known CQs such as MFCQ and the constant rank CQ [12], and plays an important role not only in MPEC but also in GNEP [19]. It can be shown that if the function g is affine and $X_1, \dots, X_N$ are polyhedral sets, then SBCQ holds [19].

Theorem 3.4. *Assume that SBCQ holds. For any bounded set $C \subset \Re^n$, there exists a bounded set $\Lambda \subset \Re^m_+$ such that*

$$\bigcup_{\Delta \in \Lambda} \mathrm{SOL}\,(F, X) \;\supset\; C \cap \mathrm{SOL}^*.$$

Proof. Fix any bounded set $C \subset \Re^n$. We claim that there exists a bounded set $\Lambda \subset \Re^m_+$ such that

$$\mathcal{M}(x) \cap \Lambda \neq \emptyset \qquad \forall x \in C \cap \mathrm{SOL}^*. \tag{13}$$

If this were not true, then there would exist a sequence $\{x^k\} \subset C \cap \mathrm{SOL}^*$ such that $\min_{\lambda \in \mathcal{M}(x^k)} \|\lambda\| \to \infty$. (Here we use the convention that $\min_{\lambda \in \mathcal{M}(x^k)} \|\lambda\| = \infty$ whenever $\mathcal{M}(x^k) = \emptyset$.) However, since $\{x^k\}$ lies in a bounded set C, SBCQ would imply that $\min_{\lambda \in \mathcal{M}(x^k)} \|\lambda\|$ is bounded, a contradiction.

Fix any $x^* \in C \cap \mathrm{SOL}^*$. By (13), there exists $\lambda^* \in \mathcal{M}(x^*) \cap \Lambda$. By Theorem 3.2, $x^* \in \mathrm{SOL}\,(F^{\lambda^*}, X)$. Thus every element of $C \cap \mathrm{SOL}^*$ belongs to $\mathrm{SOL}\,(F^{\lambda^*}, X)$ for some $\lambda^* \in \Lambda$. This proves the desired inclusion. □

If SOL^* is bounded, then we can take $C = \mathrm{SOL}^*$. Unfortunately, Theorem 3.4 does not say how large Λ should be. This is a question for further study.

3.2 Resource-Directed Parametrization

In the case where g is affine and $X_1, \ldots, X_N$ are polyhedral, it is known that $\mathcal{M}(x) \neq \emptyset$ for every GNE x. Otherwise, $\mathcal{M}(x)$ could be empty for some GNE x, and the price-directed dual parametrization approach of Section 3.1 would not be able to find this GNE. In this section we consider a resource-directed primal parametrization that does not rely on the existence of a Lagrange multiplier vector. We motivate this with an example.

Example 2. Consider a modification of Example 1 where the shared constraint $x_1 + x_2 \leq 1$ is changed to $x_1^2 + x_2^2 \leq 1$. It can be seen that the GNEs are the vectors

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix}, \qquad 0 \leq t \leq \tfrac{4}{5}.$$

The corresponding $\mathrm{VI}\,(F, X)$ still has only one solution since F is unchanged and remains strongly monotone. At the GNE $x = (0,1)^T$, $\mathcal{M}(x) = \emptyset$ (due to the constraint $x_1^2 \leq 0$ in the problem of player 1). Thus the approach of Section 3.1 would not find this GNE.

The shared constraint function in Example 2 is separable, which we will exploit in developing our primal, or resource-directed, parametrization. Specifically, we make the following blanket assumption in this subsection.

Assumption C. *The shared constraint function g has the form* (10)*, where $\hat{g}_\nu = (\hat{g}_{\nu,i})_{i=1}^{\bar{m}} : \Re^{n_\nu} \to \Re^{\bar{m}}$, $\nu = 1, \ldots, N$, and each $\hat{g}_{\nu,i}$ is a differentiable convex function.*

Now, we construct a family of VIs that contains $\mathrm{VI}\,(F, X)$ as a particular instance. Let $\beta = (\beta_\nu)_{\nu=1}^N \in \Re^m$ with $\beta_\nu \in \Re^{\bar{m}}$, $\nu = 1, \ldots, N$, be a vector of parameters satisfying $\sum_{\nu=1}^N \beta_\nu = 0$. Define the set $X^\beta \subset X$ by

$$X^\beta := X_1^{\beta_1} \times \cdots \times X_N^{\beta_N} \quad \text{with} \quad X_\nu^{\beta_\nu} := \{x_\nu \in X_\nu \mid \hat{g}_\nu(x_\nu) \leq \beta_\nu\},\ \nu = 1, \ldots, N,$$

and consider $\mathrm{VI}\,(F, X^\beta)$. By Assumption C, X^β is closed and convex (possibly empty). Intuitively, we parametrize the division of resources among the players, reminiscent of Bender's decomposition.

Now, we investigate the relationship between $\mathrm{VI}\,(F, X^\beta)$ and GNEP. The following result is easy to see.

Theorem 3.5. *For any GNE x^*, we have $x^* \in \mathrm{SOL}(F, X^\beta)$ and $\sum_{\nu=1}^N \beta_\nu = 0$, where we let $\beta_\nu = \hat{g}_\nu(x_\nu^*) - \alpha_\nu g(x^*)$ with arbitrary real numbers α_ν such that $\sum_{\nu=1}^N \alpha_\nu = 1$ and $\alpha_\nu > 0$, $\nu = 1, \ldots, N$.*

Corollary 3.3.

$$\bigcup_{\sum_{\nu=1}^N \beta_\nu = 0} \mathrm{SOL}(F, X^\beta) \supset \mathrm{SOL}^*.$$

In Example 2, if we choose $\hat{g}_1(x_1) = x_1^2$ and $\hat{g}_2(x_2) = x_2^2 - 1$, then $\mathrm{SOL}(F, X^\beta) = \{(0,1)^T\}$ for $\beta = (0,0)^T$. In general, $\mathrm{VI}(F, X^\beta)$ need not have a solution or a solution need not be a GNE. The next result gives a sufficient condition for a solution of $\mathrm{VI}(F, X^\beta)$ to be a GNE.

