Semidefinite programming reformulation for a class of robust optimization problems and its application to robust Nash equilibrium problems

Guidance

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Abstract

In a real situation, optimization problems often involve uncertain parameters. Robust optimization is one of distribution-free methodologies based on worst-case analyses for handling such problems. In the model, we first assume that the uncertain parameters belong to some *uncertainty sets*. Then we deal with the *robust counterpart* associated with the uncertain optimization problem. The robust optimization problem is in its original form a semi-infinite program. Under some assumptions, it can be reformulated as an efficiently solvable problem, such as a semidefinite program (SDP) or a second-order cone program (SOCP). During the last decade, not only has robust optimization made significant progress in theory, but it has been applied to a large number of problems in various fields. Game theory is one of such fields. For non-cooperative games with uncertain parameters, several researchers have proposed a model in which each player makes a decision according to the robust optimization policy. The resulting equilibrium is called a robust Nash equilibrium, and the problem of finding such an equilibrium is called the robust Nash equilibrium problem. It is known that the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem under certain assumptions.

In this paper, we focus on a class of uncertain linear programs. We reformulate the robust counterpart as an SDP and show that those problems are equivalent under the spherical uncertainty assumption. In the reformulation, the strong duality for nonconvex quadratic programs plays a significant role. Also, by using the same technique, we reformulate the robust counterpart of an uncertain SOCP as an SDP under some assumptions. Furthermore, we apply this idea to the robust Nash equilibrium problem. Under mild assumptions, we show that each player's optimization problem can be rewritten as an SDP and the robust Nash equilibrium problem reduces to a semidefinite complementarity problem (SDCP). We finally give some numerical results to show that those SDP and SDCP are efficiently solvable.

Contents

1 Introduction

In constructing a mathematical model from a real-world problem, we cannot always determine the objective function or the constraint functions precisely. For example, when parameters in the functions are obtained in a statistical or simulative manner, they usually involve uncertainty (e.g. statistical error, etc.) to some extent. To deal with such situations, we need to incorporate the uncertain data in a mathematical model.

Generally, the mathematical programming problem with uncertain data is expressed as follows:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x,u) \\ \text{subject to} & f_i(x,u) \in K_i, \quad (i = 1, \ldots, m) \end{array} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the decision variable, $u \in \mathbb{R}^d$ is the uncertain data, $f_0 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $f_i : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{k_i}$ $(i = 1, \ldots, m)$ are given functions, and $K_i \subseteq \mathbb{R}^{k_i}$ $(i = 1, \ldots, m)$ are given nonempty sets. Since problem (1.1) cannot be defined uniquely due to u, it is difficult to handle in a straightforward manner.

Robust optimization [13] is one of distribution-free methodologies for handling the mathematical programming problem with uncertain data. In robust optimization, the uncertain data are assumed to belong to some set $\mathcal{U} \subseteq \mathbb{R}^d$, and then, the objective function is minimized (or maximized) with taking the worst possible case into consideration. That is, the following *robust counterpart* is solved instead of the original problem (1.1):

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \sup_{u \in \mathcal{U}} f_0(x,u) \\ \text{subject to} & f_i(x,u) \in K_i, \quad (i = 1, \ldots, m), \quad \forall u \in \mathcal{U}. \end{array} \tag{1.2}$$

Recently, robust optimization has been studied by many researchers. Ben-Tal and Nemirovski [9, 10, 12], Ben-Tal, Nemirovski and Roos [14], and El Ghaoui, Oustry and Lebret [21] showed that certain classes of robust optimization problems can be reformulated as efficiently solvable problems such as a semidefinite program (SDP) [36] or a second-order cone program (SOCP) [3] under the assumptions that uncertainty set is ellipsoidal and functions f_i $(i = 0, 1, \ldots, m)$ in the problem (1.2) are expressed as

$$f_i(x,u) = g_i(x) + F_i(x)u$$

with $g_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ and $F_i : \mathbb{R}^n \to \mathbb{R}^{k_i \times d}$. El Ghaoui and Lebret [19] showed that the robust least-squares problem can be reformulated as an SOCP. Bertsimas and Sim [16] gave another robust formulation and some properties of the solution. Also, the robust optimization techniques have been applied to many practical problems such as portfolio selection [7, 20, 23, 29, 30, 39], classification problem [38], structural design [8] and inventory management problem [1, 17].

On the other hand, in game theory, there have been a large number of studies on games with uncertain data. Among them, the new concept of *robust Nash equilibrium* attracts attention recently.

Hayashi, Yamashita and Fukushima [26], and Aghassi and Bertsimas [2]*[1] have proposed the model in which each player makes a decision according to the idea of robust optimization. Aghassi et al. [2] considered the robust Nash equilibrium for N-person games in which each player solves a linear programming (LP) problem. Moreover, they proposed a method for solving the robust Nash equilibrium problem with convex polyhedral uncertainty sets. Hayashi et al. [26] defined the concept of robust Nash equilibria for bimatrix games. Under the assumption that uncertainty sets are expressed by means of the Euclidean or the Frobenius norm, they showed that each player's problem reduces to an SOCP and the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) [22, 25]. In addition, Hayashi et al. [26] studied robust Nash equilibrium problems in which the uncertainty is contained in both opponents' strategies and each player's cost parameters, whereas Aghassi et al. [2] studied only the latter case. More recently, Nishimura, Hayashi and Fukushima [33] extended the definition of robust Nash equilibria in [2] and [26] to the N-person non-cooperative games with nonlinear cost functions. In particular, they showed existence of robust Nash equilibria under the milder assumptions and gave some sufficient conditions for uniqueness of the robust Nash equilibrium. In addition, they reformulated certain classes of robust Nash equilibrium problems to SOCCPs. However, Hayashi et al. [26] and Nishimura et al. [33] have only dealt with the case where the uncertainty is contained in either opponents' strategies or each player's cost parameters, in reformulating the robust Nash equilibrium problem as an SOCCP.

In this paper, we first focus on a special class of linear programs (LPs) with uncertain data. To such a problem, we apply the strong duality in nonconvex quadratic optimization problems with two quadratic constrains studied by Beck and Eldar [5], and reformulate its robust counterpart as an SDP. Especially, when the uncertainty sets are spherical, we further show that those two problems are equivalent. Also, by using the same technique, we reformulate the robust counterpart of SOCP with uncertain data as an SDP. In this reformulation, we emphasize that the uncertainty set is different from what was considered by Ben-Tal et al. [14]. We apply these ideas to game theory, too. Particularly, we show that the robust Nash equilibrium problem in which uncertainty is contained in both opponents' strategies and each player's cost parameters can be reduced to a semidefinite complementarity problem (SDCP) [18, 37]. Finally, we give some numerical results to see that those SDP and SDCP are efficiently solvable.

This paper is organized as follows. In Section 2, we review the strong duality in nonconvex quadratic optimization problems with two quadratic constraints, which plays a key role in the SDP reformulation of the robust counterpart. In Section 3, we reformulate the robust counterpart of some LP with uncertain data as an SDP. In Section 4, we reformulate the robust counterpart of SOCP with uncertain data as an SDP. In Section 5, we first formulate the robust Nash equilibrium problem, and show that it reduces to an SDCP under appropriate assumptions. In Section 6, we give some numerical

*[1] In [2] a robust Nash equilibrium is called a robust-optimization equilibrium.

results to show the validity of our reformulation and the behavior of obtained solutions.

Throughout the paper, we use the following notations. For a set X, $\mathcal{P}(X)$ denotes the set consisting of all subsets of X. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0\ (i = 1, \ldots, n)\}$. $\mathcal{S}^n$ denotes the set of $n \times n$ real symmetric matrices. $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \mathbb{R}^{m\times n}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^m \sum_{j=1}^n (M_{ij})^2)^{1/2}$, $\|M\|_2$ is the ℓ_2-norm defined by $\|M\|_2 := \max_{x\neq 0} \|Mx\|/\|x\|$, and $\ker M$ denotes the kernel of matrix M, i.e., $\ker M := \{x \in \mathbb{R}^n \mid Mx = 0\}$. $B(x, r)$ denotes the closed sphere with center x and radius r, i.e., $B(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$. For a problem (P), val(P) denotes the optimal value.

2 Strong duality in nonconvex quadratic optimization with two quadratic constraints

In this section, we study the duality theory in nonconvex quadratic programming problems with two quadratic constrains. This concept plays a significant role in reformulating the robust optimization problem as an SDP. Especially, we give sufficient conditions, shown by Beck and Eldar [5], under which there exists no duality gap.

We consider the following optimization problem:

$$\text{(QP)} \quad \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \geq 0, \quad f_2(x) \geq 0, \end{array} \tag{2.1}$$

where f_j $(j = 0, 1, 2)$ are defined by $f_j(x) := x^\top A_j x + 2{b_j}^\top x + c_j$ with symmetric matrices $A_j \in \mathbb{R}^{n\times n}$, $b_j \in \mathbb{R}^n$, and $c_j \in \mathbb{R}$.

We first consider the Lagrangian dual problem to QP (2.1). The Lagrangian function L for QP (2.1) is defined by

$$L(x, \alpha, \beta) = \begin{cases} x^\top A_0 x + 2b_0^\top x + c_0 - \alpha(x^\top A_1 x + 2b_1^\top x + c_1) - \beta(x^\top A_2 x + 2b_2^\top x + c_2), & \alpha, \beta \geq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

with Lagrange multipliers α and β. By introducing an auxiliary variable $\lambda \in \mathbb{R} \cup \{-\infty\}$, we have

$$\begin{aligned}
&\sup_{\alpha,\beta\geq 0} \inf_{x\in\mathbb{R}^n} L(x, \alpha, \beta) \\
&= \sup_{\alpha,\beta\geq 0,\lambda} \left\{\lambda \mid L(x, \alpha, \beta) \geq \lambda,\ \forall x \in \mathbb{R}^n\right\} \\
&= \sup_{\alpha,\beta\geq 0,\lambda} \left\{\lambda \;\middle|\; \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \left(\begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} - \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} - \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0,\ \forall x \in \mathbb{R}^n \right\} \\
&= \sup_{\alpha,\beta\geq 0,\lambda} \left\{\lambda \;\middle|\; \begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} - \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} - \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \succeq 0 \right\}.
\end{aligned}$$

Hence, the Lagrangian dual problem to (QP) is written as

$$
\text{(D)} \qquad \begin{array}{ll} \underset{\alpha,\beta,\lambda}{\text{maximize}} & \lambda \\ \text{subject to} & \begin{bmatrix} A_0 & b_0 \\ b_0^\top & c_0 - \lambda \end{bmatrix} \succeq \alpha \begin{bmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{bmatrix} + \beta \begin{bmatrix} A_2 & b_2 \\ b_2^\top & c_2 \end{bmatrix} \\ & \alpha \geq 0, \quad \beta \geq 0, \quad \lambda \in \mathbb{R}. \end{array} \tag{2.2}
$$

Since (D) is an SDP, its dual problem is

$$
\text{(SDR)} \qquad \begin{array}{ll} \text{minimize} & \operatorname{tr}(M_0 X) \\ \text{subject to} & \operatorname{tr}(M_1 X) \geq 0 \\ & \operatorname{tr}(M_2 X) \geq 0 \\ & X_{n+1,n+1} = 1, \\ & X \succeq 0, \end{array} \tag{2.3}
$$

where

$$
M_j = \begin{bmatrix} A_j & b_j \\ b_j^\top & c_j \end{bmatrix} \quad (j = 0, 1, 2).
$$

Now let $\chi(x)$ be a rank-one semidefinite symmetric matrix defined by $\chi(x) := \binom{x}{1}\binom{x}{1}^\top$. Then we have $f_j(x) = \binom{x}{1}^\top M_j \binom{x}{1} = \operatorname{tr}(M_j \chi(x))$ for $j = 0, 1, 2$. Thus problem (2.1) is rewritten as

$$
\begin{array}{ll} \text{minimize} & \operatorname{tr}(M_0 \chi(x)) \\ \text{subject to} & \operatorname{tr}(M_1 \chi(x)) \geq 0 \\ & \operatorname{tr}(M_2 \chi(x)) \geq 0. \end{array} \tag{2.4}
$$

Actually, problem (2.3) can be seen as a relaxation of problem (2.4) since the rank-one condition on $\chi(x)$ is removed. In other words, problem (2.3) is the so-called semidefinite relaxation [11] of (2.4). From the above argument, we have val (QP) $\leq$ val (SDR). Hence, by using the weak duality theorem, we have

$$
\text{val (QP)} \leq \text{val (SDR)} \leq \text{val (D)}.
$$

Finally, we study the strong duality. Beck and Eldar [5] considered a nonconvex quadratic optimization problem in the complex space and its dual problem, and showed that they have zero duality gap under strict feasibility and boundedness assumptions. Furthermore, they extended the idea to the nonconvex quadratic optimization problem in the real space, and provided sufficient conditions for zero duality gap among (QP), (D) and (SDR).

