A Gap Function Approach to the Generalized Nash Equilibrium Problem

Guidance

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Abstract

The generalized Nash equilibrium problem (GNEP) is a generalization of the Nash equilibrium problem (NEP), in which each player's strategy set depends on the other players' strategies. The GNEP has many applications such as electric power models and river basin pollution games. It is known that a GNEP can be reformulated as a quasi-variational inequality problem (QVIP) under some assumptions.

We consider an optimization reformulation approach for GNEP through QVIP with the regularized gap function. There are still some difficulties with this approach. The regularized gap function for QVIP is in general not differentiable, but only directionally differentiable. Moreover, a simple condition has yet to be established, under which any stationary point of the regularized gap function solves the QVIP. We tackle these issues for the GNEP in which the shared constraints are given by equalities, while the individual constraints are given by inequalities. First we formulate the minimization problem involving the regularized gap function, and show the equivalence to GNEP. Next, we establish the differentiability of the regularized gap function under some suitable assumptions, and show that any stationary point of the minimization problem solves the original GNEP. Then, by using a barrier technique, we propose an algorithm which sequentially solves minimization problems obtained from GNEPs with the shared equality constraints only. Further, we discuss the case of shared inequality constraints and present an approach that relies on the transformation to the equality constraints by means of slack variables. We present some results of numerical experiments to show the validity of the proposed approach.

Contents

1 Introduction

A multi-player non-cooperative game is called the Nash equilibrium problem (NEP), if the goal is to find a solution in which no player has any motivation to change his/her own strategy unilaterally. The generalized Nash equilibrium problem (GNEP) is a generalization of the NEP, in which each player's strategy set depends on the other players' strategies. A solution of the GNEP is called a generalized Nash equilibrium (GNE). The GNEP has many applications such as electric power models [14, 19] and river basin pollution games [12, 15, 16].

It is well known that a NEP can be reformulated as a variational inequality problem (VIP) if each player's problem is a convex programming problem [4, 9, 11]. It is also known that a GNEP can be reformulated as a quasi-variational inequality problem (QVIP) under some assumptions [10, 19]. The relationship between the GNEP and QVIP in Hilbert space was studied by Bensoussan [2]. Harker [10] obtained some results for the problems in the finite-dimensional Euclidean space. However, compared with the VIP, the study of the QVIP is still in its infancy. In particular, just a few algorithms have been proposed to solve the QVIP numerically. Therefore, it is desired to develop a useful solution method for QVIP.

As solution methods for GNEP or QVIP, some penalty methods have been proposed. Pang and Fukushima [19] proposed an approach for GNEP that solves a sequence of VIPs corresponding to NEPs, obtained by approximating the original GNEP, by means of an exterior penalty function technique. Nabetani [17] proposed a similar approach that uses an interior penalty function technique. Facchinei and Pang [5] proposed an exact penalty method for GNEP. Fukushima [8] proposed another penalty method to find a particular GNE called a restricted GNE.

Several algorithms have been proposed for GNEP or QVIP besides penalty methods. Fukushima [7] proposed an optimization reformulation approach with the regularized gap function for QVIP. Von Heusinger and Kanzow [13] proposed a regularized Nikaido-Isoda function and reformulation of a GNEP as a smooth optimization problem. More recently, Nabetani et al. [18] proposed parametrized VI approaches to GNEP.

In this paper, we consider an optimization reformulation approach with the regularized gap function. There are still some difficulties with this approach. Unlike the case of VIP, the regularized gap function for QVIP is in general not differentiable, but only directionally differentiable [7]. Moreover, for VIP, under some monotonicity assumptions, it is proved that any stationary point of the regularized gap function solves the VIP [7]. However, such a simple condition for the QVIP has yet to be established. It is quite important to investigate these properties from the practical viewpoint.

In this paper, we consider the GNEP in which the shared constraints are given by equalities, while the individual constraints are given by inequalities. First we formulate the minimization problem involving the regularized gap function, and show the equivalence to GNEP. Next, we establish the differentiability of the regularized gap function under some suitable assumptions, and show that any stationary point of the minimization problem solves the original GNEP. Then, by using a barrier technique, we propose an algorithm which sequentially solves minimization problems obtained from GNEPs with the shared equality constraints only. Further, we discuss the case of shared inequality constraints and present an approach that relies on the transformation to the equality constraints by means of slack variables. Finally, we present some results of numerical experiments to show the validity of the proposed approach.

We use the following notation throughout the paper. For vectors $x, y \in \mathbb{R}^n$, the inner product is defined by $\langle x, y\rangle \equiv x^\top y$. For a vector $x \in \mathbb{R}^n$, the Euclidean norm is defined by $\|x\| \equiv \sqrt{\langle x, x\rangle}$. For a transposed vector, we use a simplified notation $(x^1, x^2, \ldots, x^N)^\top$ instead of $((x^1)^\top, (x^2)^\top, \ldots, (x^N)^\top)^\top$.

2 Generalized Nash Equilibrium Problem

An N-person non-cooperative game in which each player's strategy set depends on the other players' strategies is called the ***G**eneralized **N**ash **E**quilibrium **P**roblem* (**GNEP**) [3]. In the GNEP, if no player has any motivation to change his/her strategy, then the vector formed by their strategies is called a ***G**eneralized **N**ash **E**quilibrium* (**GNE**). In the GNEP, each player ν solves the following optimization problem:

$$P^\nu(x^{-\nu}): \quad \begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^{-\nu}, x^\nu) \\ \text{subject to} & x^\nu \in S^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}, \end{array}$$

where

$$\begin{aligned} n &\equiv \sum_{\nu=1}^N n_\nu, \\ n_{-\nu} &\equiv n - n_\nu, \\ x &\equiv (x^\nu)_{\nu=1}^N \in \mathbb{R}^n, \\ x^{-\nu} &\equiv (x^{\nu'})_{\nu'\neq\nu} \in \mathbb{R}^{n_{-\nu}}. \end{aligned}$$

Here, $x^\nu \in \mathbb{R}^{n_\nu}$ denotes the strategy of player ν, and $x^{-\nu} \in \mathbb{R}^{n_{-\nu}}$ denotes the vector formed by the strategies of all players except player ν. The objective function $\theta^\nu : \mathbb{R}^n \to \mathbb{R}$ of player ν is assumed to be a differentiable convex function for any fixed $x^{-\nu}$. Player ν's strategy set $S^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ is a convex set, and depends on the other player's strategies. Thus, each player's problem is a differentiable convex programming problem.