Theorem 3.6. *For any $\beta \in \Re^m$ with $\sum_{\nu=1}^N \beta_\nu = 0$ and any $x^* \in \mathrm{SOL}(F, X^\beta)$, a sufficient condition for x^* to be a GNE is that*

$$\text{for each } i = 1, \ldots, \bar{m}, \quad \left\{ \begin{array}{lll} \text{either} & \hat{g}_{\nu,i}(x_\nu^*) = \beta_{\nu,i} & \nu = 1, \ldots, N \\ \text{or} & \hat{g}_{\nu,i}(x_\nu^*) < \beta_{\nu,i} & \nu = 1, \ldots, N. \end{array} \right\} \tag{14}$$

Proof. Since x^* is a solution of $\mathrm{VI}(F, X^\beta)$, for each $\nu = 1, \ldots, N$, we have that x_ν^* is a solution of $\mathrm{VI}(\nabla_{x_\nu}\theta_\nu(\cdot, x_{-\nu}^*), X_\nu^{\beta_\nu})$ or, equivalently, x_ν^* is an optimal solution of the convex programming problem:

$$\begin{array}{ll} \text{minimize} & \theta_\nu(x_\nu, x_{-\nu}^*) \\ \text{subject to} & x_\nu \in X_\nu, \quad \hat{g}_\nu(x_\nu) \le \beta_\nu. \end{array} \tag{15}$$

By (14) and Assumption C, for each i, we have either $\hat{g}_{\nu,i}(x_\nu^*) = \beta_{\nu,i} = -\sum_{\nu' \neq \nu} \beta_{\nu',i} = -\sum_{\nu' \neq \nu} \hat{g}_{\nu',i}(x_{\nu'}^*)$, implying $g_i(x^*) = 0$, or $\hat{g}_{\nu,i}(x_\nu^*) < \beta_{\nu,i} = -\sum_{\nu' \neq \nu} \beta_{\nu',i} < -\sum_{\nu' \neq \nu} \hat{g}_{\nu',i}(x_{\nu'}^*)$, implying $g_i(x^*) < 0$. Thus, for each ν, x_ν^* is also an optimal solution of the problem:

$$\begin{array}{ll} \text{minimize} & \theta_\nu(x_\nu, x_{-\nu}^*) \\ \text{subject to} & x_\nu \in X_\nu, \quad g(x_\nu,\ x_{-\nu}^*) \le 0. \end{array} \tag{16}$$

In particular, the active inequality constraints at x_ν^* in (15) coincide with those in (16). In view of Assumption B and (3), the above problem is exactly $P_\nu(x_{-\nu}^*)$. This shows that (2) holds and hence x^* is a GNE. □

Notice that, for any GNE x^*, the β given in Theorem 3.5 satisfies the sufficient condition (14). Thus, we can refine Corollary 3.3 to

$$\bigcup_{\substack{\sum_{\nu=1}^N \beta_\nu = 0 \\ \text{(14) holds for some } x^* \in \mathrm{SOL}(F, X^\beta)}} \mathrm{SOL}(F, X^\beta) \supset \mathrm{SOL}^*.$$

If there exist $a_\nu \in \Re^{\bar{m}}$, $\nu = 1, \ldots, N$, such that

$$\hat{g}_\nu(x_\nu) \ge a_\nu, \quad \forall x_\nu \in X_\nu,\ \nu = 1, \ldots, N,$$

then we can further restrict β to the bounded set

$$\left\{ \beta \in \Re^m \;\middle|\; \sum_{\nu=1}^N \beta_\nu = 0,\ \beta_\nu \ge a_\nu,\ \nu = 1, \ldots, N \right\}.$$

In Example 2, if we choose $\hat{g}_1(x_1) = x_1^2$ and $\hat{g}_2(x_2) = x_2^2 - 1$, then we can take as lower bounds $a_1 = 0$ and $a_2 = -1$.

If a solution x^* of VI (F, X^β) has a Lagrange multiplier λ_ν^* associated with each constraint $\hat{g}_\nu(x_\nu) \leq \beta_\nu$, i.e., $0 \leq \lambda_\nu^* \perp \hat{g}_\nu(x_\nu^*) - \beta_\nu \leq 0$, then letting

$$\pi^* = \min_{\nu=1,\ldots,N} \lambda_\nu^*, \qquad \Delta_\nu = \lambda_\nu^* - \pi^*, \ \nu = 1, \ldots, N,$$

where the "min" is taken componentwise, we see that (x^*, π^*) satisfies the KKT condition (11) for VI (F^Δ, X). Thus, VI (F^Δ, X) may be viewed as the dual of VI (F, X^β), with the former requiring separability of shared constraints and the latter requiring a CQ to ensure existence of Lagrange multipliers.

3.3 Relation between some GNE concepts and VI approaches

In this subsection, we discuss the *normalized equilibrium* introduced by Rosen [23] in relation to the solution sets obtained by the VI approaches discussed in the previous subsections. We make the following assumption through this subsection.

Assumption D. *A suitable CQ holds at every GNEs.*

Assumption D implies that for every GNE x^* there exists a Lagrange multiplier $\lambda^* = (\lambda_\nu^*)_{\nu=1}^N \in \Re^m$ that satisfies the KKT system (5).

The normalized equilibrium is defined as follows [23]:

Definition 3.2. *Let a GNE x^* together with a Lagrange multiplier $\lambda^* = (\lambda_\nu^*)_{\nu=1}^N \in \Re^m$ satisfies the KKT system* (5). *We call x^* a normalized equilibrium if there exists vectors $r \in \Re^N_{++}$ and $\lambda \in \Re^{\overline{m}}$ such that*

$$\lambda_\nu^* = \lambda / r_\nu \quad \nu = 1, \ldots, N.$$

Rosen [23] proves that there exists a normalized equilibrium for any given $r = (r_\nu)_{\nu=1}^N \in \Re^N_{++}$ if the set X is compact. He defines the weighted Nikaido-Isoda-type function $\rho : \Re^n \times \Re^n \times \Re^N \to \Re$ by

$$\rho(x, y, r) = \sum_{\nu=1}^N r_\nu \theta_\nu(x_1, \ldots, y_\nu, \ldots, x_N),$$

and shows that there exists a point $x^* \in X$ such that

$$\rho(x^*, x^*, r) = \max_{y \in X} \{\rho(x^*, y, r)\} \tag{17}$$

to establish the existence of a normalized equilibrium. Note that it is easy to see that the KKT condition for (17) is equivalent to the KKT condition for the VI $(\tilde{F}^r, X)$ where $\tilde{F}^r : \Re^n \to \Re^n$ is given by

$$\tilde{F}^r(x) = (r_\nu \nabla_{x_\nu} \theta_\nu(x))_{\nu=1}^N .$$

Rosen's results may be summarized with our notation as follows.

Theorem 3.7. *[23, Theorem 3] For each $r \in \Re^N_{++}$, any solution of* VI $(\tilde{F}^r, X)$ *is a normalized equilibrium.*

Theorem 3.8. *[23, Theorem 4] Assume that X is bounded and $\tilde{F}^r$ is strictly monotone for every $r \in Q$, where Q is a subset of $\Re^N_{++}$. Then there is a unique normalized equilibrium for each $r \in Q$.*

Note that the monotonicity of F does not imply the monotonicity of $\tilde{F^r}$. In fact, consider the mapping $F : \Re^2 \to \Re^2$ defined by

$$F(x_1, x_2) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}$$

is monotone. However, a mapping $\tilde{F^r} : \Re^2 \to \Re^2$ with weight $(r_1, r_2) = (15, 1)$ is given by

$$\tilde{F^r}(x_1, x_2) = \begin{pmatrix} 30x_1 + 15x_2 \\ x_1 + 2x_2 \end{pmatrix},$$

which is not monotone because for vectors $x = (1, -4)$ and $y = (0, 0)$ we have

$$\langle \tilde{F^r}(x) - \tilde{F^r}(y), x - y \rangle = -2 < 0.$$

Now we investigate the difference between normalized equilibrium and GNE, and besides, we discuss the relation with the solution set of the VI formulations. Below, the set of normalized equilibria is denoted by SOL^N.