Theorem 2.1. [5, Theorem 3.5] *Suppose that both (QP) and (D) are strictly feasible and that*

$$
\exists \hat{\alpha},\ \hat{\beta} \in \mathbb{R} \quad \textit{such that} \quad \hat{\alpha} A_1 + \hat{\beta} A_2 \succ 0.
$$

Let $(\bar{\lambda}, \bar{\alpha}, \bar{\beta})$ *be an optimal solution of the dual problem* (D). *If*

$$
\dim(\ker(A_0 - \bar{\alpha} A_1 - \bar{\beta} A_2)) \neq 1,
$$

then val (QP) $=$ val (D) $=$ val (SDR).

3 SDP reformulation for a class of robust linear programming problems

In this section, we focus on the following uncertain LP:

$$
\begin{array}{ll}
\underset{x}{\text{minimize}} & (\hat{\gamma}^{0})^\top(\hat{A}^0 x + \hat{b}^0) \\
\text{subject to} & (\hat{\gamma}^{i})^\top(\hat{A}^i x + \hat{b}^i) \le 0 \quad (i = 1, \dots, K) \\
& x \in \Omega,
\end{array}
\tag{3.1}
$$

where Ω is a given closed convex set with no uncertainty. Let $\mathcal{U}_i$ and $\mathcal{V}_i$ be the uncertainty sets for $\hat{\gamma}^{i} \in \mathbb{R}^{m_i}$ and $(\hat{A}^i, \hat{b}^i) \in \mathbb{R}^{m_i \times (n+1)}$, respectively. Then, the robust counterpart (RC) for (3.1) can be written as

$$
\begin{array}{ll}
\underset{x}{\text{minimize}} & \displaystyle\sup_{(\hat{A}^0,\hat{b}^0)\in\mathcal{U}_0,\ \hat{\gamma}^{0}\in\mathcal{V}_0} (\hat{\gamma}^{0})^\top(\hat{A}^0 x + \hat{b}^0) \\
\text{subject to} & (\hat{\gamma}^{i})^\top(\hat{A}^i x + \hat{b}^i) \le 0 \quad \forall (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \forall \hat{\gamma}^{i} \in \mathcal{V}_i \quad (i = 1, \dots, K) \\
& x \in \Omega.
\end{array}
\tag{3.2}
$$

The main purpose of this section is to show that RC (3.2) can be reformulated as an SDP [36], which can be solved by existing algorithms such as the primal-dual interior-point method. One may think that the structures of LP (3.1) and its RC (3.2) are much more special than the existing robust optimization models for LP [10]. However, we note that the robust optimization technique in this section plays an important role in considering the robust SOCPs and the robust Nash equilibrium problems in the subsequent sections. We also note that the uncertain LP (3.1) is equivalent to the LP considered by Ben-Tal et. al [10, 11], when $\mathcal{V}_i$ is a finite set given by $\mathcal{V}_i := \{e_1^{(m_i)}, \dots, e_{m_i}^{(m_i)}\}$ where $e_k^{(m_i)}$ is a unit vector with 1 at k-th element and 0 elsewhere.

We first make the following assumptions in the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$:

Assumption 1. *For $i = 0, 1, \dots, K$, the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are expressed as*

$$
\mathcal{U}_i := \left\{ (\hat{A}^i, \hat{b}^i) \;\middle|\; (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{s_i} u_j^i (A^{ij}, b^{ij}),\ (u^i)^\top u^i \le 1 \right\},
$$

$$
\mathcal{V}_i := \left\{ \hat{\gamma} \;\middle|\; \hat{\gamma} = \gamma^{i0} + \sum_{j=1}^{t_i} v_j^i \gamma^{ij},\ (v^i)^\top v^i \le 1 \right\},
$$

respectively, where $A^{ij} \in \mathbb{R}^{m_i \times n}$, $b^{ij} \in \mathbb{R}^{m_i}$ $(j = 0, 1, \dots, s_i)$ and $\gamma^{ij} \in \mathbb{R}^{m_i}$ $(j = 1, \dots, t_i)$ are given matrices and vectors.

Moreover, we introduce the following proposition, which plays a crucial role in reformulating RC (3.2) to an SDP.

Proposition 3.1. *Consider the following optimization problem:*

$$\begin{aligned} \underset{u\in\mathbb{R}^s,\ v\in\mathbb{R}^t}{\text{maximize}} \quad & \xi(v)^\top M(u)\eta \\ \text{subject to} \quad & u^\top u \le 1, \quad v^\top v \le 1, \end{aligned} \tag{3.3}$$

where $\eta \in \mathbb{R}^n$ is a given constant, and $M : \mathbb{R}^s \to \mathbb{R}^{m\times n}$ and $\xi : \mathbb{R}^t \to \mathbb{R}^m$ are defined by

$$M(u) = M^0 + \sum_{j=1}^{s} u_j M^j, \quad \xi(v) = \xi^0 + \sum_{j=1}^{t} v_j \xi^j \tag{3.4}$$

with given constants $M^j \in \mathbb{R}^{m\times n}(j = 0, 1, \dots, s)$ and $\xi^j \in \mathbb{R}^m(j = 0, 1, \dots, t)$. Then, the following two statements hold:

(a) *The Lagrangian dual problem of* (3.3) *is written as*

$$\begin{aligned} \underset{\alpha,\beta,\lambda}{\text{minimize}} \quad & -\lambda \\ \text{subject to} \quad & \begin{bmatrix} P_0 & q \\ q^\top & r-\lambda \end{bmatrix} \succeq \alpha \begin{bmatrix} P_1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} P_2 & 0 \\ 0 & 1 \end{bmatrix}, \\ & \alpha \ge 0, \quad \beta \ge 0, \quad \lambda \in \mathbb{R} \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} & P_0 = -\frac{1}{2}\begin{bmatrix} 0 & (\Xi^\top\Phi)^\top \\ \Xi^\top\Phi & 0 \end{bmatrix}, \quad q = -\frac{1}{2}\begin{bmatrix} \Phi^\top\xi^0 \\ \Xi^\top M^0\eta \end{bmatrix}, \\ & r = -(\xi^0)^\top M^0\eta \quad P_1 = \begin{bmatrix} -I_s & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & -I_t \end{bmatrix}, \\ & \Xi = \begin{bmatrix} \xi^1 & \cdots & \xi^t \end{bmatrix}, \quad \Phi = \begin{bmatrix} M^1\eta & \cdots & M^t\eta \end{bmatrix}. \end{aligned} \tag{3.6}$$

Moreover, it always holds that val(3.3) $\le$ val(3.5).

(b) *If*

$$\dim(\ker(P_0 - \alpha^* P_1 - \beta^* P_2)) \ne 1 \tag{3.7}$$

for the optimum $(\alpha^*, \beta^*, \lambda^*)$ of the dual problem (3.5)*, then it holds that* val(3.3) $=$ val(3.5).

Proof. From the definition of $M(u)$ and $\xi(v)$, the objective function of problem (3.3) can be rewritten as

$$\begin{aligned} \xi(v)^\top M(u)\eta &= (\xi^0 + \Xi v)^\top (M^0\eta + \Phi u) \\ &= v^\top \Xi^\top \Phi u + (\xi^0)^\top \Phi u + (M^0\eta)^\top \Xi v + (\xi^0)^\top M^0\eta \\ &= -y^\top P_0 y - 2q^\top y - r, \end{aligned}$$

where $y := \binom{u}{v}$. Hence, problem (3.3) is equivalent to the following optimization problem:

$$\begin{aligned} \underset{y\in\mathbb{R}^{s+t}}{\text{maximize}} \quad & -y^\top P_0 y - 2q^\top y - r \\ \text{subject to} \quad & y^\top P_1 y + 1 \ge 0, \quad y^\top P_2 y + 1 \ge 0. \end{aligned} \tag{3.8}$$

Now, notice that problem (3.8) is a nonconvex quadratic optimization problem with two quadratic constrains since P_0 is indefinite in general. Hence, from the results stated in Section 2, problem (3.5) serves as the Lagrangian dual problem of (3.3).

Next we show (b). From Theorem 2.1, it suffices to show that the following three statements hold:

(i) Both problems (3.3) and (3.5) are strictly feasible.

(ii) There exist $\hat{\alpha} \in \mathbb{R}$ and $\hat{\beta} \in \mathbb{R}$ such that $\hat{\alpha} P_1 + \hat{\beta} P_2 \succ O$.

(iii) $\dim(\ker(P_0 - \alpha^* P_1 - \beta^* P_2)) \neq 1$ for the optimum $(\alpha^*, \beta^*, \lambda^*)$ of problem (3.5).

Problem (3.3) is obviously strictly feasible since $(u, v) = (0, 0)$ is an interior point of the feasible region. Also, problem (3.5) is strictly feasible since the inequalities in the constrains hold strictly when we choose sufficiently large α, β, and sufficiently small λ. Thus, we have (i). We can readily see (ii) since $\hat{\alpha} P_1 + \hat{\beta} P_2 \succ 0$ for any $\hat{\alpha}, \hat{\beta}$ such that $\hat{\alpha}, \hat{\beta} < 0$. We also have (iii) from the assumption of the theorem. Hence, the optimal values of (3.3) and (3.5) are equal. $\square$

Next, by using the above proposition, we reformulate RC (3.2) as an SDP. Note that RC (3.2) is rewritten as the following optimization problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) := \displaystyle\max_{(\hat{A}^0, \hat{b}^0) \in \mathcal{U}_0,\ \hat{\gamma}^0 \in \mathcal{V}_0} (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & f_i(x) := \max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\} \le 0 \\ & \quad (i = 1, \dots, K), \\ & x \in \Omega. \end{array} \tag{3.9}$$

Now for any fixed $x \in \mathbb{R}^n$, we evaluate $\max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\}$ for $i = 0, 1, \dots, K$. By letting $\eta := \binom{x}{1}$, $M^j = (A^{ij}, b^i)$, and $\xi^j := \gamma^{ij}$ in Proposition 3.1, we have the following inequality for each $i = 0, 1, \dots, K$:

$$\begin{aligned} &\max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\} \\ &\qquad \le \min \left\{ -\lambda_i \;\middle|\; \begin{array}{l} \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) - \lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_i \ge 0, \quad \beta_i \ge 0, \quad \lambda_i \in \mathbb{R} \end{array} \right\}, \end{aligned} \tag{3.10}$$

where $P_0^i(x)$, $q^i(x)$ and $r^i(x)$ are defined by

$$\begin{aligned} &P_0^i(x) = -\frac{1}{2} \begin{bmatrix} 0 & (\Gamma_i^\top \Phi_i(x))^\top \\ \Gamma_i^\top \Phi_i(x) & 0 \end{bmatrix}, \quad q^i(x) = -\frac{1}{2} \begin{bmatrix} \Phi_i(x)^\top \gamma^i \\ \Gamma_i^\top (A^{i0} x + b^{i0}) \end{bmatrix}, \\ &r^i(x) = -(\gamma^i)^\top (A^{i0} x + b^{i0}), \quad P_1^i = \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^i = \begin{bmatrix} 0 & 0 \\ 0 & -I_{t_i} \end{bmatrix}, \\ &\Gamma_i = \begin{bmatrix} \gamma^{i1} & \cdots & \gamma^{it} \end{bmatrix}, \quad \Phi_i(x) = \begin{bmatrix} A^{i1} x + b^{i1} & \cdots & A^{i s_i} x + b^{i s_i} \end{bmatrix}. \end{aligned} \tag{3.11}$$

Moreover, we consider the following problem in which $\max\{(\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\}$ in (3.9) is replaced by the right-hand side of (3.10):

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & g_0(x) := \min\left\{ -\lambda_0 \;\middle|\; \begin{array}{l} \begin{bmatrix} P_0^0(x) & q^0(x) \\ q^0(x)^\top & r^0(x)-\lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P_1^0 & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^0 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_0 \geq 0, \quad \beta_0 \geq 0, \quad \lambda_0 \in \mathbb{R} \end{array} \right\} \\
\text{subject to} \quad & g_i(x) := \min\left\{ -\lambda_i \;\middle|\; \begin{array}{l} \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x)-\lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha_i \geq 0, \quad \beta_i \geq 0, \quad \lambda_i \in \mathbb{R} \end{array} \right\} \leq 0 \\
& (i = 1, \dots, K), \\
& x \in \Omega,
\end{aligned} \tag{3.12}
$$

which is equivalent to the following SDP:

$$
\begin{aligned}
\underset{x,\alpha,\beta,\lambda}{\text{minimize}} \quad & -\lambda_0 \\
\text{subject to} \quad & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x)-\lambda_i \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \; (i = 0, 1, \dots, K), \\
& \alpha = (\alpha_0, \alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^{K+1}, \quad \beta = (\beta_0, \beta_1, \dots, \beta_K) \in \mathbb{R}_+^{K+1}, \\
& \lambda = (\lambda_0, \lambda_1, \dots, \lambda_K) \in \mathbb{R} \times \mathbb{R}_+^K, \\
& x \in \Omega.
\end{aligned} \tag{3.13}
$$

Here, notice that, if the matrix inequalities in (3.13) hold with some $\lambda_i \geq 0$ $(i = 1, \dots, K)$, then they also hold for $\lambda_i = 0$. Hence, we can set $\lambda_i = 0$ $(i = 1, \dots, K)$ without changing the optimal value of (3.13). That is, SDP (3.13) is equivalent to the following SDP:

$$
\begin{aligned}
\underset{x,\alpha,\beta,\lambda_0}{\text{minimize}} \quad & -\lambda_0 \\
\text{subject to} \quad & \begin{bmatrix} P_0^0(x) & q^0(x) \\ q^0(x)^\top & r^0(x)-\lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P_1^0 & 0 \\ 0 & 1 \end{bmatrix} + \beta_0 \begin{bmatrix} P_2^0 & 0 \\ 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \; (i = 1, \dots, K), \\
& \alpha = (\alpha_0, \alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^{K+1}, \quad \beta = (\beta_0, \beta_1, \dots, \beta_K) \in \mathbb{R}_+^{K+1}, \\
& \lambda_0 \in \mathbb{R}, \quad x \in \Omega.
\end{aligned} \tag{3.14}
$$

Consequently, we have $\mathrm{val}\,(3.9) \leq \mathrm{val}\,(3.12) = \mathrm{val}\,(3.13) = \mathrm{val}\,(3.14)$ where the inequality is due to $f_i(x) \leq g_i(x)$ for any $x \in \mathbb{R}^n$ and $i = 0, 1, \dots, K$. Moreover, we can show $\mathrm{val}\,(3.9) = \mathrm{val}\,(3.12)$, under the following assumption.