GNEP is to find a tuple $x^* \equiv (x^{*,\nu})_{\nu=1}^N$ such that for each $\nu = 1, \ldots, N$, $x^{*,\nu}$ is an optimal solution of the following optimization problem, where $x^{-\nu}$ is fixed at $x^{*,-\nu}$:

$$P^\nu(x^{*,-\nu}): \quad \begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^{*,-\nu}, x^\nu) \\ \text{subject to} & x^\nu \in S^\nu(x^{*,-\nu}). \end{array} \tag{1}$$

This tuple x^* is a GNE. In particular, if each player's strategy set does not depend on the other players' strategies, then GNEP reduces to a Nash equilibrium problem.

3 Reformulation of GNEP as Quasi-Variational Inequality Problem

Define the vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ and the point-to-set mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as follows:

$$\begin{aligned} F(x) &\equiv \left(F^\nu(x)\right)_{\nu=1}^N \equiv \left(\nabla_{x^\nu}\theta^\nu(x^{-\nu}, x^\nu)\right)_{\nu=1}^N \in \mathbb{R}^n, \qquad (2) \\ S(x) &\equiv \prod_{\nu=1}^N S^\nu(x^{-\nu}) \subseteq \mathbb{R}^n. \end{aligned}$$

By assumption, problem (1) is a convex programming problem. So $x^{*,\nu}$ is an optimal solution of (1) if and only if $x^{*,\nu}$ is a stationary point of the function $\theta^\nu(x^{*,-\nu}, \cdot)$ on the set $S(x^{*,-\nu})$, that is,

$$x^{*,\nu} \in S^\nu(x^{*,-\nu})$$

and

$$\langle \nabla_{x^\nu}\theta^\nu(x^{*,-\nu}, x^{*,\nu}), x^\nu - x^{*,\nu}\rangle \geq 0, \qquad \forall x^\nu \in S(x^{*,-\nu})$$

are satisfied. Thus, the GNEP defined in Section 2 is rewritten as follows:

$$\begin{array}{ll} \text{Find} & x^* \in S(x^*) \\ \text{such that} & \langle F(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in S(x^*). \end{array} \tag{3}$$

This problem is a ***Q**uasi-**V**ariational **I**nequality **P**roblem* (**QVIP**). In particular, if $S(x) = \hat{S}$ for all x, where $\hat{S}$ is a nonempty closed convex set, then QVIP (3) reduces to a ***V**ariational **I**nequality **P**roblem* (**VIP**).

4 A Merit Function for QVIP

For a general equilibrium problem, a *merit function* is a real-valued function f such that x is a solution of the problem if and only if $f(x) = 0$ and x satisfies the constraints. The equilibrium problem can be reformulated as an equivalent optimization problem by using a merit function.

For VIP, there have been several proposals of merit functions, such as the gap function [1] and the *regularized gap function* [6], and the properties of these functions have been studied extensively. Consider the following VIP:

$$\begin{array}{ll} \text{Find} & x^* \in \hat{S} \\ \text{such that} & \langle F(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in \hat{S}. \end{array} \tag{4}$$

The regularized gap function $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}$ for the VIP (4) is defined by

$$\hat{f}(x) = -\inf_y \left\{ \langle F(x), y - x\rangle + \frac{1}{2}\langle y - x, H(y - x)\rangle \;\middle|\; y \in \hat{S} \right\}, \tag{5}$$

where H is an $n \times n$ symmetric positive definite matrix. The minimization problem in (5), i.e.,

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \langle F(x), y - x\rangle + \frac{1}{2}\langle y - x, H(y - x)\rangle \\ \text{subject to} & y \in \hat{S}, \end{array}$$

is a convex programming problem, and it has a unique optimal solution for any given x. Denote this optimal solution by $\hat{y}(x)$. The regularized gap function $\hat{f}$ has the following properties [6].

Theorem 4.1 *For each $x \in \hat{S}$, we have $\hat{f}(x) \geq 0$. Moreover, x solves VIP* (4) *if and only if $\hat{f}(x) = 0$ and $x \in \hat{S}$.*

These properties indicate that VIP (4) can be reformulated as the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \hat{f}(x) \\ \text{subject to} & x \in \hat{S}. \end{array} \tag{6}$$

Moreover, the function $\hat{f}$ possesses some favorable properties as shown in the next theorem [6].

Theorem 4.2 *Suppose that the function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then the regularized gap function $\hat{f}$: $\mathbb{R}^n \to \mathbb{R}$ defined by* (5) *is continuous. Moreover, if F is continuously differentiable, then $\hat{f}$ is also continuously differentiable, and the gradient of $\hat{f}$ at x is given by*

$$\nabla \hat{f}(x) = F(x) - (\nabla F(x) - H)(\hat{y}(x) - x).$$

In particular, when $\nabla F(x)$ is positive definite, any stationary point of the equivalent minimization problem (6)*, i.e. any point that satisfies the first-order optimality condition, solves the VIP* (4)*.*

For the QVIP (3), an extension of the regularized gap function for the VIP (4) is proposed [7]. This function, called the regularized gap function for the QVIP, is defined by

$$f(x) \equiv -\inf_{y} \left\{ \langle F(x), y - x \rangle + \frac{1}{2}\langle y - x, H(y - x) \rangle \;\middle|\; y \in S(x) \right\}, \tag{7}$$

where H is a symmetric positive definite matrix. This function is a natural extension of the regularized gap function for the VIP (5).

Let the set $X \subseteq \mathbb{R}$ be defined by

$$X \equiv \{x \in \mathbb{R}^n \mid x \in S(x)\},$$

which is called the feasible set of QVIP (3). Similarly to VIP, for any $x \in X$, the minimization problem in (7), i.e.,

$$\begin{array}{ll} \underset{y}{\text{minimize}} & \langle F(x), y - x \rangle + \frac{1}{2}\langle y - x, H(y - x) \rangle \\ \text{subject to} & y \in S(x), \end{array} \tag{8}$$

is a convex programming problem, so it has a unique optimal solution for any x. We denote the optimal solution of problem (8) by $y(x)$. Then the regularized gap function f is written as

$$f(x) = -\langle F(x), y(x) - x \rangle - \frac{1}{2}\langle y(x) - x, H(y(x) - x) \rangle.$$

The following result holds [7].