From the definition of these concepts among with Theorems 3.1, 3.7 and Corollary 3.1, we have the following relation:

$$\begin{array}{ccc} & & \mathrm{SOL}\,(F, X) \\ & & \rotatebox{90}{$\subseteq$} \\ \mathrm{SOL}^N & = & \bigcup_{r \in \Re^N_{++}} \mathrm{SOL}\,(\tilde{F^r}, X) \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \mathrm{SOL}^* & \subseteq & \bigcup_{\Delta \in \Re^m_+} \mathrm{SOL}\,(F^\Delta, X). \end{array}$$

For any (x, λ) satisfying the KKT system (5), we say strict complementarity holds at (x, λ) if $g_i(x) = 0$ implies $\lambda_{\nu,i} > 0$ for any $\nu = 1, \ldots, N$ and $i = 1, \ldots, m$.

The next proposition states that, when there is only one shared constraint, all GNE are normalized equilibrium as long as the strict complementarity holds.

Proposition 3.2. *Suppose that there is only one shared constraint and assume that $\mathcal{M}(x^*) \neq \emptyset$ for every GNE x^*. If strict complementarity holds at (x^*, λ^*) for every GNE x^* and some $\lambda^* \in \mathcal{M}(x^*)$, then every GNE x^* is a normalized equilibrium.*

Proof. If the shared constraint is inactive, i.e., $g(x^*) < 0$, then we have $\lambda^*_\nu = 0,\ \nu = 1, \ldots, N$ and it is obvious that x^* is a normalized equilibrium.

Suppose that $g(x^*) = 0$. Then it implies $\lambda^*_\nu > 0$ for all ν by the strict complementarity assumption. Let $\lambda \in \Re$ and a weight vector $r \in \Re^N$ be given by

$$\begin{aligned} \lambda &:= 1 \\ r_\nu &:= \frac{1}{\lambda^*_\nu}, \quad \nu = 1, \ldots, N. \end{aligned}$$

Then we have $\lambda^*_\nu = \frac{\lambda}{r_\nu}$ for $\nu = 1, \ldots, N$. This indicates that x^* is a normalized equilibrium. ☐

Corollary 3.4. *Suppose that the assumptions of Proposition 3.2 hold. Then we have*

$$\bigcup_{r \in \Re^N_{++}} \mathrm{SOL}\,(\tilde{F^r}, X) = \mathrm{SOL}^N = \mathrm{SOL}^*,$$

and hence the following relation holds:

$$
\begin{array}{ccc}
 & & \mathrm{SOL}\,(F,X) \\
 & & \rotatebox{90}{$\subseteq$} \\
\mathrm{SOL}^{N} & = & \bigcup_{r\in\Re^N_{++}} \mathrm{SOL}\,(\tilde{F^r},X) \\
\| & & \rotatebox{90}{$\subseteq$} \\
\mathrm{SOL}^* & \subseteq & \bigcup_{\Delta\in\Re^m_+} \mathrm{SOL}\,(F^{\Delta},X).
\end{array}
$$

From the above results, in the single shared constraint case, we may expect to obtain all GNE by means of parametrization with weights r_ν on the objective functions θ_ν.

In the case of multiple shared constraints, however, such parametrization may fail to yield the set of GNE, since a GNE is not necessarily a normalized equilibrium in general. This is illustrated in the next example.

Example 3. Consider the following two-person game, where player 1 and player 2 solve the minimization problems

$$
\begin{array}{lll}
P_1(z): & \underset{x,y}{\text{minimize}} & x^2+xy+y^2+(x+y)z-25x-38y \\
 & \text{subject to} & x\geq 0 \\
 & & y\geq 0 \\
 & & x+2y-z\leq 14 \\
 & & 3x+2y+z\leq 30, \\
P_2(x,y): & \underset{z}{\text{minimize}} & z^2+(x+y)z-25z \\
 & \text{subject to} & z\geq 0 \\
 & & x+2y-z\leq 14 \\
 & & 3x+2y+z\leq 30,
\end{array}
$$

respectively. This is a GNEP with two shared constraints.

The solution set of this GNEP consists of vectors

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 11-t \\ 8-t \end{pmatrix}, \quad t\in[0,2].
$$

Hence the shared constraints are both active at each GNE.

Note that the VI (F,X) has a single solution

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \\ 8 \end{pmatrix}.
$$

The Lagrange multipliers $\lambda=(\lambda_{\nu,i})\in\Re^4$ corresponding to the GNE $(x(t),y(t),z(t))=(t,11-t,8-t)$ is given by

$$
\begin{pmatrix} \lambda_{1,1}(t) \\ \lambda_{1,2}(t) \\ \lambda_{2,1}(t) \\ \lambda_{2,2}(t) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}t+3 \\ 1-\frac{t}{2} \\ \overline{s}+2-2t \\ \overline{s} \end{pmatrix}, \quad \overline{s}\geq\max\{0,2t-2\}.
$$

Thus the tuple $(\lambda_1,\lambda_2,r_1,r_2)\in\Re^4_+$ satisfying

$$
\begin{aligned}
\frac{\lambda_1}{r_1} &= \lambda_{1,1}(t), & \frac{\lambda_2}{r_1} &= \lambda_{1,2}(t), \\
\frac{\lambda_1}{r_2} &= \lambda_{2,1}(t), & \frac{\lambda_2}{r_2} &= \lambda_{2,2}(t)
\end{aligned}
$$

exists only if $0 \le t < 1$. Hence, the following relation holds in this example:

$$\left\{ \begin{pmatrix} 0 \\ 11 \\ 8 \end{pmatrix} \right\} = \mathrm{SOL}\,(F, X)$$

$$\subsetneq \left\{ \begin{pmatrix} t \\ 11-t \\ 8-t \end{pmatrix} \middle| \ 0 \le t < 1 \right\} = \mathrm{SOL}^{N}$$

$$\subsetneq \left\{ \begin{pmatrix} t \\ 11-t \\ 8-t \end{pmatrix} \middle| \ 0 \le t \le 2 \right\} = \mathrm{SOL}^{*}.$$

This example indicates that, when there are more than one shared constraints, some GNE may not be obtained by the VI approach using parametrization with weights on the objective functions.