Assumption 2. *Let $z^* := (x^*, \alpha^*, \beta^*, \lambda_0^*)$ be an optimum of SDP* (3.14). *Then, there exists $\varepsilon > 0$ such that*

$$\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \; (i = 0, 1, \dots, K)$$

for all $(x, \alpha, \beta, \lambda_0^*) \in B(z^*, \varepsilon)$.

Theorem 3.2. *Suppose that Assumption 1 holds, and* $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ *be the optimum of SDP* (3.14)*, then* x^* *is feasible in RC* (3.2) *and* val (3.14) *is an upper bound of* val (3.2)*. Moreover,* x^* *solves RC* (3.2) *if Assumption 2 further holds.*

Proof. Since the first part is trivial from $f_i(x) \leq g_i(x)$ for any $x \in \mathbb{R}^n$ and $i = 0, 1, \ldots, K$, we only show the last part.

Define $X, Y \subseteq \mathbb{R}^n$ and $\theta, \omega : \mathbb{R}^n \to (-\infty, \infty]$ by

$$\begin{aligned} X &= \{x \in \mathbb{R}^n \mid f_i(x) \leq 0 \ (i = 1, \ldots, K)\} \cap \Omega, \\ Y &= \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ (i = 1, \ldots, K)\} \cap \Omega, \\ \theta(x) &= f_0(x) + \delta_X(x), \\ \omega(x) &= g_0(x) + \delta_Y(x), \end{aligned}$$

where δ_X and δ_Y denote the indicator functions [34] of X and Y, respectively. Then, we can see that RC(3.2) and SDP(3.14) are equivalent to the unconstrained minimization problems with objective functions θ and ω, respectively. In addition, since functions f_i, g_i $(i = 0, 1, \ldots, K)$ are proper and convex [15, Proposition 1.2.4(c)], θ and ω are proper and convex, too.

Let $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ be an arbitrary solution to SDP (3.14). Then, it is obvious that x^* minimizes ω. Moreover, from Proposition 3.1(b) and Assumption 2, there exists a closed neighborhood $B(x^*, \varepsilon)$ of x^* such that $\theta(x) = \omega(x)$ for all $x \in B(x^*, \varepsilon)$. Hence, we have

$$\theta(x^*) = \omega(x^*) \leq \omega(x) = \theta(x), \quad \forall x \in B(x^*, \varepsilon). \tag{3.15}$$

Now, for contradiction, assume that x^* is *not* a solution to RC(3.2). Then, there must exist $\bar{x} \in \mathbb{R}^n$ such that $\theta(\bar{x}) < \theta(x^*)$. Moreover, we have $\bar{x} \notin B(x^*, \varepsilon)$ from (3.15). Set $\alpha := \varepsilon / \|\bar{x} - x^*\|$ and $\tilde{x} := (1 - \alpha)x^* + \alpha\bar{x}$. Then, $\alpha \in (0, 1)$ since $\bar{x} \notin B(x^*, \varepsilon)$, i.e., $\|\bar{x} - x^*\| > \varepsilon$. Thus, we have

$$\begin{aligned} \theta(\tilde{x}) &= \theta((1 - \alpha)x^* + \alpha\bar{x}) \\ &\leq (1 - \alpha)\theta(x^*) + \alpha\theta(\bar{x}) \\ &< (1 - \alpha)\theta(x^*) + \alpha\theta(x^*) = \theta(x^*), \end{aligned}$$

where the first inequality follows from the convexity of θ, and the last inequality follows from $\theta(\bar{x}) < \theta(x^*)$ and $\alpha > 0$. However, since $\|\tilde{x} - x^*\| = \alpha\|\bar{x} - x^*\| = \varepsilon$, we have $\tilde{x} \in B(x^*, \varepsilon)$, which implies $\theta(x^*) \leq \theta(\tilde{x})$ from (3.15). Hence, x^* is an optimum of RC (3.2). □

In order to see whether Assumption 2 holds or not, we generally have to check every function value in the neighborhood of the optimum. However, in some situations, it can be guaranteed more easily. For example, suppose that at the optimum $z^* = (x^*, \alpha^*, \beta^*, \lambda_0^*)$,

$$\dim(\ker(P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i)) = 0 \ (i = 0, 1, \ldots, K),$$

equivalently $P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succ 0$[*2]. Then, by the continuity of $P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i$, it also follows $P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i \succ 0$ for any z sufficiently close to z^*.

Moreover, when the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are spherical, Assumption 2 also holds automatically. We will show this fact in the remainder of this section.

Assumption 3. *Suppose that Assumption 1 holds. Moreover, for each* $i = 0, 1, \dots, K$*, matrices* (A^{ij}, b^{ij}) $(j = 1, \dots, m_i(n+1))$ *and vectors* γ^{ij} $(j = 1, \dots, t_i)$ $(t_i \geq 2)$ *satisfy the following.*

- *For* $(k, l) \in \{1, \dots, m_i\} \times \{1, \dots, n+1\}$*,*

$$(A^{ij}, b^{ij}) = \rho_i e_k^{(m_i)} (e_l^{(n+1)})^\top \quad \textit{with } j := m_i l + k,$$

  where ρ_i *is a given nonnegative constant, and* $e_r^{(p)}$ *is a unit vector with 1 at* r*-th element and 0 elsewhere.*
- *For any* $(k, l) \in \{1, \dots, t_i\} \times \{1, \dots, t_i\}$*,*

$$(\gamma^{ik})^\top \gamma^{il} = \sigma_i^2 \delta_{kl},$$

  where σ_i *is a given nonnegative constant, and* δ_{kl} *denotes Kronecker's delta, i.e.,* $\delta_{kl} = 0$ *for* $k \neq l$ *and* $\delta_{kl} = 1$ *for* $k = l$*.*

Assumption 3 claims that $\mathcal{U}_i$ is an $m_i(n+1)$-dimensional sphere with radius ρ_i in the $m_i(n+1)$-dimensional space and $\mathcal{V}_i$ is a t_i-dimensional sphere with radius σ_i in the m_i-dimensional space, i.e.,

$$\begin{aligned}
\mathcal{U}_i &= \{(\hat{A}^i, \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + (\delta A^i, \delta b^i),\ \|(\delta A^i, \delta b^i)\|_F \leq \rho_i\} \subset \mathbb{R}^{m_i(n+1)}, \\
\mathcal{V}_i &= \{\hat{\gamma}^i \mid \hat{\gamma}^i = \gamma^{i0} + \delta\gamma^i,\ \|\delta\gamma^i\| \leq \sigma_i, \delta\gamma^i \in \operatorname{span}\{\gamma^{ij}\}_{j=1}^{t_i}\} \subset \mathbb{R}^{m_i}.
\end{aligned}$$

The following proposition provides sufficient conditions under which condition (3.7) in Proposition 3.1 holds. It also plays an important role in showing that Assumption 3 implies Assumption 2.

Proposition 3.3. *Consider the optimization problem* (3.3) *with a given constant* $\eta \in \mathbb{R}^n$ *and functions* $M : \mathbb{R}^s \to \mathbb{R}^{m \times n}$ *and* $\xi : \mathbb{R}^t \to \mathbb{R}^m$ *defined by* (3.4)*. Moreover, suppose that there exist nonnegative constants* ρ *and* σ *such that the following statements hold.*

- $t, n \geq 2$.
- $s = m(m+1)$*. Moreover,* M^j $(j = 1, \dots, s)$ *are given by*

$$M^j = \rho e_k^{(m)} (e_l^{(n)})^\top \quad \textit{with} \quad j := ml + k,$$

  for each $k = 1, \dots, m$ *and* $l = 1, \dots, n$*.*
- *For any* $(k, l) \in \{1, \dots, t\} \times \{1, \dots, t\}$*,* $(\xi^k)^\top \xi^l = \sigma^2 \delta_{kl}$*.*

[*2] By the constraints of SDP (3.14), $P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succeq 0$ always holds at the optimum $(x^*, \alpha^*, \beta^*, \lambda_0^*,)$.

Then, for P_0, P_1 and P_2 defined by (3.6)*, it holds that*

$$\dim(\ker(P_0 - \alpha P_1 - \beta P_2)) \neq 1$$

for any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$*, and hence* $\mathrm{val}\,(3.3) = \mathrm{val}\,(3.5)$.

Proof. Let $P(\alpha, \beta) := P_0 - \alpha P_1 - \beta P_2$. Then, since $P(\alpha, \beta)$ is symmetric, it suffices to show that the multiplicity of zero eigenvalues of $P(\alpha, \beta)$ can *never* be 1.

We first define matrices Ξ and Φ by (3.6). By Assumption 3, we have the following equalities:

$$\Xi^\top \Xi = \begin{bmatrix} \xi^1 & \cdots & \xi^t \end{bmatrix}^\top \begin{bmatrix} \xi^1 & \cdots & \xi^t \end{bmatrix} = \sigma^2 I,$$

$$\begin{aligned} \Phi &= \begin{bmatrix} M^1 \eta & \cdots & M^t \eta \end{bmatrix} \\ &= \rho \begin{bmatrix} e_1^{(m)} (e_1^{(n)})^\top \eta & e_2^{(m)} (e_1^{(n)})^\top \eta & \cdots & e_m^{(m)} (e_n^{(n)})^\top \eta \end{bmatrix} \\ &= \rho \begin{bmatrix} \begin{bmatrix} \eta_1 e_1^{(m)} & \eta_1 e_2^{(m)} & \cdots & \eta_1 e_m^{(m)} \end{bmatrix} & \cdots & \begin{bmatrix} \eta_n e_1^{(m)} & \eta_n e_2^{(m)} & \cdots & \eta_n e_m^{(m)} \end{bmatrix} \end{bmatrix} \\ &= \rho \begin{bmatrix} \eta_1 I_m & \cdots & \eta_n I_m \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Xi^\top \Phi \Phi^\top \Xi &= \Xi^\top (\rho^2 \|\eta\|^2 I_m) \Xi \\ &= \rho^2 \sigma^2 \|\eta\|^2 I_t. \end{aligned}$$

Now we consider the eigenvalue equation $\det(P(\alpha, \beta) - \zeta I) = 0$. If $\zeta \neq \alpha$, then we have

$$\begin{aligned} \det(P(\alpha, \beta) - \zeta I) &= \det\left(\begin{bmatrix} (\alpha - \zeta) I_{mn} & -\frac{1}{2} (\Xi^\top \Phi)^\top \\ -\frac{1}{2} \Xi^\top \Phi & (\beta - \zeta) I_t \end{bmatrix} \right) \\ &= \det\left[(\alpha - \zeta) I_{mn} \right] \cdot \det\left[(\beta - \zeta) I_t - \frac{1}{4(\alpha - \zeta)} \Xi^\top \Phi \Phi^\top \Xi \right] \\ &= (\alpha - \zeta)^{mn - t} \det\left[\left((\alpha - \zeta)(\beta - \zeta) - \frac{1}{4} \rho\sigma \|\eta\|^2 \right) I_t \right] \\ &= (\alpha - \zeta)^{mn - t} \left((\alpha - \zeta)(\beta - \zeta) - \frac{1}{4} \rho\sigma \|\eta\|^2 \right)^t, \end{aligned} \tag{3.16}$$

where the second equality follows from the Schur complement [24, Theorem 13.3.8]. Moreover, since $\det(P(\alpha, \beta) - \zeta I)$ is continuous at any (α, β, ζ), equality (3.16) is valid at $\zeta = \alpha$[*3]. Since we have $mn - t \geq 2$ from $t, n \geq 2$ and $t \leq m$, (3.16) indicates that the multiplicity of all eigenvalues of $P(\alpha, \beta)$ is greater than 2. Hence, even if $P(\alpha, \beta)$ has zero eigenvalue, the multiplicity cannot be 1. □

By the above proposition, we obtain the following theorem.