Theorem 4.3 *For each $x \in X$, we have $f(x) \geq 0$. Moreover, x solves QVIP* (3) *if and only if $f(x) = 0$ and $x \in X$.*

This theorem indicates that QVIP (3) can be reformulated as the following optimization problem:

$$Q: \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X. \end{array}$$

That is to say, the function f is a merit function of QVIP (3). Unfortunately, unlike the case of VIP, the regularized gap function f for QVIP is in general not differentiable, but only directionally differentiable when the function F is differentiable. Moreover, even if $\nabla F(x)$ is positive definite at any stationary point x of problem Q, it does not imply that x solves the original QVIP. In the next section, we will explore the possibility of avoiding these difficulties with the gap function approach for a class of GNEPs.

5 GNEP with Shared Equality Constraints and Barrier Method

Consider the GNEP with player ν's problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta^\nu(x^{-\nu}, x^\nu) \\ \text{subject to} \quad & \langle a_i^\nu, x^\nu \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle,\ i = 1, \ldots, m, \\ & h_j^\nu(x^\nu) \leq 0,\ j = 1, \ldots, l_\nu, \end{aligned} \tag{9}$$

where the shared constraints are given by equalities, while the individual constraints are given by inequalities. We denote this GNEP as P.

In the remainder of this paper, we make the following assumptions:

Assumption 5.1 $\theta^\nu(x^{-\nu}, \cdot)\colon \mathbb{R}^{n_\nu} \to \mathbb{R}$ is a twice continuously differentiable convex function for any $x^{-\nu} \in \mathbb{R}^{n_{-\nu}}$, and $h_j^\nu\colon \mathbb{R}^{n_\nu} \to \mathbb{R},\ j = 1, \ldots, l_\nu$ are twice continuously differentiable convex functions.

In this section, we reformulate problem P as the GNEP with the shared equality constraints only, by applying a barrier technique. We then propose an optimization approach using the regularized gap function for the QVIP derived from the latter GNEP. Note that the proposed barrier method incorporates each player's individual constraints in the objective function by using the barrier function. This is different from the common approach where the penalty technique is applied to the shared constraints [17, 19].

By adding the barrier term associated with the individual constraints to the objective function, problem (9) is approximated by the following problem:

$$\begin{aligned} \underset{x^\nu}{\text{minimize}} \quad & \theta^\nu(x^{-\nu}, x^\nu) - \rho \sum_{j=1}^{l_\nu} \log(-h_j^\nu(x^\nu)) \\ \text{subject to} \quad & \langle a_i^\nu, x^\nu \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle,\ i = 1, \ldots, m, \end{aligned} \tag{10}$$

where $\rho > 0$ is a parameter. Let P_ρ denote the GNEP with each player's problem given by problem (10). Since problem (10) is a convex programming problem, the GNEP P_ρ can be reformulated as the following QVIP:

$$\begin{aligned} \text{Find} \quad & x \in S_0(x) \cap \Sigma_0 \\ \text{such that} \quad & \langle F(x) - \rho E(x), y - x \rangle \geq 0, \quad \forall y \in S_0(x), \end{aligned} \tag{11}$$

where the function F is defined by (2), and $\Sigma_0 \subseteq \mathbb{R}^n$, $E\colon \Sigma_0 \to \mathbb{R}^n$ and $S_0\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are defined by

$$\begin{aligned} \Sigma_0 &\equiv \prod_{\nu=1}^{N} \left\{ x^\nu \;\middle|\; h_j^\nu(x^\nu) < 0,\ j = 1, \ldots, l_\nu \right\}, \\ E(x) &\equiv \left(\sum_{j=1}^{l_\nu} \frac{\nabla h_j^\nu(x^\nu)}{h_j^\nu(x^\nu)} \right)_{\nu=1}^{N}, \\ S_0(x) &\equiv \prod_{\nu=1}^{N} \left\{ y^\nu \;\middle|\; \langle a_i^\nu, y^\nu \rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle,\ i = 1, \ldots, m \right\}, \end{aligned} \tag{12}$$

respectively. Then, the regularized gap function for this problem is given by

$$f_\rho(x) = -\inf\left\{ \langle F(x) - \rho E(x), y - x\rangle + \frac{1}{2}\langle y - x, H(y-x)\rangle \;\middle|\; y \in S_0(x) \right\}. \tag{13}$$

Note that the function f_ρ is defined only on the open set Σ_0. Thus, by letting $f_\rho(x) \equiv +\infty \; \forall x \notin \Sigma_0$, the GNEP P_ρ is reformulated as the minimization problem

$$Q_\rho: \quad \begin{array}{ll} \text{minimize} & f_\rho(x) \\ \text{subject to} & x \in X_0, \end{array} \tag{14}$$

where the set $X_0 \subseteq \mathbb{R}^n$ is defined by

$$X_0 \equiv \left\{ x \;\middle|\; b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu\rangle = 0, \; i = 1, \ldots, m \right\}.$$

This fact is formally stated as follows.

Theorem 5.2 *For each $x \in X_0$, we have $f_\rho(x) \geq 0$. Moreover, x solves QVIP* (11) *if and only if $f_\rho(x) = 0$ and $x \in X_0$.*

Now we consider the differentiability of the function f_ρ. Let $y(x)$ denote the unique solution of the optimization problem on the right-hand side of (13).

Lemma 5.3 *If $x \in X_0$, then $y(x) \in X_0$.*

Proof. Since $y(x) \in S_0(x)$, for each $\nu = 1, \ldots, N$, $y^\nu(x)$ satisfies

$$\langle a_i^\nu, y^\nu(x)\rangle = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'}\rangle, \quad i = 1, \ldots, m. \tag{15}$$

Since $x \in X_0$ by the assumption , we have

$$b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu\rangle = 0, \quad i = 1, \ldots, m. \tag{16}$$

Hence, by (15) and (16), for each $\nu = 1, \ldots, N$, we obtain

$$\langle a_i^\nu, x^\nu\rangle = \langle a_i^\nu, y^\nu(x)\rangle, \quad i = 1, \ldots, m. \tag{17}$$

Therefore, by (16) and (17), we have

$$0 = b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu\rangle = b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, y^\nu(x)\rangle, \quad i = 1, \ldots, m,$$

which implies $y(x) \in X_0$. ■

The following theorem shows that the function f_ρ is directionally differentiable in general; moreover it is differentiable under suitable assumptions.