4 Numerical Experiments

In this section, we show some numerical experiments with our VI approaches. We give a brief description of the GNEPs taken from the literature in the next subsection and present numerical results of our approaches in Subsection 4.2.

4.1 Examples

Example 4 (Harker's example). This problem is taken from [8]. There are two players and they solve the following problems:

$$\begin{array}{llll} P_1(x_2): & \text{minimize} & x_1^2 + \frac{8}{3}x_1x_2 - 34x_1 \\ & \text{subject to} & 0 \le x_1 \le 10 \\ & & x_1 + x_2 \le 15, \end{array} \qquad \begin{array}{llll} P_2(x_1): & \text{minimize} & x_2^2 + \frac{5}{4}x_1x_2 - 24.25x_2 \\ & \text{subject to} & 0 \le x_2 \le 10 \\ & & x_1 + x_2 \le 15. \end{array}$$

This is a GNEP with one shared constraint and the solution set is given by

$$\mathrm{SOL}^{*} = \left\{ \begin{pmatrix} 5 \\ 9 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} t \\ 15-t \end{pmatrix} \middle| \ 9 \le t \le 10 \right\}. \tag{18}$$

The corresponding $F : \Re^2 \to \Re^2$ and $X \subseteq \Re^2$ are represented as

$$F(x) = \begin{pmatrix} 2x_1 + \frac{8}{3}x_2 - 34 \\ \frac{5}{4}x_1 + 2x_2 - 24.25 \end{pmatrix}, \tag{19}$$
$$X = \left\{ x \in \Re^2 \ \mid x_1 + x_2 \le 15, \ 0 \le x_\nu \le 15, \ \nu = 1, \ 2 \right\}.$$

Since F is strongly monotone on $\Re^2$, VI (F, X) has a unique solution, which is given by $x = (5, 9)^T$. Note that this solution lies in the interior of the set X.

Example 5 (River basin pollution game). Consider the 3-person river basin pollution game studied in [13, Section 5.3], where the problem of player $\nu \in \{1, 2, 3\}$ is defined by

$$\begin{array}{lll} P_\nu(x_\nu): & \text{minimize} & (\alpha_\nu x_\nu + \beta(x_1 + x_2 + x_3) - \chi_\nu)x_\nu \\ & \text{subject to} & x_\nu \ge 0, \\ & & 3.25x_1 + 1.25x_2 + 4.125x_3 \le 100, \\ & & 2.2915x_1 + 1.5625x_2 + 2.8125x_3 \le 100, \end{array}$$

with $\alpha_1 = 0.01$, $\alpha_2 = 0.05$, $\alpha_3 = 0.01$, $\beta = 0.01$, $\chi_1 = 2.9$, $\chi_2 = 2.88$, $\chi_3 = 2.85$. This GNEP has two shared constraints. The corresponding $F : \Re^3 \to \Re^3$ and $X \subseteq \Re^3$ are given by

$$F(x) = \begin{pmatrix} 0.04 & 0.01 & 0.01 \\ 0.01 & 0.12 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{pmatrix} x - \begin{pmatrix} 2.9 \\ 2.88 \\ 2.85 \end{pmatrix}, \tag{20}$$

$$X = \left\{ x \in \Re^3_+ \;\middle|\; \begin{pmatrix} 3.25 & 1.25 & 4.125 \\ 2.2915 & 1.5625 & 2.8125 \end{pmatrix} x \le \begin{pmatrix} 100 \\ 100 \end{pmatrix} \right\}.$$

Since F is strongly monotone on $\Re^3$, VI(F, X) has a single solution, which given by $x = (\frac{4673}{221}, \frac{5754}{359}, \frac{567}{208})^T \approx (21.14, 16.03, 2.73)^T$. Note that, at this solution, the first inequality defining X is active, while the second inequality is inactive.

Example 6 (Electric power market model). We take a model in electric power markets with endogenous arbitrage from [19, Section 5.3]. The model consists of several electricity firms competing on a spatial network of markets along with an arbitrager who attempts to make a profit by exploiting price differentials between regions. In the original model [11], each firm maximizes its profit with anticipating the arbitragers' optimal response, resulting in a multi-leader-follower-game. In [19, Section 5.3], the arbitrager is removed from the model and the price differentials are assumed to be less than the shipping costs. In this setting, the model can be formulated as a GNEP as described in the following.

The regions are represented by the nodes in a network and each firm has electric plants at those nodes. Each firm determines how much it should produce at each plant and how much it should sell at each node to maximize its profit.

We introduce the notations to formulate the problem.

Parameters

$\mathcal{N}$	:	set of nodes
$\mathcal{F}$	:	set of firms
c_i^f	:	cost per unit generation at node i by firm f
P_i	:	price intercept of sales function at node i
Q_i	:	quantity intercept of sales function at node i
e_{ij}	:	unit cost of shipping from node i to j
CAP_i^f	:	production capacity at node i for firm f

Variables

s_{ij}^f : amount produced at node i and sold at node j by firm f

S_j : amount of total sales at node j

$$S_j := \sum_{t \in \mathcal{F}} \sum_{i \in \mathcal{N}} s_{ij}^t, \quad \forall j \in \mathcal{N}$$

p_j : market price at node j

$$p_j(S_j) := P_j - \frac{P_j}{Q_j} S_j, \quad \forall j \in \mathcal{N}$$

Each firm f's problem is to find s_{ij}^f, $(i, j) \in \mathcal{N} \times \mathcal{N}$ that solve the following maximization problem: Given $s_{ij}^f,\ t(\neq f) \in \mathcal{F},\ (i,j) \in \mathcal{N} \times \mathcal{N}$,

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j \in \mathcal{N}} \left[p_j(S_j) \sum_{i \in \mathcal{N}} s_{ij}^f \right] - \sum_{(i,j) \in \mathcal{N} \times \mathcal{N}} e_{ij} s_{ij}^f - \sum_{i \in \mathcal{N}} c_i^f \sum_{j \in \mathcal{N}} s_{ij}^f \\ \text{subject to} & \displaystyle\sum_{j \in \mathcal{N}} s_{ij}^f \leq \mathrm{CAP}_i^f, \quad \forall i \in \mathcal{N} \\ & p_j(S_j) - p_i(S_i) - e_{ij} \leq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N} \\ & s_{ij}^f \geq 0, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}. \end{array}$$

Note that these problems yield a GNEP with shared constraints, because the price differential constraints $p_j(S_j) - p_i(S_i) - e_{ij} \leq 0$ depend on the total sales at each node $S_j = \sum_{t \in \mathcal{F}} \sum_{i \in \mathcal{N}} s_{ij}^t$.

4.2 Numerical Results

Now we show numerical results with our parametrized VI approaches for the following three examples: Harker's example, the river basin pollution game and the electric power market model. All parametrized VI (F^{Δ}, X) and VI (F, X^{β}) are converted to equivalent linear complementarity problems (LCPs) and solved by the MATLAB code PATHLCP.M [7].