Theorem 3.4. *Suppose Assumption* 3 *holds. Then, x^* solves RC* (3.2) *if and only if there exists* $(\alpha^*, \beta^*, \lambda_0^*)$ *such that* $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ *is an optimal solution of SDP* (3.14).

[*3] From the continuity, we have $\lim_{\tilde{\zeta} \to \alpha, \tilde{\zeta} \neq \alpha} \det(P(\alpha, \beta) - \tilde{\zeta} I) = \det(P(\alpha, \beta) - \alpha I)$

Proof. In a way similar to the proof of Theorem 3.2, we evaluate $\max\{(\hat{\gamma}^i)^\top(\hat{A}^i x + \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \hat{\gamma}^i \in \mathcal{V}_i\}$ for each $i = 0, 1, \ldots, K$, in (3.9). In Proposition 3.3, let $\eta := \binom{x}{1}$ and $M^j := (A^{ij}, b^{ij})$, and $\xi^j := \gamma^{ij}$. Then, for all $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we can see that

$$\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1,\ (i = 0, 1, \ldots, K).$$

From Proposition 3.1, we have $f_i(x) = g_i(x)$ for all $x \in \mathbb{R}^n$. Hence, problems (3.9) is identical to (3.12). This completes the proof. □

In Theorem 3.2, the optimality of SDP (3.14) is nothing more than a sufficient condition for the optimality of RC (3.2) under appropriate assumptions. However, Theorem 3.4 shows not only the sufficiency but also the necessity. This is due to the fact that Assumption 3 guarantees $f_i(x) = g_i(x)$ for all $x \in \mathbb{R}^n$, though Assumption 2 guarantees it only in a neighborhood of the SDP solution.

4 Robust second-order cone programming problems with ellipsoidal uncertainty

The second-order cone programming problem (SOCP) is expressed as follows:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & M^i x + q^i \in \mathcal{K}^{n_i} \quad (i = 1, \ldots, K), \\ & x \in \Omega, \end{array} \tag{4.1}$$

where $\mathcal{K}^{n_i}$ denotes the n_i-dimensional second-order cone defined by $\mathcal{K}^{n_i} := \{(x_0, \bar{x}^\top)^\top \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid x_0 \leq \|\bar{x}\|\}$ and Ω is a given closed convex set. SOCP is applicable to many practical problems such as the antenna array weight design problems and the truss design problems [3, 31]. We note that the second-order cone constraints $M^i x + q^i \in \mathcal{K}^{n_i}$ $(i = 1, \ldots, K)$ in (4.1) are rewritten as $\|A^i x + b^i\| \leq (c^i)^\top x + d^i$ with $M^i = \binom{(c^i)^\top}{A^i}$ and $q^i = \binom{d^i}{b^i}$.

In this section, we consider the following uncertain SOCP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & \|\hat{A}^i x + \hat{b}^i\| \leq (\hat{c}^i)^\top x + \hat{d}^i \quad (i = 1, \ldots, K), \\ & x \in \Omega, \end{array} \tag{4.2}$$

where $\hat{A}^i \in \mathbb{R}^{m_i \times n}$, $\hat{b}^i \in \mathbb{R}^{m_i}$, $\hat{c}^i \in \mathbb{R}^n$ and $\hat{d}^i \in \mathbb{R}$ are uncertain data with uncertainty set $\mathcal{U}_i$. Then, the robust counterpart (RC) for (4.2) can be written as

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f^\top x \\ \text{subject to} & \|\hat{A}^i x + \hat{b}^i\| \leq (\hat{c}^i)^\top x + \hat{d}^i, \quad \forall (\hat{A}^i, \hat{b}^i, \hat{c}^i, \hat{d}^i) \in \mathcal{U}_i, \\ & \quad (i = 1, \ldots, K), \\ & x \in \Omega. \end{array} \tag{4.3}$$

Throughout this section, we assume $m_i \geq 2$ for all $i = 1, \ldots, K$[*4].

Ben-Tal and Nemirovski [9] showed that RC (4.3) can be reformulated as an SDP in the case where the uncertainty sets for $(\hat{A}^i, \hat{b}^i)$ and $(\hat{c}^i, \hat{d}^i)$ are independent and can be represented with two ellipsoids as

$$\mathcal{U}_{L_i} = \{(\hat{A}^i, \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{l} u^i_j (A^{ij}, b^{ij}),\ (u^i)^\top u^i \leq 1\},$$
$$\mathcal{U}_{R_i} = \{(\hat{c}^i, \hat{d}^i) \mid (\hat{c}^i, \hat{d}^i) = (c^{i0}, d^{i0}) + \sum_{j=1}^{r} v^i_j (c^{ij}, d^{ij}),\ (v^i)^\top v^i \leq 1\},$$

with given constants A^{ij}, b^{ij}, c^{ij} and d^{ij}. However, according to Ben-Tal and Nemirovski [13], it was an open problem until quite recently whether or not RC (4.3) can be reformulated as an SDP under the following assumption (*one* ellipsoid case).

Assumption 4. *The uncertainty sets $\mathcal{U}_i$ $(i = 1, \ldots, K)$ in RC (4.3) are given by*

$$\mathcal{U}_i = \left\{ \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} \middle| \begin{bmatrix} \hat{A}^i & \hat{b}^i \\ (\hat{c}^i)^\top & \hat{d}^i \end{bmatrix} = \begin{bmatrix} A^{i0} & b^{i0} \\ (c^{i0})^\top & d^{i0} \end{bmatrix} + \sum_{j=1}^{s_i} u^i_j \begin{bmatrix} A^{ij} & b^{ij} \\ (c^{ij})^\top & d^{ij} \end{bmatrix},\ (u^i)^\top u^i \leq 1 \right\},$$

where A^{ij}, b^{ij}, c^{ij} and d^{ij} $(i = 1, \ldots, K,\ j = 0, 1, \ldots, s_i)$ are given constants.

In this section, we show that the robust counterpart can be reformulated as an explicit SDP under this assumption, using the results in the previous section[*5].

We first rewrite RC (4.2) in the form RC (3.1). To this end, we introduce the following result in semi-infinite programming [32, Section 4].

Proposition 4.1. *Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ be given. Then $x \in \mathbb{R}^n$ satisfies the inequality $\|Ax + b\| \leq c^\top x + d$ if and only if x satisfies $\tilde{y}^\top (Ax + b) \leq c^\top x + d$ for all $\tilde{y} \in \mathbb{R}^m$ such that $\|\tilde{y}\| \leq 1$.*

[*4] If $m_i = 1$ for some i, then the constraint can be rewritten as two linear inequalities $-(\hat{c}^i)^\top x + \hat{d}^i \leq \hat{A}^i x + \hat{b}^i \leq (\hat{c}^i)^\top x + \hat{d}^i$. So existing frameworks can be applied. (See Ben-Tal and Nemirovski [10]).

[*5] Fairly recently, it has been shown that another SDP reformulation is possible by Hildebrand's Lorentz-positivity results [27, 28]. However, our approach has an advantage in terms of computational complexity. We state the details at the end of this section.

By this proposition, RC (4.3) can be rewritten as follows:

$$
\begin{array}{ll}
\underset{x}{\text{minimize}} & f^\top x \\
\text{subject to} & (\hat{\gamma}^i)^\top \left(\begin{bmatrix} \hat{A}^i \\ -(\hat{c}^i)^\top \end{bmatrix} x + \begin{bmatrix} \hat{b}^i \\ -\hat{d}^i \end{bmatrix} \right) \le 0, \\
& \forall (\hat{A}^i, \hat{b}^i, \hat{c}^i, \hat{d}^i) \in \mathcal{U}_i, \quad \forall \hat{\gamma}^i \in \mathcal{V}_i := \left\{ ((\tilde{\gamma}^i)^\top, 1)^\top \,\middle|\, \|\tilde{\gamma}^i\| \le 1 \right\} \\
& \quad (i = 1, \dots, K), \\
& x \in \Omega.
\end{array}
\tag{4.4}
$$

Clearly, problem (4.4) belongs to the class of problems of the form RC (3.2). In addition, when Assumption 4 holds, Assumption 1 also holds by setting $\mathcal{V}_i := \{\hat{\gamma}^i \mid \hat{\gamma}^i = \gamma^{i0} + \sum_{j=1}^{m_i} v_j^i \gamma^{ij},\ (v^i)^\top v^i \le 1\}$ with $\gamma^{i0} = e_{m_i+1}^{(m_i+1)}$, $\gamma^{ij} = e_j^{(m_i)}$ $(j = 1, \dots, m_i)$. Thus, we have the following theorem, whose proof is omitted since it readily follows from Theorem 3.2.

Theorem 4.2. *Suppose that Assumption* 4 *holds. Let* (x^*, α^*, β^*) *be an optimal solution of the following SDP:*

$$
\begin{array}{ll}
\underset{x,\alpha,\beta}{\text{minimize}} & f^\top x \\
\text{subject to} & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \ (i = 1, \dots, K), \\
& \alpha = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^K, \quad \beta = (\beta_1, \dots, \beta_K) \in \mathbb{R}_+^K, \\
& x \in \Omega,
\end{array}
\tag{4.5}
$$

where

$$
\begin{aligned}
P_0^i(x) &= -\frac{1}{2} \begin{bmatrix} 0 & \Psi_i(x)^\top \\ \Psi_i(x) & 0 \end{bmatrix}, \quad q^i(x) = -\frac{1}{2} \begin{bmatrix} -\psi_i(x) \\ A^{i0}x + b^{i0} \end{bmatrix}, \\
r^i(x) &= (c^{i0})^\top x + d^{i0}, \quad P_1^i = \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^i = \begin{bmatrix} 0 & 0 \\ 0 & -I_{m_i-1} \end{bmatrix}, \\
\psi_i(x) &= \begin{bmatrix} (c^{i1})^\top x + d^{i1} & \cdots & (c^{is_i})^\top x + d^{is_i} \end{bmatrix}^\top, \\
\Psi_i(x) &= \begin{bmatrix} A^{i1}x + b^{i1} & \cdots & A^{is_i}x + b^{is_i} \end{bmatrix}.
\end{aligned}
\tag{4.6}
$$

Then, x^* *solves RC* (4.3) *if*

$$
\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \quad (i = 0, 1, \dots, K) \tag{4.7}
$$

in an neighborhood of (x^*, α^*, β^*).

We can easily see that condition (4.7) is guaranteed to hold if

$$
P_0^i(x^*) - \alpha_i^* P_1^i - \beta_i^* P_2^i \succ 0, \tag{4.8}
$$

by using similar arguments to those just after Theorem 3.2. Also when the uncertainty sets are spherical, condition (4.7) is satisfied and hence the following theorem holds.

Assumption 5. *The uncertainty sets $\mathcal{U}_i$ in RC* (4.3) *are given by*

$$\mathcal{U}_i = \left\{ (\hat{A}^i, \hat{b}^i, \hat{c}^i, \hat{d}^i) = (A^{i0} + \delta A^i, b^{i0} + \delta b^i, c^{i0} + \delta c^i, d^{i0} + \delta d^i) \;\middle|\; \left\| \begin{bmatrix} \delta A^i & \delta b^i \\ (\delta c^i)^\top & \delta d^i \end{bmatrix} \right\|_F \le \rho_i \right\}.$$

Theorem 4.3. *Suppose Assumption* 5 *holds. Then, x^* solves RC* (4.4) *if and only if there exists* (α^*, β^*) *such that* (x^*, α^*, β^*) *is an optimal solution of SDP* (4.5)*.*

Proof. Problem (4.4) and Assumption 5 reduce to RC (3.2) and Assumption 3, respectively. Hence, the theorem readily follows from Theorem 3.4. □

By the correspondence between problem (4.4) and the robust LP (3.2), Assumption 5 is equivalent to Assumption 3. Thus, we have the following theorem, whose proof is omitted since it readily follows from Theorem 3.4.