Theorem 5.4 *The function f_ρ defined by* (13) *is directionally differentiable at every $x \in \Sigma_0$ along any direction*

$d \in \mathbb{R}^n$, and the directional derivative is given by

$$f'_\rho(x; d) = \min_{\mu \in M(x)} \Bigg\{ \langle (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x), d \rangle - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu \left\langle (a_i^1, \ldots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \ldots, a_i^N)^\top, d \right\rangle \Bigg\},$$

where $M(x) \subseteq \mathbb{R}^{Nm}$ consists of all vectors $\mu \equiv \left((\mu_i^\nu)_{i=1}^m\right)_{\nu=1}^N \in \mathbb{R}^{Nm}$ satisfying

$$F(x) - \rho E(x) + H(y(x) - x) + \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu (0, \ldots, 0, a_i^\nu, 0, \ldots, 0)^\top = 0. \tag{18}$$

In particular, if $M(x)$ is a singleton, i.e.,

$$M(x) = \{\mu(x)\},$$

then f_ρ is differentiable at x and the gradient of f_ρ at x is given by

$$\nabla f_\rho(x) = (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x) - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu(x) (a_i^1, \ldots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \ldots, a_i^N)^\top.$$

Proof. The regularized gap function f_ρ is defined by substituting $F(x) - \rho E(x)$ for $F(x)$ in the definition (7) of the regularized gap function f. Since $F(x) - \rho E(x)$ is differentiable, the assertion of this theorem immediately follows from [7, Theorem 3]. ■

The next assumption ensures that the set $M(x)$ is a singleton for any x.

Assumption 5.5 For each $\nu = 1, \ldots, N$, the vectors $a_i^\nu,\ i = 1, \ldots, m$ are linearly independent.

Theorem 5.6 *Let Assumption 5.5 hold. Then the function f_ρ is differentiable at any point $x \in \Sigma_0$, and its gradient is given by*

$$\nabla f_\rho(x) = -(\nabla F(x) - \rho \nabla E(x))(y(x) - x) - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu(x) a_i,$$

where $a_i \equiv (a_i^1, \ldots, a_i^N)^\top \in \mathbb{R}^n$.

Proof. By Assumption 5.5, the vectors $a_i^\nu,\ i = 1, \ldots, m$ are linearly independent for each $\nu = 1, \ldots, N$. Therefore, the vectors $(0, \ldots, 0, a_i^\nu, 0, \ldots, 0)^\top \in \mathbb{R}^n,\ i = 1, \ldots, m,\ \nu = 1, \ldots, N$ are linearly independent, and $M(x)$ has only one element $\mu(x)$. Hence, by Theorem 5.4, the function f_ρ is differentiable at any point $x \in \Sigma_0$.

Moreover, by Theorem 5.4, the gradient of f_ρ at x is given by

$$\nabla f_\rho(x) = (F(x) - \rho E(x)) - (\nabla F(x) - \rho \nabla E(x) - H)(y(x) - x) - \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu(x) (a_i^1, \ldots, a_i^{\nu-1}, 0, a_i^{\nu+1}, \ldots, a_i^N)^\top. \tag{19}$$

The last term on the right-hand side of (19) is rewritten as

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)(a_i^1,\dots,a_i^{\nu-1},0,a_i^{\nu+1},\dots,a_i^N)^\top \\
&=\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)(a_i^1,\dots,a_i^{\nu-1},a_i^\nu,a_i^{\nu+1},\dots,a_i^N)^\top-\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)(0,\dots,0,a_i^\nu,0,\dots,0)^\top \\
&=\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)a_i+F(x)-\rho E(x)+H(y(x)-x), \qquad (20)
\end{aligned}
$$

where the last equality follows from (18). By using (20), the formula (19) can be rewritten as

$$\nabla f_\rho(x)=-(\nabla F(x)-\rho\nabla E(x))(y(x)-x)-\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)a_i.$$

■

The following theorem gives a condition under which any point that satisfies the first-order optimality condition for problem (14) is a solution of GNEP P_ρ.

Theorem 5.7 *Suppose Assumption 5.5 holds. Let $x\in X_0$ be a stationary point of problem* (14)*, and $\nabla F(x)-\rho\nabla E(x)$ be positive definite. Then the point x is a solution of QVIP* (11)*, that is to say, x is a solution of GNEP P_ρ.*

Proof. First, note that $x\in\Sigma_0$. By Theorem 5.4, the function f_ρ is differentiable at the point x under the given assumptions. Thus, when x is a stationary point of problem (14), by making use of the fact that the feasible set X_0 is an affine set, we have

$$\langle\nabla f_\rho(x), y-x\rangle=0,\quad \forall y\in X_0. \qquad (21)$$

Note that $\langle a_i,x\rangle-b_i=0$ holds by $x\in X_0$. Moreover, we have $y(x)\in X_0$ by Lemma 5.3, i.e.,

$$b_i-\langle a_i,y(x)\rangle=0,\quad i=1,\dots,m.$$

Hence, we have

$$\langle a_i, y(x)-x\rangle=0,\ i=1,\dots,m. \qquad (22)$$

Thus, it follows from Theorem 5.6 together with (21) and (22) that

$$
\begin{aligned}
0&=\langle\nabla f_\rho(x),y(x)-x\rangle\\
&=-\left\langle(\nabla F(x)-\rho\nabla E(x))(y(x)-x)+\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)a_i,\ y(x)-x\right\rangle\\
&=-\langle(\nabla F(x)-\rho\nabla E(x))(y(x)-x),y(x)-x\rangle-\sum_{i=1}^{m}\sum_{\nu=1}^{N}\mu_i^\nu(x)\langle a_i,y(x)-x\rangle\\
&=-\langle(\nabla F(x)-\rho\nabla E(x))(y(x)-x),y(x)-x\rangle. \qquad (23)
\end{aligned}
$$

Moreover, since $\nabla F(x)-\rho\nabla E(x)$ is positive definite by assumption, we must have $y(x)=x$ from (23). Then, the definition (13) of f_ρ yields $f_\rho(x)=0$, and x is a solution of QVIP (11) according to Theorem 5.2. ■

Corollary 5.8 *Suppose Assumption 5.5 holds and $\nabla F(x)$ is positive definite at any point $x \in X_0 \cap \Sigma_0$. Then, for any $\rho > 0$, a stationary point x of problem* (14) *is a solution of GNEP P_ρ.*

Proof. By direct calculation, it follows from the definition (12) of $E(x)$ that

$$\nabla E(x) = \mathrm{Diag}\left[\sum_{j=1}^{l_\nu}\left(\frac{\nabla^2 h_j^\nu(x^\nu)}{h_j^\nu(x^\nu)} - \frac{\nabla h_j^\nu(x^\nu)\nabla h_j^\nu(x^\nu)^\top}{h_j^\nu(x^\nu)^2}\right)\right]_{\nu=1}^N,$$

where $\mathrm{Diag}[B_\nu]_{\nu=1}^N$ denotes the block diagonal matrix whose block diagonal elements are $B_\nu,\ \nu = 1,\dots,N$. Notice that each $\nabla^2 h_j^\nu(x^\nu)$ is positive semidefinite since h_j^ν is convex. This implies that $\nabla E(x)$ is negative semidefinite for any $x \in \Sigma_0$, since $h_j^\nu(x^\nu) < 0$.