Example 4 (Harker's example). In the price-directed parametrization, by (9) and (19), the mapping F^{Δ} is given by

$$F^{\Delta}(x) = \begin{pmatrix} 2x_1 + \frac{8}{3}x_2 + \Delta_1 - 34 \\ \frac{5}{4}x_1 + 2x_2 + \Delta_2 - 24.25 \end{pmatrix},$$

with parameters $\Delta = (\Delta_1, \Delta_2)^T \in \Re_+^2$. The KKT condition of VI (F^{Δ}, X) is then written as the following LCP:

$$\begin{array}{ccccl} 0 & \leq & x_1 & \perp & 2x_1 + \frac{8}{3}x_2 + \Delta_1 + \pi + \mu_1 - 34 \geq 0, \\ 0 & \leq & x_2 & \perp & \frac{5}{4}x_1 + 2x_2 + \Delta_2 + \pi + \mu_2 - 24.25 \geq 0, \\ 0 & \leq & \mu_1 & \perp & 10 - x_1 \geq 0, \\ 0 & \leq & \mu_2 & \perp & 10 - x_2 \geq 0, \\ 0 & \leq & \pi & \perp & 15 - x_1 - x_2 \geq 0. \end{array}$$

In view of Corollary 3.2, we restrict the parameter space to the union of two line segments $\{\Delta \in \Re_+^2 \mid 0 \leq \Delta_1 \leq 2,\ \Delta_2 = 0\} \cup \{\Delta \in \Re_+^2 \mid \Delta_1 = 0,\ 0 \leq \Delta_2 \leq 2\}$, on which we pick grid points. Specifically, we generate $512 = 2 \times 256$ grid points on the segments and check whether an obtained solution x^* of each parametrized VI satisfies the condition (12) approximately, i.e.,

$$|\Delta_\nu(x_1^* + x_2^* - 15)| < 10^{-6}, \quad \nu = 1, 2.$$

By solving 512 LCPs, we obtained 98 GNEP solutions, as plotted in Figure 1. We observe that the solutions obtained by this approach are widely distributed in its solution set given by (18), while the VI approach of [26, 3, 4] can only find the unique equilibrium $x = (5,\ 9)^T$.

Next we test the resource-directed parametrization approach. We let $\hat{g}_1(x_1) = x_1 - \frac{15}{2}$ and $\hat{g}_2(x_2) = x_2 - \frac{15}{2}$. Then the parametrized feasible set X^{β} is defined by

$$X^{\beta} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, x_\nu - \frac{15}{2} \leq \beta_\nu,\ 0 \leq x_\nu \leq 10,\ \nu = 1,\ 2 \right\},$$

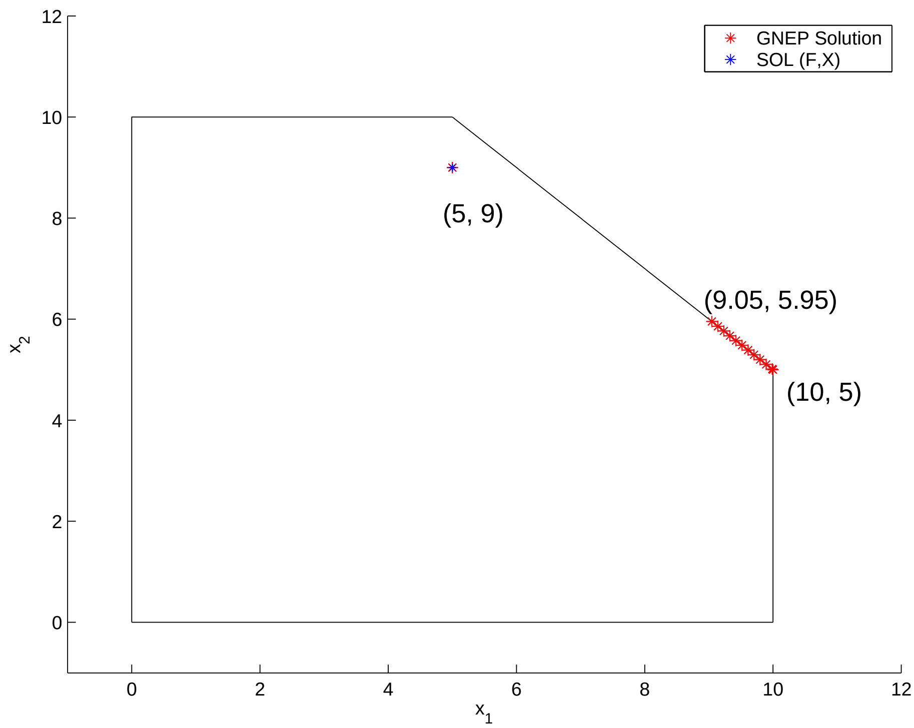

Figure 1: GNEs obtained by the price-directed parametrization approach for Harker's example

where $\beta \in \{(\beta_1,\ \beta_2)^T \in \Re^2 \mid \beta_1 + \beta_2 = 0\}$. The KKT condition of VI (F, X^β) is written as the following LCP:

$$\begin{array}{lllll} 0 & \leq & x_1 & \perp & 2x_1 + \frac{8}{3}x_2 + \lambda_1 + \mu_1 - 34 \geq 0, \\ 0 & \leq & x_2 & \perp & \frac{5}{4}x_1 + 2x_2 + \lambda_2 + \mu_2 - 24.25 \geq 0, \\ 0 & \leq & \mu_1 & \perp & 10 - x_1 \geq 0, \\ 0 & \leq & \mu_2 & \perp & 10 - x_2 \geq 0, \\ 0 & \leq & \lambda_1 & \perp & \frac{15}{2} - \beta_1 - x_1 \geq 0, \\ 0 & \leq & \lambda_2 & \perp & \frac{15}{2} - \beta_2 - x_2 \geq 0. \end{array}$$

Note that the function $\hat{g}_\nu$ is bounded below on $X_\nu := \{x_\nu \mid 0 \leq x_\nu \leq 10\}$, that is,

$$\hat{g}_\nu(x_\nu) \geq -\frac{15}{2} \quad \forall x_\nu \in X_\nu.$$

Hence we can restrict β to the bounded set

$$\left\{ \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \Re^2 \;\middle|\; \beta_1 + \beta_2 = 0,\ \beta_\nu \geq -\frac{15}{2} \right\} = \left\{ \begin{pmatrix} \beta \\ -\beta \end{pmatrix} \in \Re^2 \;\middle|\; -\frac{15}{2} \leq \beta \leq \frac{15}{2} \right\}.$$

We choose 256 points from the interval $[-\frac{15}{2}, \frac{15}{2}]$ and for each β, check whether an obtained solution x^* of the parametrized VI satisfies the condition (14) approximately, i.e.,

$$\begin{array}{ll} \text{either} & |x_\nu - \frac{15}{2} - \beta_\nu| < 10^{-6}, \quad \nu = 1, 2 \\ \text{or} & x_\nu - \frac{15}{2} - \beta_\nu < -10^{-6}, \quad \nu = 1, 2. \end{array}$$

As a result of solving 256 LCPs, we obtained 34 GNEP solutions, among which 16 GNEP solutions correspond to the equilibrium $x = (5,\ 9)^T$, and the shared constraint $x_1 + x_2 \leq 15$ is active at all the rest of computed GNEP solutions. Figure 2 shows those computed solutions. We see that the solutions obtained by this approach are also widely distributed in the solution set.