Finally, we mention another SDP reformulation approach based on Hildebrand's recent results. Hildebrand [27, 28] showed that the cone of "*Lorentz-positive*" matrices is represented by an explicit SDP, and then, Ben-Tal, El Ghaoui and Nemirovski [6] pointed out that problem (4.3) can be reformulated as an explicit SDP under Assumption 4 by applying Hildebrand's idea. Specifically, Ben-Tal et al. [6] state that the following statement holds:

$$\|\hat{A}^i x + \hat{b}^i\| \le (\hat{c}^i)^\top x + \hat{d}^i, \quad \forall (\hat{A}_i, \hat{b}^i, \hat{c}^i, \hat{d}^i) \in \mathcal{U}_i$$
$$\Updownarrow$$
$$\exists X_i \in \mathcal{A}^{m_i} \otimes \mathcal{A}^{s_i}, \quad (\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) \left(\begin{bmatrix} (c^{i0})^\top x + d^{i0} & \psi_i(x)^\top \\ A^{i0} x + b^{i0} & \Psi_i(x) \end{bmatrix} \right) + X_i \succeq 0$$

where $\mathcal{A}^p$ denotes the set of $p \times p$ real skew-symmetric matrices, $\otimes$ denotes the tensor product, and functions Ψ_i and ψ_i are defined by (4.6). Moreover, $(\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) : \mathbb{R}^{(m_i+1)\times(s_i+1)} \to \mathcal{S}^{m_i} \otimes \mathcal{S}^{s_i}$ is the tensor product of the linear mapping $\mathcal{W}_r : \mathbb{R}^r \to \mathcal{S}^{r-1}$ defined by

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r-1} \end{bmatrix} \mapsto \begin{bmatrix} x_0 + x_1 & x_2 & \cdots & x_{r-1} \\ x_2 & x_0 - x_1 & & 0 \\ \vdots & & \ddots & \\ x_{r-1} & 0 & & x_0 - x_1 \end{bmatrix}.$$

Thus, we obtain the following SDP equivalent to RC (4.3) under Assumption 4:

$$\begin{array}{ll} \underset{x,\alpha,\beta}{\text{minimize}} & f^\top x \\ \text{subject to} & (\mathcal{W}_{m_i+1} \otimes \mathcal{W}_{s_i+1}) \left(\begin{bmatrix} (c^{i0})^\top x + d^{i0} & \psi_i(x)^\top \\ A^{i0} x + b^{i0} & \Psi_i(x) \end{bmatrix} \right) + X_i \succeq 0, \\ & X_i \in \mathcal{A}^{m_i} \otimes \mathcal{A}^{s_i} \quad (i = 1, \ldots, K), \\ & x \in \Omega. \end{array} \tag{4.9}$$

The Hildebrand-based SDP reformulation (SDP (4.9)) has some advantages and disadvantages compared with our approach (SDP (4.5)). They are summarized as follows:

Advantage

- Without any additional assumption, the equivalence between SDP (4.5) and RC (4.3) under Assumption 4 is guaranteed. (Our approach requires condition (4.7).)

Disadvantage

- The size of matrix inequalities is large in (4.9). Actually, in SDP (4.9), the matrix size is $(m_i s_i) \times (m_i s_i)$ for each i, while it is only $(m_i + s_i + 1) \times (m_i + s_i + 1)$ in SDP (4.5).
- The size of decision variables is also large in (4.9). Essentially, SDP (4.9) has $n + \sum_{i=1}^{K} m_i s_i (m_i - 1)(s_i - 1)/4$ decision variables, while SDP (4.5) has only $n + 2K$ variables.

In the subsequent numerical experiments, we will observe the above advantage and disadvantage, by comparing those two SDP reformulations.

5 SDCP reformulation of robust Nash equilibrium problems

In this section, we apply the idea discussed in Section 3 to the robust Nash equilibrium problem, and show that it can be reduced to a semidefinite complementarity problem (SDCP) under some assumptions.

5.1 Robust Nash equilibrium and its existence

In this subsection, we study the concept of a robust Nash equilibrium and its existence [33]. We consider an N-person non-cooperative game in which each player tries to minimize his own cost. Let $x^i \in \mathbb{R}^{m_i}$, $S_i \subseteq \mathbb{R}^{m_i}$, and $f_i : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N} \to \mathbb{R}$ be player i's strategy, strategy set, and cost function, respectively. Moreover, we denote

$$\begin{aligned}
&\mathcal{I} := \{1, \ldots, N\}, \quad \mathcal{I}_{-i} := \mathcal{I} \setminus \{i\}, \quad m := \sum_{j \in \mathcal{I}} m_j, \quad m_{-i} := \sum_{j \in \mathcal{I}_{-i}} m_j, \\
&x := (x^j)_{j \in \mathcal{I}} \in \mathbb{R}^m, \quad x^{-i} := (x^j)_{j \in \mathcal{I}_{-i}} \in \mathbb{R}^{m_{-i}}, \\
&S := \prod_{j \in \mathcal{I}} S_j \subseteq \mathbb{R}^m, \quad S_{-i} := \prod_{j \in \mathcal{I}_{-i}} S_j \subseteq \mathbb{R}^{m_{-i}}.
\end{aligned}$$

When the complete information is assumed, each player i decides his own strategy by solving the following optimization problem with the opponents' strategies x^{-i} fixed:

$$\begin{aligned}
&\underset{x^i}{\text{minimize}} \quad f_i(x^i, x^{-i}) \\
&\text{subject to} \quad x^i \in S_i.
\end{aligned} \tag{5.1}$$

A tuple $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ satisfying $\bar{x}^i \in \operatorname{argmin}_{x^i \in S_i} f_i(x^i, \bar{x}^{-i})$ for each player $i = 1, \ldots, N$ is called a Nash equilibrium. In other words, if each player i chooses the strategy $\bar{x}^i$, then no player has an incentive to change his own strategy. The Nash equilibrium is well-defined only when each player can estimate his opponents' strategies and can evaluate his own cost exactly. In the real situation, however, any information may contain uncertainty such as observation errors or estimation errors. Thus, we focus on games with uncertainty.

To deal with such uncertainty, we introduce uncertainty sets U_i and $X_i(x^{-i})$, and assume the following statements for each player $i \in \mathcal{I}$:

(A) Player i's cost function involves a parameter $\hat{u}^i \in \mathbb{R}^{s_i}$, i.e., it can be expressed as $f_i^{\hat{u}^i} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to \mathbb{R}$. Although player i does not know the exact value of $\hat{u}^i$ itself, he can estimate that it belongs to a given nonempty set $U_i \subseteq \mathbb{R}^{s_i}$.

(B) Although player i knows his opponents' strategies x^{-i}, his actual cost is evaluated with x^{-i} replaced by $\hat{x}^{-i} = x^{-i} + \delta x^{-i}$, where δx^{-i} is a certain error or noise. Player i cannot know the exact value of $\hat{x}^{-i}$. However, he can estimate that $\hat{x}^{-i}$ belongs to a certain nonempty set $X_i(x^{-i})$.

Under these assumptions, each player encounters the difficulty of addressing the following family of problems involving uncertain parameters $\hat{u}^i$ and $\hat{x}^{-i}$:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \\ \text{subject to} & x^i \in S_i, \end{array} \tag{5.2}$$

where $\hat{u}^i \in U_i$ and $\hat{x}^{-i} \in X_i(x^{-i})$. To overcome such a difficulty, we further assume that each player chooses his strategy according to the following criterion of rationality:

(C) Player i tries to minimize his worst cost under assumptions (A) and (B).

From assumption (C), each player considers the worst cost function $\tilde{f}_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to (-\infty, +\infty]$ defined by

$$\tilde{f}_i(x^i, x^{-i}) := \sup\{f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \mid \hat{u}_i \in U_i, \hat{x}^{-i} \in X_i(x^{-i})\}, \tag{5.3}$$

and then solves the following worst cost minimization problem:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \tilde{f}_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in S_i. \end{array} \tag{5.4}$$

Note that, for fixed x^{-i}, (5.4) is nothing other than the robust counterpart of the uncertain cost minimization problem (5.2). Also, (5.4) can be regarded as a complete information game with cost functions $\tilde{f}_i$. Based on the above discussions, we define the robust Nash equilibrium.

Definition 5.1. Let $\tilde{f}_i$ be defined by (5.3) for $i = 1, \dots, N$. A tuple $(\overline{x}^i)_{i \in \mathcal{I}}$ is called a robust Nash equilibrium of game (5.2), if $\overline{x}^i \in \operatorname{argmin}_{x^i \in S_i} \tilde{f}_i(x^i, \overline{x}^{-i})$ for all i, i.e., a Nash equilibrium of game (5.4). The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.

Finally, we give sufficient conditions for the existence of robust Nash equilibria. Since the following theorem follows directly from Nash's equilibrium existence theorem [4, Theorem 9.1.1], we omit the proof.

Theorem 5.2. *Suppose that, for every player $i \in \mathcal{I}$,* (i) *the strategy set S_i is nonempty, convex and compact,* (ii) *the worst cost function $\tilde{f}_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to \mathbb{R}$ is continuous, and* (iii) *$\tilde{f}_i(\cdot, x^{-i})$ is convex for any $x^{-i} \in S_{-i}$. Then, game* (5.4) *has at least one Nash equilibrium, i.e., game* (5.2) *has at least one robust Nash equilibrium.*

5.2 SDCP reformulation of robust Nash equilibrium problems

In this subsection, we focus on the games in which each player takes mixed strategy and minimizes a convex quadratic cost function with respect to his own strategy. For such games, we show that each player's optimization problem can be reformulated as an SDP, and the robust Nash equilibrium problem reduces to an SDCP.

Originally, SDCP [18, 37] is a problem of finding, for a given mapping $F : \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m \to \mathcal{S}^n \times \mathbb{R}^m$, a triple $(X, Y, z) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ such that

$$\mathcal{S}^n_+ \ni X \perp Y \in \mathcal{S}^n_+, \quad F(X, Y, z) = 0,$$

where $X \perp Y$ means $\mathrm{tr}(XY) = 0$. SDCP can be solved by some modern algorithms such as a non-interior continuation method [18].

Throughout this subsection, the cost functions and the strategy sets satisfy the followings.

(i) Player i's cost function $f_i^{\hat{u}^i}$ is defined by*6

$$f_i(x^i, x^{-i}) = \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij} \hat{x}^j, \tag{5.5}$$

where $\hat{A}_{ij} \in \mathbb{R}^{m_i \times m_j}$ $(j \in \mathcal{I}_{-i})$ are given constants involving uncertainties.

(ii) Player i takes mixed strategy, i.e.,

$$S_i = \{x^i \in \mathbb{R}^{m_i} \mid x^i \geq 0,\ \mathbf{1}_{m_i}^\top x^i = 1\} \tag{5.6}$$

where $\mathbf{1}_{m_i}$ denotes $(1, 1, \ldots, 1)^\top \in \mathbb{R}^{m_i}$.

(iii) $m_i \geq 3$ for all $i \in \mathcal{I}$.

We call $\hat{A}_{ij}$ a cost matrix. Note that these constants correspond to the cost function parameter $\hat{u}^i$, i.e.,

$$\hat{u}^i = \mathrm{vec}\begin{bmatrix} \hat{A}_{i1} & \ldots, \hat{A}_{iN} \end{bmatrix} \in \mathbb{R}^{m_i m}$$

*6 Although we can consider the additional term $c^\top x$, for simplicity, we omit the term.

where vec denotes the vectorization operator that creates an nm-dimensional vector $[(p_1^c)^\top \cdots (p_m^c)^\top]^\top$ from a matrix $P \in \Re^{n\times m}$ with column vectors $p_1^c, \ldots, p_m^c \in \mathbb{R}^n$.

For the robust Nash equilibrium problem with the above cost functions and strategy sets, Hayashi et al. [26] and Nishimura et al. [33] showed that it can be reformulated as an SOCCP. Since the SOCCP can be solved by some existing algorithms, we can calculate the robust Nash equilibria efficiently. However, they have only dealt with the case where the uncertainty is contained in either opponents' strategies or each player's cost matrices and vectors.

In this subsection, we consider the case where each player cannot exactly estimate both the cost matrices and the opponents' strategies. For such a case, we first show the existence of a robust Nash equilibrium, and then, prove that the robust Nash equilibrium problem can be reformulated as an SDCP. To this end, we make the following assumption.