Hence, $\nabla F(x) - \rho\nabla E(x)$ is positive definite at any $x \in \Sigma_0$, whenever so is $\nabla F(x)$. Therefore, Theorem 5.7 ensures that, for any $\rho > 0$, a stationary point x of (14) is a solution of GNEP P_ρ. ■

6 Convergence of the Barrier Method

In the previous section, we have shown that, for every fixed $\rho > 0$, a solution of GNEP P_ρ can be obtained by solving the minimization problem (14). Here we present an algorithm for solving the GNEP P by solving the problems (14) sequentially by letting the parameter ρ tend to zero.

Algorithm 6.1 Let $\{\rho_k\} \subset \mathbb{R}$ be a decreasing sequence of positive scalars tending to 0. Generate a sequence of iterates $\{x_k\}$ as follows: For each k, find a stationary point x_k of the minimization problem

$$Q_{\rho_k}: \quad \begin{array}{ll} \text{minimize} & f_{\rho_k}(x) \\ \text{subject to} & x \in X_0. \end{array}$$

By imposing an appropriate condition, we show that the generated sequence $\{x_k\}$ converges to a solution of the GNEP P as $\rho_k \to 0$.

Assumption 6.2 $\nabla F(x)$ is positive definite at any point $x \in X_0 \cap \Sigma_0$.

Theorem 6.3 *Suppose that Assumption 6.2 holds. Let x_∞ be any accumulating point of the sequence $\{x_k\}$ generated by Algorithm 6.1. Assume that $\Sigma^\nu \equiv \{x^\nu \in \mathbb{R}^{n_\nu} \mid h^\nu(x^\nu) \le 0\}$ is a bounded set, and define the index sets $\gamma_\infty^\nu \equiv \{j \mid h_j^\nu(x_\infty^\nu) = 0\} \subseteq \{1,2,\dots,l_\nu\}$ for each $\nu = 1,\dots,N$. Suppose the following Mangasarian-Fromovotz constraint qualification (MFCQ) holds:*

$$\left.\begin{array}{c} \displaystyle\sum_{j\in\gamma_\infty^\nu} \lambda_j^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^m \mu_i^\nu a_i^\nu = 0 \\ \lambda_j^\nu \ge 0, \quad j \in \gamma_\infty^\nu \\ \mu_i^\nu \in \mathbb{R}, \quad i = 1,\dots,m \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \lambda_j^\nu = 0, & j \in \gamma_\infty^\nu, \\ \mu_i^\nu = 0, & i = 1,\dots,m. \end{array}\right.$$

Then x_∞ is a solution of the GNEP P.

Proof. From Corollary 5.8, x_k is a solution of the following QVIP:

$$\begin{array}{ll} \text{Find} & x \in S_0(x) \\ \text{such that} & \langle F(x) - \rho_k E(x), y - x \rangle \geq 0, \quad \forall y \in S_0(x). \end{array} \tag{24}$$

Let $\{x_k\}_{k\in\kappa}$ be a convergent subsequence whose limit is x_∞. By Assumption 5.5, the linear independence constraint qualification holds for problem P_{ρ_k}. Thus, it follows from the Karush-Kuhn-Tucker (KKT) condition for problem (24) that for any k there exist Lagrange multipliers $\left(\mu^\nu_{k,i}\right)_{i=1}^m$ such that

$$F^\nu(x_k) - \rho_k \sum_{j=1}^{l_\nu} \frac{1}{h^\nu_j(x^\nu_k)} \nabla h^\nu_j(x^\nu_k) + \sum_{i=1}^m \mu^\nu_{k,i} a^\nu_i = 0, \qquad \nu = 1, \ldots, N. \tag{25}$$

Put $\lambda^\nu_{k,j} = -\rho_k / h^\nu_j(x^\nu_k) \geq 0,\ j = 1, \ldots, n_l$, and define the vectors

$$\phi^\nu_k = \begin{pmatrix} \lambda^\nu_k \\ \mu^\nu_k \end{pmatrix},$$

where $\lambda^\nu_k \equiv \left(\lambda^\nu_{k,j}\right)_{j=1}^{l_\nu}$ and $\mu^\nu_k \equiv \left(\mu^\nu_{k,i}\right)_{i=1}^m$. Let us show that the sequence $\{\phi^\nu_k\}$ is bounded for each ν. In fact, if $\{\phi^\nu_k\}$ is unbounded, then there exists a subsequence $\{\phi^\nu_k\}_{k\in\kappa'}$ such that

$$\lim_{\kappa' \ni k \to \infty} \|\phi^\nu_k\| = \infty.$$

By dividing both sides of (25) by $\|\phi^\nu_k\|$, we have

$$\frac{1}{\|\phi^\nu_k\|} F^\nu(x_k) + \sum_{j=1}^{l_\nu} \frac{\lambda^\nu_{k,j}}{\|\phi^\nu_k\|} \nabla h^\nu_j(x^\nu_k) + \sum_{i=1}^m \frac{\mu^\nu_{k,i}}{\|\phi^\nu_k\|} a^\nu_i = 0, \qquad \nu = 1, \ldots, N.$$

Since $\{\lambda^\nu_{k,j}/\|\phi^\nu_k\|\}$ and $\{\mu^\nu_{k,i}/\|\phi^\nu_k\|\}$ are bounded, these sequences have accumulation points $\bar\lambda^\nu_j$ and $\bar\mu^\nu_i$, respectively. Therefore, we have

$$\sum_{j=1}^{l_\nu} \bar\lambda^\nu_j \nabla h^\nu_j(x^\nu_\infty) + \sum_{i=1}^m \bar\mu^\nu_i a^\nu_i = 0, \qquad \nu = 1, \ldots, N. \tag{26}$$