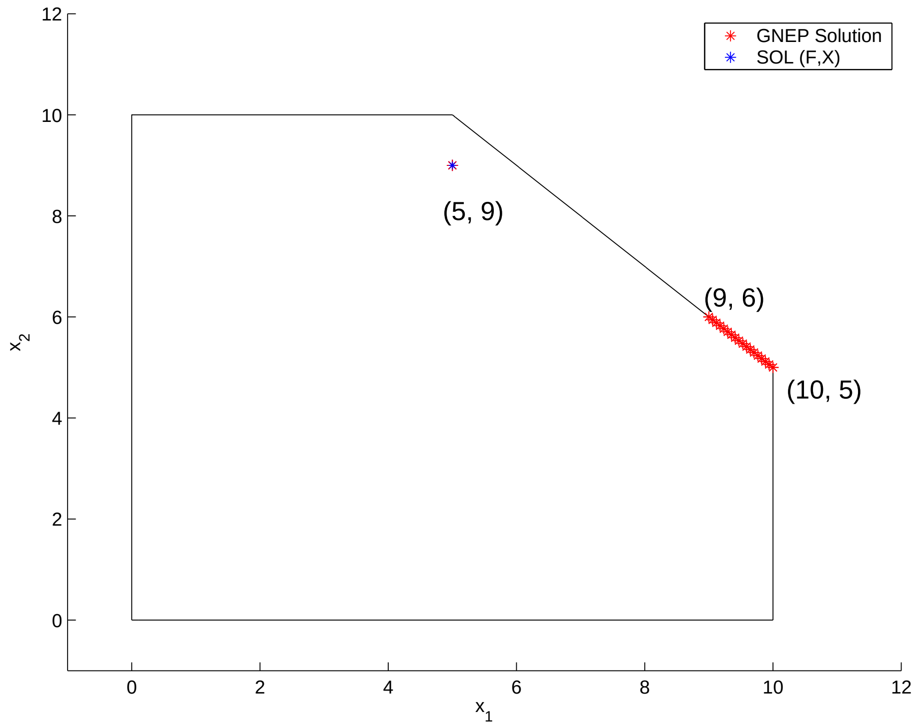

Figure 2: GNEs obtained by the resource-directed parametrization approach for Harker's example

Example 5 (River basin pollution game). In the price-directed parametrization approach, by (9) and (20), the mapping F^{Δ} is given by

$$F^{\Delta}(x) = \begin{pmatrix} 0.04 & 0.01 & 0.01 \\ 0.01 & 0.12 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{pmatrix} x + \sum_{\nu=1}^{3} \begin{pmatrix} 3.25 \\ 1.25 \\ 4.125 \end{pmatrix} \Delta_{\nu,1} + \sum_{\nu=1}^{3} \begin{pmatrix} 2.2915 \\ 1.5625 \\ 2.8125 \end{pmatrix} \Delta_{\nu,2} - \begin{pmatrix} 2.9 \\ 2.88 \\ 2.85 \end{pmatrix}$$

with parameters $\Delta = \left((\Delta_{\nu,1},\ \Delta_{\nu,2})^T\right)_{\nu=1}^{3} \in \Re^6_+$.

In view of Corollary 3.2, we restrict the parameter space to the set

$$\mathcal{P} := \prod_{i=1}^{2} \left(\bigcup_{\nu=1}^{3} \mathcal{P}_i(\nu) \right),$$

where for $i = 1, 2$ and $\nu = 1, 2, 3$,

$$\mathcal{P}_i(\nu) := \left\{ \Delta_i \in \Re^3_+ \mid \Delta_{\nu,i} = 0,\ 0 \le \Delta_{\nu',i} \le 2,\ \nu' \in \{1,2,3\} \backslash \{\nu\} \right\},$$

on which we pick points randomly. Specifically, we generate

(a) 10,000 points from the set $\mathcal{P}_1(\nu_1) \times \mathcal{P}_2(\nu_2)$ for each $(\nu_1,\ \nu_2) \in \{1,\ 2,\ 3\} \times \{1,\ 2,\ 3\}$,

(b) 1,000 points from the $\mathcal{P}_1(\nu) \times \{(0,0,0)^T\}$ for each $\nu \in \{1,\ 2,\ 3\}$,

(c) 1,000 points from the $\{(0,0,0)^T\} \times \mathcal{P}_2(\nu)$ for each $\nu \in \{1,\ 2,\ 3\}$ and

(d) the origin $(0,0,0,0,0,0)^T$.

Thus we generate 96,001 points in total. Roughly speaking, these four cases aim to find GNEs such that (a) both of the shared constraints are active, (b) the constraint $3.25x_1+1.25x_2+4.125x_3 \le 100$ is active, (c) the constraint $2.2915x_1+1.5625x_2+2.8125x_3 \le 100$ is active and (d) neither of the shared constraints is active, respectively. Then we check whether an obtained solution x^* of each parametrized VI satisfies the condition (12) approximately, i.e.,

$$\begin{cases} |\Delta_{\nu,1}(3.25x_1^* + 1.25x_2^* + 4.125x_3^* - 100)| < 10^{-6}, & \nu = 1,2,3 \\ |\Delta_{\nu,2}(2.2915x_1^* + 1.5625x_2^* + 2.8125x_3^* - 100)| < 10^{-6}, & \nu = 1,2,3. \end{cases}$$

By solving 96,001 LCPs, we obtained 2,445 GNEP solutions as shown in Figure 3. Note that the polyhedral set in Figure 3 represents the feasible region X. We obtain the equilibrium $x^* = (21.14, 16.03, 2.73)^T$ by setting $\Delta = (0,0,0,0,0,0)^T$ and the remaining 2,444 GNEP solutions are obtained when we pick the points from the $\mathcal{P}_1(\nu) \times \{(0,0,0)^T\}$ for $\nu = 1,2,3$. Thus, the shared constraint $3.25x_1 + 1.25x_2 + 4.125x_3 \le 100$ is active at any of these GNEP solutions.