Assumption 6. *For each $i \in \mathcal{I}$, the uncertainty sets $X_i(\cdot)$ and U_i are given as follows.*

(a) $X_i(x^{-i}) = \prod_{j\in\mathcal{I}_{-i}} X_{ij}(x^j)$*, where* $X_{ij}(x^j) = \{x^j + \delta x^{ij} \mid \|\delta x^{ij}\| \le \sigma_{ij},\ \mathbf{1}_{m_j}^\top \delta x^j = 0\ (j \in \mathcal{I}_{-i})\}$ *for some nonnegative scalar σ_{ij}.*

(b) $U_i = \prod_{j\in\mathcal{I}_{-i}} D_{ij}$*, where* $D_{ij} := \{A_{ij} + \delta A_{ij} \in \mathbb{R}^{m_i\times m_j} \mid \|\delta A_{ij}\|_F \le \rho_{ij}\}$ *for some nonnegative scalar ρ_{ij}. Moreover, $A_{ii} + \rho_{ii} I$ is symmetric and positive semidefinite.*

Assumption 6 claims that $X_{ij}(x^j)$ is the closed sphere with center x^j and radius σ_{ij} in the subspace $\{x \in \mathbb{R}^{m_j} \mid \mathbf{1}_{m_j}^\top x = 0\}$, and D_{ij} is also the closed sphere with center A_{ij} and radius ρ_{ij}. Note that Assumption 6 is milder than the assumptions made by Hayashi et al. [26] and Nishimura et al. [33]. Indeed, Assumption 6 with either $\rho_{ij} = 0$ or $\sigma_{ij} = 0$ for all $(i, j) \in \mathcal{I} \times \mathcal{I}$ corresponds to their assumptions.

Under Assumption 6, we rewrite each player i's optimization problem (5.4). Note that the worst cost function $\tilde{f}_i$ can be written as

$$
\begin{aligned}
&\tilde{f}_i(x^i, x^{-i}) \\
&\quad = \max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij}\hat{x}^j \;\middle|\; \begin{array}{l} \hat{A}_{ii} \in D_{ii}, \\ \hat{A}_{ij} \in D_{ij},\ \hat{x}^j \in X_{ij}(x^j)\ (j \in \mathcal{I}_{-i}) \end{array} \right\} \\
&\quad = \max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \;\middle|\; \hat{A}_{ii} \in D_{ii} \right\} + \sum_{j\in\mathcal{I}_{-i}} \max\left\{ (x^i)^\top \hat{A}_{ij}\hat{x}^j \;\middle|\; \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\} \\
&\quad = \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j\in\mathcal{I}_{-i}} \max\left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \;\middle|\; \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\}, \qquad (5.7)
\end{aligned}
$$

where the last equality holds since

$$\begin{aligned}
\max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \,\middle|\, \hat{A}_{ii} \in D_{ii} \right\} &= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max\left\{ \frac{1}{2}(x^i)^\top \delta A_{ii} x^i \,\middle|\, \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max\left\{ \frac{1}{2}(x^i \otimes x^i)\,\mathrm{vec}(\delta A_{ii}) \,\middle|\, \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \frac{1}{2}\rho_{ii}\|x^i\|^2 \\
&= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i .
\end{aligned}$$

Hence, each player i's optimization problem (5.4) can be rewritten as follows:

$$\begin{aligned}
\underset{x^i}{\text{minimize}} \quad & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j \in \mathcal{I}_{-i}} \max\left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \,\middle|\, \hat{A}_{ij} \in D_{ij},\, \hat{x}^j \in X_{ij}(x^j) \right\} \\
\text{subject to} \quad & \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \ge 0.
\end{aligned} \tag{5.8}$$

Now we show the existence of a robust Nash equilibrium under Assumption 6.

Theorem 5.3. *Suppose that the cost functions and the strategy sets are given by* (5.5) *and* (5.6)*, respectively. Suppose further that Assumption* 6 *holds. Then, there exists at least one robust Nash equilibrium.*

Proof. It suffices to show that the worst cost function $\tilde{f}_i$ and the strategy set S_i satisfy the three conditions given in Theorem 5.2. From (5.6), S_i is obviously nonempty, convex and compact. From (5.7), $\tilde{f}_i$ is continuous. Moreover, $\tilde{f}_i(\cdot, x^{-i})$ is convex for arbitrarily fixed $x^{-i} \in S_{-i}$ since we have (5.7), $A_{ii} + \rho_{ii} I \succeq 0$, and [15, Proposition 1.2.4(c)]. □

Next we show that problem (5.8) can be rewritten as an SDP. We note that problem (5.8) has a structure analogous to problem (3.2), and $X_{ij}(x^j)$ and D_{ij} satisfy Assumption 3. Indeed, $X_{ij}(x^j)$ can be constructed by the vectors γ^{ijk} ($k = 1, \ldots, m_j - 1$) which from orthogonal bases of the subspace $\{x \mid \mathbf{1}_{m_j}^\top x = 0\}$ with $\|\gamma^{ijk}\| = \sigma_{ij}$ for all k. Thus, by Theorem 3.4, problem (5.8) can be rewritten as the following SDP:

$$\begin{aligned}
\underset{x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}}{\text{minimize}} \quad & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i - \sum_{j \in \mathcal{I}_{-i}} \lambda_{ij} \\
\text{subject to} \quad & \begin{bmatrix} P_0^{ij}(x^i) & q^{ij}(x^i, x^j) \\ q^{ij}(x^i, x^j)^\top & r^{ij}(x^i, x^j) - \lambda_{ij} \end{bmatrix} \succeq \alpha_{ij} \begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1 \end{bmatrix} + \beta_{ij} \begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1 \end{bmatrix}, \ (j \in \mathcal{I}_{-i}) \\
& \alpha^{-i} = (\alpha_{ij})_{j \in \mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \quad \beta^{-i} = (\beta_{ij})_{j \in \mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \\
& \lambda^{-i} = (\lambda_{ij})_{j \in \mathcal{I}_{-i}} \in \mathbb{R}^{N-1}, \\
& \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \ge 0,
\end{aligned} \tag{5.9}$$

where

$$
\begin{aligned}
P_0^{ij}(x^i) &= -\frac{1}{2}\begin{bmatrix} 0 & \rho_{ij}(\Xi_{ij}^\top((x^i)^\top \otimes I_{m_j}))^\top \\ \rho_{ij}\,\Xi_{ij}^\top((x^i)^\top \otimes I_{m_j}) & 0 \end{bmatrix},\\
q^{ij}(x^i, x^j) &= -\frac{1}{2}\begin{bmatrix} \rho_{ij}((x^i)^\top \otimes I_{m_j})^\top x^j \\ \Xi_{ij}^\top A_{ij}^\top x^i \end{bmatrix}, \quad r^{ij}(x^i, x^j) = -(x^j)^\top A_{ij}^\top x^i, \\
P_1^{ij} &= \begin{bmatrix} -I_{m_i m_j} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -I_{m_j - 1} \end{bmatrix},\\
\Xi_{ij} &= \begin{bmatrix} \xi^{ij1} & \cdots & \xi^{ij(m_j-1)} \end{bmatrix}.
\end{aligned}
\tag{5.10}
$$

Finally, we show that the robust Nash equilibrium problem reduces to an SDCP. Since the semidefinite constraints in (5.9) are linear with respect to x^i, α^{-i}, β^{-i} and λ^{-i}, we can rewrite the constraints as

$$
\sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij} M_\lambda^{ij} \succeq \alpha_{ij} M_\alpha^{ij} + \beta_{ij} M_\beta^{ij}, \quad (j \in \mathcal{I}_{-i}),
$$

with $M_k^{ij} \in \mathcal{S}^{m_j(m_i+1)}$ $(k = 1, \ldots, m_i)$, $M_\lambda^{ij}, M_\alpha^{ij}, M_\beta^{ij} \in \mathcal{S}^{m_j(m_i+1)}$ defined by

$$
\begin{gathered}
M_k^{ij}(x^j) := \begin{bmatrix} P_0^{ij}(e_k^{(m_i)}) & q^{ij}(e_k^{(m_i)}, x^j) \\ q^{ij}(e_k^{(m_i)}, x^j)^\top & r^{ij}(e_k^{(m_i)}, x^j) \end{bmatrix},\\
M_\lambda^{ij} := -e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)} \left(e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)}\right)^\top, \quad M_\alpha^{ij} := \begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1 \end{bmatrix}, \quad M_\beta^{ij} := \begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1 \end{bmatrix},
\end{gathered}
$$

respectively. Then, the Karush-Kuhn-Tucker (KKT) conditions for (5.9) are given by

$$
\begin{gathered}
((A_{ii} + \rho_{ii} I)x^i)_k - \sum_{j \in \mathcal{I}_{-i}} \operatorname{tr}(Z^{ij} M_k^{ij}(x^j)) - (\mu_x^i)_k + \nu^i = 0, \quad (k = 1, \ldots, m_i),\\
\operatorname{tr}(Z^{ij} M_\alpha^{ij}) - (\mu_\alpha^i)_j = 0, \quad (j \in \mathcal{I}_{-i}),\\
\operatorname{tr}(Z^{ij} M_\beta^{ij}) - (\mu_\beta^i)_j = 0, \quad (j \in \mathcal{I}_{-i}),\\
\operatorname{tr}(Z_{ij} M_\lambda^{ij}) + 1 = 0, \quad (j \in \mathcal{I}_{-i}),\\
\operatorname{tr}\left(Z^{ij} \Big(\sum_{k=1}^{m_1} x_k^i M_k^{ij}(x^j) + \lambda_{ij} M_\lambda^{ij} - \alpha_{ij} M_\alpha^{ij} - \beta_{ij} M_\beta^{ij} \Big) \right) = 0,\\
(\mu_\alpha^i)^\top \alpha^{-i} = 0, \quad (\mu_\beta^i)^\top \beta^{-i} = 0, \quad (\mu_x^i)^\top x^i = 0,\\
\sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij} M_\lambda^{ij} \succeq \alpha_{ij} M_\alpha^{ij} + \beta_{ij} M_\beta^{ij}, \quad (j \in \mathcal{I}_{-i}),\\
\mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \geq 0, \quad \alpha^{-i} \geq 0, \quad \beta^{-i} \geq 0,\\
Z^{ij} \succeq 0, \quad \mu_x^i \geq 0, \quad \mu_\alpha^i \geq 0, \quad \mu_\beta^i \geq 0,
\end{gathered}
$$

where $Z^{ij} \in \mathcal{S}^{m_j(m_i+1)}$, $\mu_x^i \in \mathbb{R}^{m_i}$, $\mu_\alpha^i, \mu_\beta^i \in \mathbb{R}^{N-1}$ and $\nu^i \in \mathbb{R}$ are Lagrange multipliers. Eliminating

μ_x^i, μ_α^i and μ_β^i, we obtain the following conditions for each $i \in \mathcal{I}$:

$$
\begin{aligned}
&\mathcal{S}_+^{m_i(m_j+1)} \ni Z^{ij} \perp \sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij} M_\lambda^{ij} - \alpha_{ij} M_\alpha^{ij} - \beta_{ij} M_\beta^{ij} \in \mathcal{S}_+^{m_i(m_j+1)}, \quad (j \in \mathcal{I}_{-i}),\\
&\qquad \mathbb{R}_+^{m_i} \ni x^i \perp \Big(((A_{ii} + \rho_{ii} I)x^i)_k - \sum_{j \in \mathcal{I}_{-i}} \operatorname{tr}(Z^{ij} M_k^{ij}(x^j)) + \nu^i \Big)_{k=1,\dots,m_i} \in \mathbb{R}^{m_i},\\
&\quad \mathbb{R}_+^{N-1} \ni \alpha^{-i} \perp \operatorname{tr}(Z^{ij} M_\alpha^{ij})_{j \in \mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \quad \mathbb{R}_+^{N-1} \ni \beta^{-i} \perp \operatorname{tr}(Z^{ij} M_\beta^{ij})_{j \in \mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1},\\
&\qquad\qquad \operatorname{tr}(Z^{ij} M_\lambda^{ij}) = -1, \ (j \in \mathcal{I}_{-i}), \quad \mathbf{1}_{m_i}^\top x^i = 1.
\end{aligned}
\tag{5.11}
$$

Noticing that the above KKT conditions hold for all players simultaneously, the robust Nash equilibrium problem can be reformulated as the problem of finding $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j \in \mathcal{I}_{-i}}, \nu^i)_{i \in \mathcal{I}}$ such that (5.11) for all $i \in \mathcal{I}$. Thus, we obtain the following theorem.

Theorem 5.4. *Suppose that the cost functions and the strategy sets are given by* (5.5) *and* (5.6), *respectively. Suppose further that Assumption* 6 *holds. Then, x^* is a robust Nash equilibrium if and only if $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j \in \mathcal{I}_{-i}}, \nu^i)_{i \in \mathcal{I}}$ is a solution of SDCP* (5.11).

6 Numerical experiments

In this section, we report some numerical results on the SDP/SDCP reformulation approaches discussed in the previous sections. Particularly, we solve the robust second-order cone programming problems and the robust Nash equilibrium problems, to observe the efficiency of our approach and the properties of obtained solutions. All programs are coded in MATLAB 7.4.0 and run on a machine with Intel® Core 2 DUO 3.00GHz CPU and 3.20GB memories.