Now notice that $\bar\lambda^\nu_j \geq 0$ for all j. In particular, since

$$\limsup_{k\to\infty} h^\nu_j(x^\nu_k) < 0, \qquad \forall j \notin \gamma^\nu_\infty,$$

we have $\lambda^\nu_{k,j} = 0$, for all $k \in \kappa'$ sufficient large, implying $\bar\lambda^\nu_j = 0$ for all $j \notin \gamma^\nu_\infty$. Thus, it follows from (26) that

$$\sum_{j\in\gamma^\nu_\infty} \bar\lambda^\nu_j \nabla h^\nu_j(x^\nu_\infty) + \sum_{i=1}^m \bar\mu^\nu_i a^\nu_i = 0, \qquad \nu = 1, \ldots, N.$$

However, this along with the fact that

$$\begin{array}{l} \bar\lambda^\nu_j \geq 0, \quad j \in \gamma^\nu_\infty, \\ \left\| \begin{pmatrix} \bar\lambda^\nu \\ \bar\mu^\nu \end{pmatrix} \right\| = 1 \end{array}$$

contradicts the assumed MFCQ. This implies that $\{\phi_k^\nu\}$ is bounded, and that $\{\lambda_k^\nu\}$ and $\{\mu_k^\nu\}$ have accumulation points λ_∞^ν and μ_∞^ν, respectively. Therefore, x_∞ satisfies

$$\left.\begin{array}{l} F^\nu(x_\infty) + \sum_{j=1}^{l_\nu} \lambda_{\infty,j}^\nu \nabla h_j^\nu(x_\infty^\nu) + \sum_{i=1}^{m} \mu_i^\nu a_i^\nu = 0 \\ h_j^\nu(x_\infty^\nu) \le 0, \quad \lambda_{\infty,j}^\nu \ge 0, \quad \lambda_{\infty,j}^\nu h_j^\nu(x_\infty^\nu) = 0, \quad j = 1, \ldots, l_\nu \\ \langle a_i^\nu, x_\infty^\nu \rangle + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x_\infty^{\nu'} \rangle - b_i = 0, \quad i = 1, \ldots, m \end{array}\right. \quad \nu = 1, \ldots, N.$$

This is nothing but the KKT condition for problem (3) with $S(x)$ defined by the constraints in problems (9). Consequently, x_∞ is a solution of the GNEP P. ■

7 Extension to GNEP with Shared Inequality Constraints

The GNEP considered in the previous section assumed that each player's shared constraints are defined by equalities only. In practice, however, the shared constraints often contain inequalities. In this section, we discuss the case of shared inequality constraints and present an approach that relies on the transformation to the equality constraints by means of slack variables.

Suppose that, for each ν, player ν's problem is given as

$$\begin{array}{ll} \underset{x^\nu}{\text{minimize}} & \theta^\nu(x^{-\nu}, x^\nu) \\ \text{subject to} & \langle a_i^\nu, x^\nu \rangle \le b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle, \ i = 1, \ldots, m, \\ & h_j^\nu(x^\nu) \le 0, \ j = 1, \ldots, l_\nu. \end{array} \tag{27}$$

Denote this GNEP as $\hat{P}$. Introducing slack variables $s^\nu = (s_1^\nu, \ldots, s_m^\nu)$ as supplementary variables for each player ν, the problem (27) is rewritten as follows:

$$\begin{array}{ll} \underset{(x^\nu, s^\nu)}{\text{minimize}} & \theta^\nu(x^{-\nu}, x^\nu) \\ \text{subject to} & \langle a_i^\nu, x^\nu \rangle + s_i^\nu = b_i - \sum_{\nu' \neq \nu} \left(\langle a_i^{\nu'}, x^{\nu'} \rangle + s_i^{\nu'} \right), \ i = 1, \ldots, m, \\ & h_j^\nu(x^\nu) \le 0, \ j = 1, \ldots, l_\nu, \\ & s_i^\nu \ge 0, \ i = 1, \ldots, m. \end{array} \tag{28}$$

Denote this GNEP as $\check{P}$. The vector consisting of all slack variables is denoted by $s \equiv (s^\nu)_{\nu=1}^N \in \mathbb{R}^{Nm}$. The next result shows that, under some conditions, a solution of the GNEP $\check{P}$ is also a solution of the GNEP $\hat{P}$.

Theorem 7.1 *Let (x, s) be a solution of the GNEP $\check{P}$. If the relation*

$$s_i^\nu = 0 \text{ for some } \nu \quad \Longrightarrow \quad s_i^\nu = 0 \text{ for all } \nu \tag{29}$$

holds for all i, then x is a solution of the GNEP $\hat{P}$.

Proof. Define the Lagrange function for the problem (28) as

$$\mathscr{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) = \theta^\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m} \mu_i^\nu \left(\langle a_i^\nu, x^\nu \rangle + s_i^\nu + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle + \sum_{\nu' \neq \nu} s_i^{\nu'} - b_i \right)$$
$$+ \sum_{j=1}^{l_\nu} \lambda_j^\nu h_j^\nu(x^\nu) - \sum_{i=1}^{m} \eta_i^\nu s_i^\nu .$$

A solution (x, s) of GNEP $\check{P}$ satisfies the following KKT conditions for all ν:

$$\nabla_{x^\nu} \mathscr{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) = \nabla_{x^\nu} \theta^\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m} \mu_i^\nu a_i^\nu + \sum_{j=1}^{l_\nu} \lambda_j^\nu \nabla h_j^\nu(x^\nu) = 0, \tag{30a}$$

$$\nabla_{s^\nu} \mathscr{L}^\nu(x^\nu, s^\nu, \lambda^\nu, \mu^\nu, \eta^\nu) = \mu^\nu - \eta^\nu = 0, \tag{30b}$$

$$\langle a_i^\nu, x^\nu \rangle + s_i^\nu + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle + \sum_{\nu' \neq \nu} s_i^{\nu'} - b_i = 0, \; i = 1, \ldots, m, \tag{30c}$$

$$\lambda_j^\nu \geq 0, \quad \lambda_j^\nu h_j^\nu(x^\nu) = 0, \quad h_j^\nu(x^\nu) \leq 0, \; j = 1, \ldots, l_\nu, \tag{30d}$$

$$\eta_i^\nu \geq 0, \quad \eta_i^\nu s_i^\nu = 0, \quad s_i^\nu \geq 0, \; i = 1, \ldots, m. \tag{30e}$$

By the relation (29), either of the following statements holds for each i:

(i) $s_i^\nu = 0$ for all ν,
(ii) $s_i^\nu > 0$ for all ν.