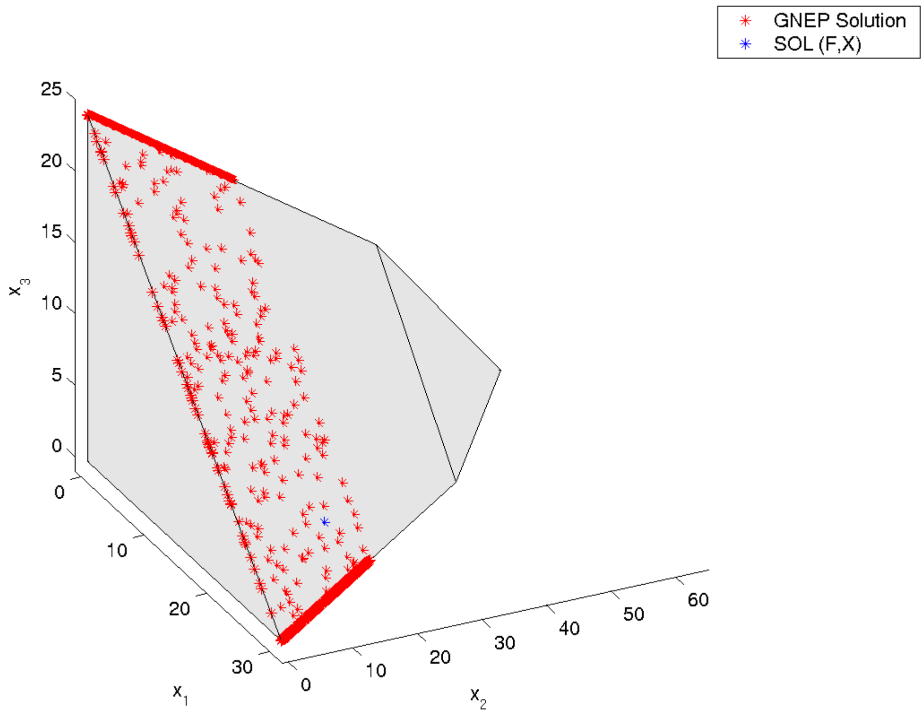

Figure 3: GNEs obtained by the price-directed parametrization approach for Example 5

Next, we test the resource-directed parametrization approach. We let $\hat{g}_{\nu,i}(x_\nu) = \gamma_{\nu,i}x_\nu - \frac{100}{3}$ for $\nu = 1,2,3$ and $i = 1,2$, where $\gamma_{1,1} = 3.25$, $\gamma_{2,1} = 1.25$, $\gamma_{3,1} = 4.125$, $\gamma_{1,2} = 2.2915$, $\gamma_{2,2} = 1.5625$, $\gamma_{3,2} = 2.8125$. Then the parametrized feasible set X^β is given by

$$X^\beta = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \ \gamma_{\nu,i}x_\nu - \frac{100}{3} \le \beta_{\nu,i},\ x_\nu \ge 0,\ i = 1,\ 2,\ \nu = 1,\ 2,\ 3 \right\},$$

where

$$\beta \in \left\{ (\beta_{\nu,i}) \in \Re^6 \mid \beta_{1,i} + \beta_{2,i} + \beta_{3,i} = 0,\ i = 1,\ 2 \right\}.$$

Note that each function $\hat{g}_{\nu,i}$ is bounded below on $X_\nu = \Re_+$, that is,

$$\hat{g}_{\nu,i}(x_\nu) \geq -\frac{100}{3}, \quad \forall x_\nu \in X_\nu.$$

So we can restrict β to the bounded set

$$\begin{aligned}
&\left\{ (\beta_{\nu,i}) \in \Re^6 \;\middle|\; \begin{array}{l} \beta_{1,i} + \beta_{2,i} + \beta_{3,i} = 0,\ \beta_{\nu,i} \geq -\frac{100}{3}, \\ \nu = 1,2,3,\ i = 1,2 \end{array} \right\} \\
&= \prod_{i=1}^{2} \left\{ \begin{pmatrix} \beta_{1,i} \\ \beta_{2,i} \\ \beta_{3,i} \end{pmatrix} \in \Re^3 \;\middle|\; \beta_{1,i} + \beta_{2,i} + \beta_{3,i} = 0,\ \beta_{\nu,i} \geq -\frac{100}{3} \right\} \\
&= \prod_{i=1}^{2} \left\{ \begin{pmatrix} \beta_{1,i} \\ \beta_{2,i} \\ -\beta_{1,i} - \beta_{2,i} \end{pmatrix} \in \Re^3 \;\middle|\; \beta_{1,i} + \beta_{2,i} \leq \frac{100}{3},\ \beta_{\nu,i} \geq -\frac{100}{3} \right\}. \qquad (21)
\end{aligned}$$

To generate points that belong to this set, we pick 100,000 points randomly from the rectangle

$$\left\{ (\beta_{1,1}, \beta_{2,1}, \beta_{2,1}, \beta_{2,2}) \in \Re^4 \;\middle|\; -\frac{100}{3} \leq \beta_{\nu,i} \leq \frac{200}{3} \right\},$$

and then check whether each point $\beta = (\beta_{1,1}, \beta_{2,1}, -\beta_{1,1} - \beta_{2,1}, \beta_{1,2}, \beta_{2,2}, -\beta_{1,2} - \beta_{2,2})^T$ satisfies the conditions in (21). In this way, we obtain 24,858 points that belong to the set given by (21).

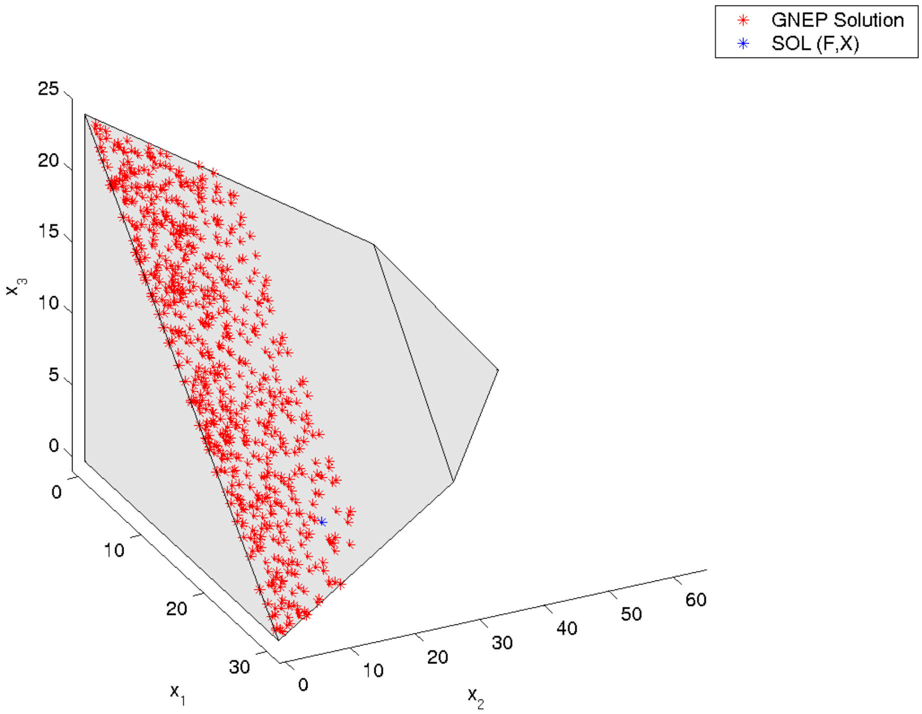

Figure 4: GNEs obtained by the resource-directed parametrization approach for Example 5

As a result of solving 24,858 LCPs, we obtained 699 GNEP solutions as shown in Figure 4. Thus, the resource-directed parametrization approach was also able to find GNEP solutions distributed widely on the face determined by the shared constraint $3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100$.