6.1 Robust second-order cone programming problems

In this subsection, we show some numerical results on the robust SOCPs discussed in Section 4. We consider the following robust SOCP with one second-order cone constraint and linear equality constraints:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f^\top x\\
\text{subject to} \quad & \|\hat{A}x + \hat{b}\| \le \hat{c}^\top x + \hat{d}, \quad \forall (\hat{A}, \hat{b}, \hat{c}, \hat{d}) \in \mathcal{U},\\
& A_{eq} x = b_{eq},
\end{aligned}
\tag{6.1}
$$

where $\hat{A} \in \mathbb{R}^{m \times n}, \hat{b} \in \mathbb{R}^m, \hat{c} \in \mathbb{R}^n$, and $\hat{d} \in \mathbb{R}$ are uncertain data with uncertainty set $\mathcal{U}$, and $A_{eq} \in \mathbb{R}^{m_{eq} \times n}$ and $b_{eq} \in \mathbb{R}^{m_{eq}}$ are given constants. Notice that the second-order cone constraint is always active if $m_{eq} < n$ and problem (6.1) is solvable.

6.1.1 Experiment 1

In the first experiment, we generate 100 random test problems with *ellipsoidal* uncertainties, and another 100 random test problems with *spherical* uncertainties. Then, we solve each problem by our SDP reformulation approach, to confirm that the obtained solution is surely the original RC solution when the sufficient condition (e.g., Assumption 4 with condition (4.8), or Assumption 5) is satisfied. For solving each SDP, we use SDPT3 [35] solver based on the infeasible path-following method.

We generate each test problem (6.1) as follows. We first let $(n, m_{eq}, m) := (5, 2, 5)$, and $A^0 \in \mathbb{R}^{m\times n}, b^0 \in \mathbb{R}^m, c^0 \in \mathbb{R}^n, d^0 \in \mathbb{R}, A_{eq} \in \mathbb{R}^{m_{eq}\times n}, b_{eq} \in \mathbb{R}^{m_{eq}}$ and $f \in \mathbb{R}^n$ be randomly chosen so that each component follows the uniform distribution in the interval $[-5, 5]$. We also choose κ randomly from the interval $[0.01, 0.1]$ according to the uniform distribution. Moreover, we determine the uncertainty set $\mathcal{U}$ by using either of the two procedures corresponding to the ellipsoidal and spherical uncertainty cases. In both cases, $\mathcal{U}$ is determined so that the *relative error* is at most κ, i.e.,

$$\max_{X\in\mathcal{U}} \operatorname{dist}\left(X, \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix}\right) = \kappa \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{matrix} \right\|_F .$$

Procedure 6.1 (Ellipsoidal uncertainty case)**.** *Generate* $(A^j, b^j, c^j, d^j)_{j=1,\ldots,(m+1)(n+1)}$ *as follows:*

1. *Generate the random matrices*

$$\begin{bmatrix} \tilde{A}^j & \tilde{b}^j \\ (\tilde{c}^j)^\top & \tilde{d}^j \end{bmatrix} \in \mathbb{R}^{(m+1)\times(n+1)}, \quad j = 1, \ldots, (m+1)(n+1)$$

so that each component follows the uniform distribution in the interval $[-1, 1]$.

2. *Let*

$$\tau := \max_{X\in\tilde{\mathcal{U}}} \operatorname{dist}\left(X, \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix}\right)$$

where

$$\tilde{\mathcal{U}} = \left\{ \begin{bmatrix} \hat{A} & \hat{b} \\ \hat{c}^\top & \hat{d} \end{bmatrix} \middle| \begin{bmatrix} \hat{A} & \hat{b} \\ \hat{c}^\top & \hat{d} \end{bmatrix} = \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix} + \sum_{j=1}^{(m+1)(n+1)} u_j \begin{bmatrix} \tilde{A}^j & \tilde{b}^j \\ (\tilde{c}^j)^\top & \tilde{d}^j \end{bmatrix},\ u^\top u \le 1 \right\}.$$

3. *Let*

$$\begin{bmatrix} A^j & b^j \\ (c^j)^\top & d^j \end{bmatrix} := \frac{\kappa}{\tau} \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{matrix} \right\|_F \begin{bmatrix} \tilde{A}^j & \tilde{b}^j \\ (\tilde{c}^j)^\top & \tilde{d}^j \end{bmatrix}, \quad j = 1, \ldots, (m+1)(n+1).$$

Then, define $\mathcal{U}$ *by*

$$\mathcal{U} := \left\{ \begin{bmatrix} \hat{A} & \hat{b} \\ \hat{c}^\top & \hat{d} \end{bmatrix} \middle| \begin{bmatrix} \hat{A} & \hat{b} \\ \hat{c}^\top & \hat{d} \end{bmatrix} = \begin{bmatrix} A^0 & b^0 \\ (c^0)^\top & d^0 \end{bmatrix} + \sum_{j=1}^{(m+1)(n+1)} u_j \begin{bmatrix} A^j & b^j \\ (c^j)^\top & d^j \end{bmatrix},\ u^\top u \le 1 \right\}.$$

Procedure 6.2 (Spherical uncertainty case)**.** *Let*

$$\rho = \kappa \left\| \begin{matrix} A^0 & b^0 \\ (c^0)^\top & d \end{matrix} \right\|_F .$$

Then, define U by

$$\mathcal{U} := \left\{ (\hat{A}, \hat{b}, \hat{c}, \hat{d}) = (A^0 + \delta A, b^0 + \delta b, c^0 + \delta c, d^0 + \delta d) \;\middle|\; \left\| \begin{bmatrix} \delta A & \delta b \\ (\delta c)^\top & \delta d \end{bmatrix} \right\|_F \leq \rho \right\}.$$

We show the obtained results in Table 1, in which "prob.", N_{suf} and N_{suc} denote the number of solvable problem instances, the number of times that condition (4.8) holds (which applies only to the ellipsoidal case), and the number of times that original RC solution is obtained, respectively. In practice, we decide that condition (4.8) holds when all eigenvalues are greater than 10^{-6}, and that the original RC solution is obtained when $\mathrm{val}\,(4.5) - \mathrm{val}\,(4.9) < 10^{-6}$ holds. (That is, we also solve the Hildebrand-based SDP (4.9) for each test problem, and compare val (4.9) with val (4.5).)

Table. 1 The number of solvable instances by the proposed reformulation

	prob.	N_{suf}	N_{suc}
ellipsoidal	100	98	98
spherical	100	–	100

Table 1 shows that, in the spherical case, the proposed SDP reformulation approach finds the original RC solution for all instances. In the ellipsoidal uncertainty case, our approach cannot find the RC optimum for two instances. However, both of them do not satisfy condition (4.8). Hence, the obtained result indicates that our SDP reformulation approach always finds the RC optimum under the sufficient conditions such as Assumption 4 with (4.8), or Assumption 5.

6.1.2 Experiment 2

In this section, we solve 200,000 problem instances with ellipsoidal uncertainties by our SDP reformulation approach. Especially, this experiment is motivated from the following three questions:

- How often does condition (4.8) hold when our SDP reformulation approach is applied?
- If condition (4.8) does *not* hold, how often does the optimum of SDP (4.5) solve the original RC?
- If the optimum of SDP (4.5) does not solve the original RC, how much is the difference between the optimal value of SDP (4.5) and that of the original RC?

We generate 200,000 test problems of the form (6.1) as follows. We first generate 1,000 nominal problems*7 such that (i) $(n, m_{eq}, m) = (5, 2, 5)$, (ii) $A^0, b^0, c^0, d^0, A_{eq}, b_{eq}$ and f are random matri-

*7 The problem where $(\hat{A}, \hat{b}, \hat{c}, \hat{d})$ is replaced by (A^0, b^0, c^0, d^0) is called nominal problem.

ces and vectors whose components follow the uniform distribution in the interval $[-5, 5]$, and (iii) each nominal problem has an optimal solution[*8]. Moreover, for each nominal problem, we generate 200 ellipsoidal uncertainty sets $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(200)}$ as follows: we generate $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \dots, \mathcal{U}^{(100)}$ by Procedure 6.1 with relative error $\kappa = 0.01$, and then, set $\mathcal{U}^{(i+100)} := 10\mathcal{U}^{(i)}$ for $i = 1, 2, \dots, 100$, i.e., $\mathcal{U}^{(101)}, \dots, \mathcal{U}^{(200)}$ correspond to the case of $\kappa = 0.1$ and their shapes are similar to $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(100)}$, respectively. Thus, we have 1,000 problem groups, each of which contains 200 instances sharing the same nominal data $A^0, b^0, c^0, d^0, A_{eq}, b_{eq}$ and f.

The obtained results are shown in Tables 2 and 3. Table 2 shows the number of times that the reformulated SDP (4.5) becomes feasible for each κ. In Table 3, we focus on only 9 problem groups, say Group 1 – Group 9, each of which contains at least one instance such that the reformulated SDP (4.5) is feasible but condition (4.8) does not hold. (In other 991 groups, every instance satisfies condition (4.8) if the reformulated SDP (4.5) is feasible.) Each column in Table 3 denotes the number of feasible instances (feas.), the number of instances that condition (4.8) holds (N_{suf}), the number of instances that the original RC solutions are obtained (N_{suc}), and the mean of the relative error, i.e.,

$$\text{Error} = \text{Mean}\left(\frac{\text{val}\,(4.5) - \text{val}\,(4.9)}{|\,\text{val}\,(4.9)|}\right)$$

where the mean value is taken among the instances violating condition (4.8). Note that the RC optimality is determined by Hildebrand-based SDP (4.9), similarly to the previous experiment.

From these tables, we can see that condition (4.8) holds in most cases. However, we can also see that, if condition (4.8) does not hold, then the optimum of SDP (4.5) often violates the optimality of the original problem (6.1). For example, in case of $\kappa = 0.01$, only 6 among 77,367 feasible instances violate condition (4.8), where the number 6 comes from the sum of (feas. $- N_{\text{suf}}$) in Table 3. However, among those 6 instances, we failed to find the optimum of (6.1) for 5 times, where the number 5 comes from the sum of (feas. $- N_{\text{suc}}$) in Table 3. On the other hand, when $\kappa = 0.1$, no less than 66 instances violate condition (4.8). This result indicates that condition (4.8) is less likely to hold as κ becomes larger. However, for all instances, the relative error of the optimal value is sufficiently small (less than 1%). In other words, our SDP reformulation approach finds almost optimal solutions even if (4.8) does not hold. In addition to the above experiments, we examined the relationship between the likelihood of (4.8) and the shape[*9] of the ellipsoid $\mathcal{U}$. However, we could not see any relevance between them. We hence expect that, whether condition (4.8) holds or not mainly depends on the nominal problem and the size of the uncertainty set.

[*8] Note that, if a nominal problem has an optimal solution, then the objective function value of problem (6.1) is bounded below. (The feasible region of problem (6.1) becomes smaller as κ becomes larger.)

[*9] More precisely, we examined the condition number of a certain matrix that characterizes the shapes of the ellipsoid $\mathcal{U}$. The condition number of matrix H is defined as (maximum singular value of H)/(minimum singular value of H). If the condition number is 1, then $\mathcal{U}$ is a sphere. If the condition number is large, then $\mathcal{U}$ becomes a distorted ellipsoid.

Table. 2 The number of feasible instances

	$\kappa = 0.01$	$\kappa = 0.1$
total	100,000	100,000
feasible	77,367	46,927

Table. 3 Detailed results for the 9 problem groups

	$\kappa = 0.01$				$\kappa = 0.1$			
	feas.	N_{suf}	N_{suc}	Error	feas.	N_{suf}	N_{suc}	Error
Group 1	100	100	100	–	100	93	93	1.70×10^{-3}
Group 2	100	100	100	–	100	94	94	5.73×10^{-4}
Group 3	100	100	100	–	100	98	98	3.21×10^{-5}
Group 4	100	99	99	1.78×10^{-4}	0	0	0	–
Group 5	2	1	1	3.85×10^{-5}	0	0	0	–
Group 6	100	96	97	9.96×10^{-6}	100	72	75	1.36×10^{-4}
Group 7	100	100	100	–	100	81	86	8.62×10^{-4}
Group 8	100	100	100	–	100	97	98	1.10×10^{-3}
Group 9	100	100	100	–	100	99	99	6.76×10^{-3}

6.1.3 Experiment 3

Finally, we compare our SDP reformulation approach with Hildebrand-based one in terms of the computation time. In this experiment, we vary the values of n and m, i.e., the dimensions of decision variables and the second-order cone in problem (6.1). We generate 100 random test problems with ellipsoidal uncertainties for each (n, m). In a way similar to the previous subsections, we let $A^0 \in \mathbb{R}^{m\times n}$, $b^0 \in \mathbb{R}^m$, $c^0 \in \mathbb{R}^n$, $d^0 \in \mathbb{R}$, $A_{eq} \in \mathbb{R}^{m_{eq}\times n}$, $b_{eq} \in \mathbb{R}^{m_{eq}}$ and $f \in \mathbb{R}^n$ be randomly chosen from the interval $[-5, 5]$, and determine the uncertainty set $\mathcal{U}$ by Procedure 6.1 with $\kappa = 0.01$. Then, we solve each test problem by our SDP reformulation approach and Hildebrand-based one and take the computation time to require in each approach.