(i) Suppose $s_i^\nu = 0$ for all ν. By (30e), we have

$$\eta_i^\nu \geq 0.$$

Then it follows from (30b) and (30c), that

$$\mu_i^\nu \geq 0$$

and

$$\langle a_i^\nu, x^\nu \rangle + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle - b_i = 0$$

for all ν.

(ii) Suppose $s_i^\nu > 0$ for all ν. Then, by (30e), we have

$$\eta_i^\nu = 0.$$

Therefore, from (30b) and (30c), we obtain

$$\mu_i^\nu = 0$$

and

$$\langle a_i^\nu, x^\nu \rangle + \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle - b_i < 0$$

for all ν.

Hence, the following complementarity conditions hold for all i:

$$\mu_i^\nu \geq 0, \quad \mu_i^\nu \left(\sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \right) = 0, \quad \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \leq 0. \tag{31}$$

Combining (30a), (30d) and (31), we have for all ν

$$\begin{aligned}
&\nabla_{x^\nu} \theta^\nu(x^{-\nu}, x^\nu) + \sum_{i=1}^{m} \mu_i^\nu a_i^\nu + \sum_{j=1}^{l_\nu} \lambda_j^\nu \nabla h_j^\nu(x^\nu) = 0, \\
&\mu_i^\nu \geq 0, \quad \mu_i^\nu \left(\sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \right) = 0, \quad \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - b_i \leq 0, \ i = 1, \ldots, m, \\
&\lambda_j^\nu \geq 0, \quad \lambda_j^\nu h_j^\nu(x^\nu) = 0, \quad h_j^\nu(x^\nu) \leq 0, \ j = 1, \ldots, l_\nu.
\end{aligned}$$

This implies that for each ν, x^ν satisfies the KKT conditions for problem (27) with given $x^{-\nu}$. Thus, $x \equiv (x^\nu)_{\nu=1}^N$ is a solution of GNEP $\hat{P}$. ■

By adding the barrier term associated with the individual constraints to the objective function, player ν's problem (28) is approximated by the following problem:

$$\begin{aligned}
\underset{(x^\nu, s^\nu)}{\text{minimize}} \quad & \theta^\nu(x^{-\nu}, x^\nu) - \rho \left(\sum_{j=1}^{l_\nu} \log(-h_j^\nu(x^\nu)) + \sum_{i=1}^{m} \log s_i^\nu \right) \\
\text{subject to} \quad & \langle a_i^\nu, x^\nu \rangle + s_i^\nu = b_i - \sum_{\nu' \neq \nu} \left(\langle a_i^{\nu'}, x^{\nu'} \rangle + s_i^{\nu'} \right), \ i = 1, \ldots, m.
\end{aligned} \tag{32}$$

Denote this GNEP as $\check{P}_\rho$. Let the vector-valued function $\check{F} \colon \mathbb{R}^{n+Nm} \to \mathbb{R}^{n+Nm}$ be defined by

$$\check{F}(x, s) \equiv \begin{pmatrix} F(x) \\ 0 \end{pmatrix} \in \mathbb{R}^{n+Nm},$$

where $F(x)$ is given by (2). Define the vector-valued function $G \colon \mathbb{R}^{Nm} \to \mathbb{R}^{Nm}$ as follows:

$$G(s) = \left(\left(1/s_i^\nu \right)_{i=1}^m \right)_{\nu=1}^N.$$

Moreover, let the vector-valued function $\check{E} \colon \mathbb{R}^{n+Nm} \to \mathbb{R}^{n+Nm}$ be defined by

$$\check{E}(x, s) \equiv \begin{pmatrix} E(x) \\ G(s) \end{pmatrix} \in \mathbb{R}^{n+Nm},$$

where $E(x)$ is given by (12).

Since problem (32) is a convex programming problem, the GNEP $\check{P}_\rho$ can be reformulated as the following QVIP:

$$\begin{aligned}
\text{Find} \quad & (x, s) \in \check{S}_0(x, s) \cap \check{\Sigma}_0 \\
\text{such that} \quad & \left\langle \check{F}(x, s) - \rho \check{E}(x, s), (y, t) - (x, s) \right\rangle \geq 0, \quad \forall (y, t) \in \check{S}_0(x, s),
\end{aligned} \tag{33}$$

where $\check{\Sigma}_0 \subseteq \mathbb{R}^{n+Nm}$ and $\check{S}_0 \colon \mathbb{R}^{n+Nm} \rightrightarrows \mathbb{R}^{n+Nm}$ are defined by

$$\check{\Sigma}_0 \equiv \prod_{\nu=1}^{N} \left\{ (x^\nu, s^\nu) \,\middle|\, h_j^\nu(x^\nu) < 0,\ j = 1, \dots, l_\nu, s_i^\nu > 0,\ i = 1, \dots, m \right\},$$

$$\check{S}_0(x, s) \equiv \prod_{\nu=1}^{N} \left\{ (y^\nu, t^\nu) \,\middle|\, \langle a_i^\nu, y^\nu \rangle + t_i^\nu = b_i - \sum_{\nu' \neq \nu} \langle a_i^{\nu'}, x^{\nu'} \rangle - \sum_{\nu' \neq \nu} s_i^\nu,\ i = 1, \dots, m \right\},$$

respectively. Then, the regularized gap function for this problem is given by

$$\check{f}_\rho(x, s) = -\inf \left\{ \langle \check{F}(x, s) - \rho \check{E}(x, s), (y, t) - (x, s) \rangle + \frac{1}{2} \langle (y, t) - (x, s), \check{H}((y, t) - (x, s)) \rangle \,\middle|\, (y, t) \in \check{S}_0(x, s) \right\},$$

where $\check{H}$ is a simmetric positive definite matrix. Note that the function $\check{f}_\rho$ is defined only on the open set $\check{\Sigma}_0$. Thus, by letting $\check{f}_\rho(x, s) \equiv +\infty\ \forall (x, s) \notin \check{\Sigma}_0$, the GNEP $\check{P}_\rho$ is reformulated as the minimization problem

$$\check{Q}_\rho : \quad \begin{array}{ll} \text{minimize} & \check{f}_\rho(x, s) \\ \text{subject to} & (x, s) \in \check{X}_0, \end{array}$$

where the set $\check{X}_0 \subseteq \mathbb{R}^{n+Nm}$ is defined by

$$\check{X}_0 \equiv \left\{ (x, s) \,\middle|\, b_i - \sum_{\nu=1}^{N} \langle a_i^\nu, x^\nu \rangle - \sum_{\nu=1}^{N} s_i^\nu = 0,\ i = 1, \dots, m \right\}.$$

This fact implies that for each $\rho > 0$, the set of optimum solutions of the problem $\check{Q}_\rho$ equals the set of GNEs of $\check{P}_\rho$. We will show that any stationary point for problem $\check{Q}_\rho$ is a solution of GNEP $\check{P}_\rho$ under some conditions.