Table 1: Generation costs c_i^f and capacities CAP_i^f

(f, i)	$(\mathrm{I}, 1)$	$(\mathrm{I}, 2)$	$(\mathrm{II}, 2)$	$(\mathrm{II}, 3)$
c_i^f	15	15	15	15
CAP_i^f	100	50	100	50

Table 2: Price function data $(P_i,\ Q_i)$

node i	1	2	3
P_i	40	35	32
Q_i	500	400	600

Example 6 (Electric market model). We apply the price-directed parametrization approach to a simple example [19] consisting of two firms, named I and II, which compete in a 3-node network with six arcs $\{(1,\ 2),\ (2,\ 1),\ (1,\ 3),\ (3,\ 1),\ (2,\ 3),\ (3,\ 2)\}$. Firm I owns electric plants at nodes 1 and 2, and firm II owns electric plants at nodes 2 and 3. The other data for the example are given in Table 1 and Table 2.

The unit cost of shipping e_{ij} are set to be

$$e_{ij} = \begin{cases} 1 & (i \neq j) \\ 0 & (i = j). \end{cases}$$

The parameter space $\{\Delta = (\Delta_{ij}^f) \in \Re_+^{18}\}$ is too large to enumerate fine grid points, so we just pick up points from the restricted space $\mathcal{P} := \bigcup_{(i,j) \in \mathcal{N} \times \mathcal{N}} \mathcal{P}_{ij}$, where

$$\mathcal{P}_{ij} := \left\{ \Delta \in \Re_+^{18} \mid 0 \leq \Delta_{ij}^f \leq 20,\ \Delta_{kl}^f = 0 \quad \forall (k,\ l) \neq (i,\ j),\ f = 1, 2 \right\}.$$

Specifically, for each $(i, j) \in \mathcal{N} \times \mathcal{N}$, we generate 100 points $(\Delta_{ij}^1,\ \Delta_{ij}^2)$ randomly from the rectangle $[0,\ 20] \times [0,\ 20]$ and hence 900 points Δ in total. Then we check whether a computed solution of each parametrized VI satisfies the condition (12) approximately, i.e.,

$$|\Delta_{i,j}^f (p_j(S_j) - p_i(S_i) - e_{ij})| < 10^{-6} \quad \forall (i, j),\ f = 1, 2. \tag{22}$$

By solving 900 LCPs, we obtained 392 GNEP solutions. Note that, for $(i, j) = (1, 1),\ (2, 2),\ (3, 3)$, the sufficient condition (22) is always satisfied for any Δ_{ij}^f. Hence, we got 300 GNEP solutions when we pick the points from $\mathcal{P}_{ii}$ $(i = 1, 2, 3)$ and all those solutions correspond to a single solution, which is nothing but the equilibrium obtained by the approach of [26, 3, 4]. All the rest of computed GNEP solutions are obtained when we pick the points from $\mathcal{P}_{31}$.

Here, instead of showing the whole picture of those GNEs, let us compare two particular GNEP solutions, i.e., the equilibrium $\bar{x}$ obtained by the approach of [26, 3, 4] and the GNE x^* obtained by the price-directed parametrization approach for VI (F^Δ, X) with $(\Delta_{3,1}^{\mathrm{I}},\ \Delta_{3,1}^{\mathrm{II}}) = (14.1819,\ 8.6348)$ and $\Delta_{ij}^f = 0$ for all f and $(i, j) \neq (3, 1)$. The sales and the nodal prices of GNEs $\bar{x}$ and x^* are summarized in Table 3 and Table 4, respectively. We observe that the nodal prices p_i in Table 3 are all identical for the two GNEs. However, comparing the firms' profit, we have

$$\theta_1(\bar{x}) = 1969.5 < 1975.1 = \theta_1(x^*)$$
$$\theta_2(\bar{x}) = 1923.6 = 1923.6 = \theta_2(x^*),$$

Table 3: Nodal prices

node i	1	2	3
$\bar{x}$	28.82	27.82	27.82
x^*	28.82	27.82	27.82

Table 4: Firms' sales

(f,i,j)	(I, 1, 1)	(I, 1, 3)	(I, 2, 2)	(I, 2, 3)	(II, 2, 1)	(II, 2, 2)	(II, 3, 1)	(II, 3, 3)
$\bar{s}_{ij}^f$	77.01	22.99	41.84	8.16	59.83	40.17	2.85	47.15
s^{*f}_{ij}	79.79	20.21	41.84	8.16	59.83	40.17	0.08	49.92

where $\theta_f(x) := \sum_{j\in\mathcal{N}} \left[p_j(S_j) \sum_{i\in\mathcal{N}} s_{ij}^f \right] - \sum_{(i,j)\in\mathcal{N}\times\mathcal{N}} e_{ij} s_{ij}^f - \sum_{i\in\mathcal{N}} c_i^f \sum_{j\in\mathcal{N}} s_{ij}^f$. Thus, the GNE $\bar{x}$ obtained by the approach of [26, 3, 4] is weakly dominated by the GNE x^*, in the sense that, at the GNE $\bar{x}$, firm I has a motivation to move to the GNE x^* by paying a small incentive $\varepsilon > 0$ to firm II. (Recall that each player is maximizing its profit.) Hence, the GNE $\bar{x}$ would not appear in the actual situation. This result indicates that the approach of [26, 3, 4] may fail to find some important GNEs, while the proposed parameterization approach has a better chance to find those GNEs.

5 Concluding Remarks

In Section 4, we have applied the proposed VI approaches to three numerical examples and observe that the GNEs obtained by proposed methods are widely distributed in the GNEP solution set while the approach of [26, 3, 4] can only find some particular GNE.

We note that, in the price-directed parameterization, we have to restrict the parameter space heuristically in the price-directed parametrization because Theorem 3.4 does not say how large the parameter space should be. This remains a question for further study. Moreover, in our numerical experiments, we used fixed grid points or generated points randomly in the parameter space. Although such a simple procedure works well for small examples, more efficient parameter search procedures will be required for practical GNEPs.

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