Lemma 7.2 *Suppose $x \in \Sigma_0$ and $s > 0$. Let $\nabla F(x) - \rho \nabla E(x)$ be positive definite for $\rho > 0$. Then, $\nabla \check{F}(x, s) - \rho \nabla \check{E}(x, s)$ is also positive definite.*

Proof. By direct calculation, we have

$$\nabla G(s) = \operatorname{diag} \left(\left(-1/(s_i^\nu)^2 \right)_{i=1}^{m} \right)_{\nu=1}^{N} \prec 0$$

for $s > 0$. Thus, by the assumption, for all $\xi \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{Nm}$ such that $(\xi, \sigma) \neq (0, 0)$, it follows that

$$\begin{aligned} (\xi, \sigma)^\top \left(\nabla \check{F}(x, s) - \rho \nabla \check{E}(x, s) \right) (\xi, \sigma) &= (\xi, \sigma)^\top \left(\begin{pmatrix} \nabla F(x) & 0 \\ 0 & 0 \end{pmatrix} - \rho \begin{pmatrix} \nabla E(x) & 0 \\ 0 & \nabla G(s) \end{pmatrix} \right) (\xi, \sigma) \\ &= \xi^\top (\nabla F(x) - \rho \nabla E(x)) \xi + \sigma^\top (-\rho \nabla G(s)) \sigma \\ &> 0 \end{aligned}$$

for $\rho > 0$. This concludes the proof. ■

Theorem 7.3 *Suppose Assumption 5.5 holds. Let $(x, s) \in \check{X}_0$ be a stationary point of problem $\check{Q}_\rho$, and $\nabla F(x) - \rho \nabla E(x)$ be positive definite for $\rho > 0$. Then the point (x, s) is a solution of QVIP* (33)*, that is to say, (x, s) is a solution of GNEP $\check{P}_\rho$.*

Proof. By Assumption 5.5, the vectors $a_i^\nu,\ i = 1, \dots, m$ are linearly independent for each $\nu = 1, \dots, N$. Therefore, the vectors $((0, \dots, 0, a_i^\nu, 0, \dots, 0)^\top, e_i^\nu) \in \mathbb{R}^{n+Nm},\ i = 1, \dots, m,\ \nu = 1, \dots, N$ are linearly independent, where $e_i^\nu \in \mathbb{R}^{Nm}$ denotes the unit vector whose element corresponding to s_i^ν is one and the others are zero. Thus, the following set is a singleton:

$$\check{M}(x,s) \equiv \left\{ \mu \;\middle|\; \check{F}(x,s) - \rho \check{E}(x,s) + H((y(x,s), t(x,s)) - (x,s)) + \sum_{i=1}^{m} \sum_{\nu=1}^{N} \mu_i^\nu ((0,\ldots,0,a_i^\nu,0,\ldots,0)^\top, e_i^\nu) = 0 \right\}$$

That is to say, $\check{M}(x,s)$ has only one element $\mu(x,s)$. Hence, by Theorem 5.4, the function $\check{f}_\rho$ is differentiable at any point $(x,s) \in \check{\Sigma}_0$.

Since $\nabla F(x) - \rho \nabla E(x)$ is positive denfinite, we have that $\nabla \check{F}(x,s) - \rho \nabla \check{E}(x,s)$ is positive definite by Lemma 7.2. Therefore, Theorem 5.7 ensures that any stationary point (x,s) of problem $\check{Q}_\rho$ solves the QVIP (33), which means (x,s) is a solution of GNEP $\check{P}_\rho$. ■

8 Numerical Results

In this section, we report our numerical experience for the examples. We implemented the Algorithm 6.1 using the Sequential Quadratic Programming (SQP) method on MATLAB®7.4.0.336 (R2007a).

Example 8.1 We consider Harker's example [10]. In this game, there are two players who solve the following problems:

$$P_1(x^2): \quad \begin{array}{ll} \underset{x^1}{\text{minimize}} & (x^1)^2 + (8/3)x^1x^2 - 34x^1 \\ \text{subject to} & 0 \le x^1 \le 10, \\ & x^1 + x^2 \le 15. \end{array} \qquad P_2(x^1): \quad \begin{array}{ll} \underset{x^2}{\text{minimize}} & (x^2)^2 + (5/4)x^1x^2 - 24.25x^2 \\ \text{subject to} & 0 \le x^2 \le 10, \\ & x^1 + x^2 \le 15. \end{array}$$

The solution set of this GNEP is

$$\left\{ \begin{pmatrix} 5 \\ 9 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} t \\ 15 - t \end{pmatrix} \;\middle|\; 9 \le t \le 10 \right\}.$$

We found the GNE $(5, 9)^\top$ as Figure 1.

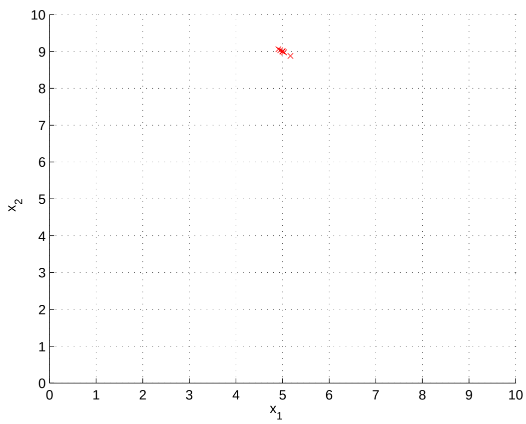

Figure 1: GNEs found by the proposed algorithm.

9 Conclusion

We have proposed a gap function approach to the GNEP in which the shared constraints are given by equalities, while the individual constraints are given by inequalities. This approach uses a barrier technique to transform the player's problems into problems involving the shared equality constraints only. Further, we have shown that the approach can be applied to the GNEP with the shared inequality constraints by using the slack variables under some assumptions. Finally, we have implemented the proposed algorithm on some examples and have confirmed its numerical behavior.

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