The result is shown in Table 4, in which "add. var" and "matrix size" denote the number of additional variables and the size of the square matrix in the semidefinite constraint, respectively. Similarly to Table 1, N_{suf} denotes the number of times that condition (4.8) holds. Also, "–" means failure due to out of memory.

Table 4 shows that our SDP reformulation approach solves all test problems within a reasonable time, whereas Hildebrand-based approach is much more expensive and does not work anymore for

Table. 4 Our approach vs. Hildebrand-based approach in terms of CPU time

dimension (n, m)	our approach			Hildebrand-based approach			N_{suf}
	add. var.	matrix size	Time [sec]	add. var.	matrix size	Time [sec]	
(3, 3)	2	20	0.3331	360	48	0.7189	100
(4, 4)	2	30	0.3638	1800	100	9.8358	100
(5, 5)	2	42	0.3927	6300	180	236.9636	100
(6, 6)	2	56	0.5615	17640	294	–	100
(10, 10)	2	132	2.3691	326700	1210	–	100
(20, 20)	2	462	39.5398	1.8×10^7	8820	–	100

$n, m \geq 6$. Particularly, the number of additional variables for Hildebrand-based approach grows explosively as n or m becomes larger. Thus, we can conclude that our SDP reformulation approach outperforms Hildebrand-based one in terms of computation time.

6.2 Robust Nash equilibrium problems

In this subsection, we solve some robust Nash equilibrium problems with uncertainties in both the cost matrices and the opponents' strategies, by using the SDCP reformulation approach proposed in Section 5. Then, we change the size of uncertainty sets variously, and observe some properties of the obtained equilibria. For solving the reformulated SDCPs, we apply the Fisher-Burmeister type merit function approach proposed by Yamashita and Fukushima [37]. In minimizing the merit function, we use `fminunc` in MATLAB Optimization toolbox.

In this experiment, we consider the two-person robust Nash equilibrium problem where the cost functions and the strategy sets are given by (5.5) and (5.6), respectively. We also suppose that Assumption 6 holds with

$$A_{11} = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 8 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 6 & -1 \\ 2 & -1 & 9 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} -1 & -9 & 11 \\ 10 & -1 & 4 \\ 3 & 10 & 1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -5 & -4 & -8 \\ -1 & 0 & 5 \\ 3 & 1 & 4 \end{bmatrix}$$

$\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} = \sigma$ and $\rho_{12} = \rho_{21} = \rho$, where (ρ, σ) is chosen from $\{0, 1, 2\} \times \{0, 0.01, 0.1\}$. Table 5 shows the obtained robust Nash equilibria with various choice of (ρ, σ). Note that the robust Nash equilibrium with $(\rho, \sigma) = (0, 0)$ corresponds to the Nash equilibrium with $\hat{A}^{ij}$ and $\hat{x}^j$ $(i, j = 1, 2)$ in (5.5) replaced by A^{ij} and x^j, respectively. Figures 1 and 2 show the trajectories of each player's strategies at the robust Nash equilibria, in which the horizontal and vertical axes denote the first and second components of three-dimensional vectors, respectively[*10]. Each figure contains three trajectories with $\rho \in \{0, 1, 2\}$, and each trajectory consists of three points corresponding to $\sigma \in \{0, 0.01, 0.1\}$ Table 5, Figures 1 and 2 indicate that the robust Nash equilibria monotonically move as σ becomes larger, and the trajectories resemble each other. Although we omit figures, the above properties hold for the trajectory with respect to ρ.

Table. 5 Size of uncertainty sets and robust Nash equilibria

ρ	σ	player 1	player 2
0	0	(0.7793, 0.0000, 0.2207)	(0.2903, 0.3243, 0.3854)
0	0.01	(0.7763, 0.0000, 0.2237)	(0.2945, 0.3275, 0.3780)
0	0.1	(0.7485, 0.0000, 0.2515)	(0.3307, 0.3570, 0.3123)
1	0	(0.7407, 0.0382, 0.2211)	(0.3272, 0.3310, 0.3418)
1	0.01	(0.7366, 0.0383, 0.2251)	(0.3297, 0.3340, 0.3362)
1	0.1	(0.6997, 0.0404, 0.2599)	(0.3521, 0.3623, 0.2856)
2	0	(0.6895, 0.0935, 0.2170)	(0.3501, 0.3398, 0.3102)
2	0.01	(0.6826, 0.0950, 0.2224)	(0.3515, 0.3415, 0.3069)
2	0.1	(0.6441, 0.0986, 0.2573)	(0.3687, 0.3682, 0.2631)

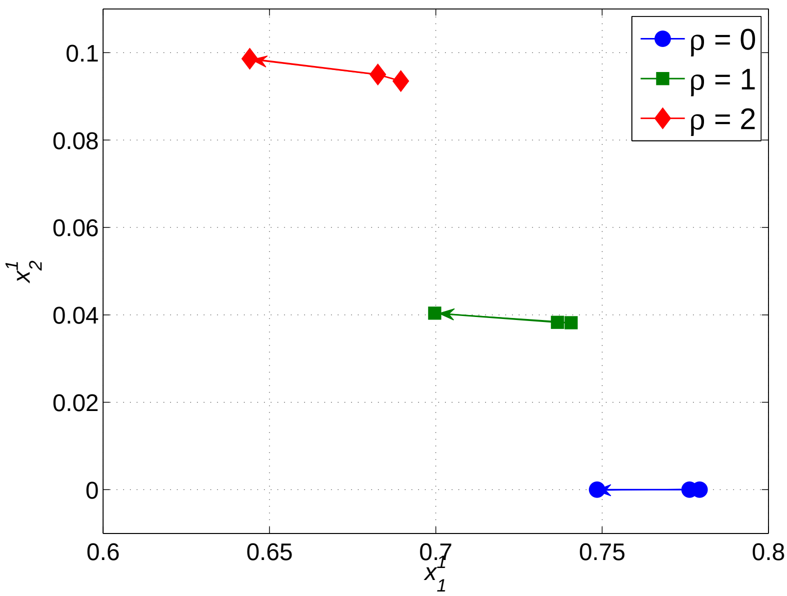

Fig. 1 Trajectory of player 1's strategy at the robust Nash equilibria with respect to σ

[*10] Since each player takes the mixed strategy, the last component is automatically determined.

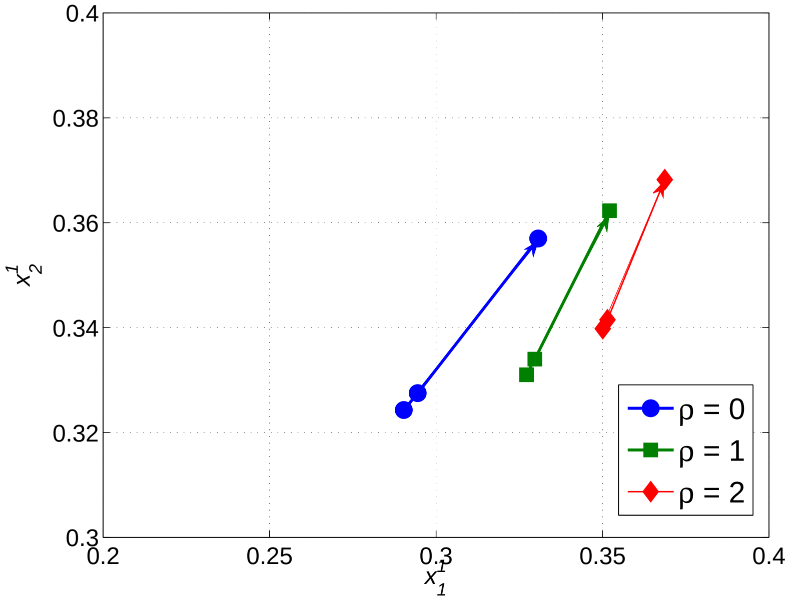

Fig. 2 Trajectory of player 2's strategy at the robust Nash equilibria with respect to σ

7 Concluding remarks

In this paper, we considered a class of LPs with ellipsoidal uncertainty, and constructed its RC as an SDP by exploiting the strong duality in nonconvex quadratic programs with two quadratic constraints. We showed that the optimum of the RC can be obtained by solving the SDP under an appropriate condition. Moreover, we showed that those two problems are equivalent when the uncertainty sets are spherical. By using the same technique, we reformulated the robust counterpart of SOCP with *one* ellipsoidal uncertainty as an SDP. We applied these ideas to the robust Nash equilibrium problem in which uncertainties are contained in both opponents' strategies and each player's cost parameters, and showed that it reduces to an SDCP. Finally, we carried out some numerical results, and investigated some empirical properties of our SDP reformulation approach and some behaviors of the robust Nash equilibria.

We still have some future issues to be addressed. (1) One important issue is to weaken the sufficient conditions for equivalence of the original RC and the proposed SDP. Especially, it seems to be interesting to study the case with some restricted classes of ellipsoids. (2) Another issue is to extend our reformulation approach to other classes of the robust optimization problems. (3) In this paper, we have reformulated the robust Nash equilibrium problem as a *nonlinear* SDCP. Since many efficient algorithms have been proposed for *linear* SDCPs, it may be useful to reduce the robust Nash equilibrium problem to a linear SDCP.

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References

[1] E. Adida and G. Perakis, *A robust optimization approach to dynamic pricing and inventory control with no backorders*, Mathematical Programming, 107 (2006), pp. 97–129.

[2] M. Aghassi and D. Bertsimas, *Robust game theory*, Mathematical Programming, 107 (2006), pp. 231–273.

[3] F. Alizadeh and D. Goldfarb, *Second-order cone programming*, Mathematical Programming, 95 (2003), pp. 3–51.

[4] J.-P. Aubin, *Mathematical Methods of Game and Economic Theory*, Dover Publications, New York, 2007.

[5] A. Beck and Y. C. Eldar, *Strong duality in nonconvex quadratic optimization with two quadratic constrains*, SIAM Journal on Optimization, 17 (2006), pp. 844–860.

[6] A. Ben-Tal, L. ElGhaoui, and A. Nemirovski, *Robust optimization.* draft, 2008.

[7] A. Ben-Tal, T. Margalit, and A. Nemirovski, *Robust modeling of multi-stage portfolio problems*, in High Performance Optimization, H. Frenk, K. Roos, T. Terlaky, and S. Zhang, eds., Dordrecht Kluwer, 2000, pp. 303–328.

[8] A. Ben-Tal and A. Nemirovski, *Stable truss topology design via semidefinite programming*, SIAM Journal on Optimization, 7 (1997), pp. 991–1016.

[9] ——, *Robust convex optimization*, Mathematics of Operations Research, 23 (1998), pp. 769–805.

[10] ——, *Robust solutions of uncertain linear programs*, Operations Research Letters, 25 (1999), pp. 1–13.

[11] ——, *Lectures on Modern Convex Optimization*, Society for Industrial & Applied Mathematics, Philadelphia, 2001.

[12] ——, *Extending scope of robust optimization: Comprehensive robust counterparts of uncertain*

problems, Mathematical Programming, 107 (2006), pp. 63–89.

[13] ——, *Selected topics in robust convex optimization*, Mathematical Programming, 112 (2008), pp. 125–158.

[14] A. Ben-Tal, A. Nemirovski, and C. Roos, *Robust solutions of uncertain quadratic and conic-quadratic problems*, SIAM Journal on Optimization, 13 (2002), pp. 535–560.

[15] D. P. Bertsekas, *Convex analysis and optimization*, Athena Scientific, 2003.

[16] D. Bertsimas and M. Sim, *The price of robustness*, Operations Research, 52 (2004), pp. 35–53.

[17] D. Bertsimas and A. Thiele, *Robust optimization approach to inventory theory*, Operations Research, 54 (2006), pp. 150–168.

[18] X. Chen and P. Tseng, *Non-interior continuation methods for solving semidefinite complementarity problems*, Mathematical Programming, 95 (2003), pp. 431–474.

[19] L. ElGhaoui and H. Lebret, *Robust solutions to least-squares problem with uncertain data*, SIAM Journal on Matrix Analysis and Applications, 18 (1997), pp. 1035–1064.

[20] L. ElGhaoui, M. Oks, and F. Oustry, *Worst-case Value-at-Risk and robust portfolio optimization: a conic programming approach*, Operations Research, 51 (2003), pp. 543–556.

[21] L. ElGhaoui, F. Oustry, and H. Lebret, *Robust solutions to uncertain semidefinite programs*, SIAM Journal on Optimization, 9 (1998), pp. 33–52.

[22] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second-order cone complementarity problems*, SIAM Journal on Optimization, 12 (2001), pp. 436–460.

[23] D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Mathematics of Operations Research, 28 (2003), pp. 1–37.

[24] D. A. Harville, *Matrix Algebra from a Statistician's Perspective*, Springer, 2007.

[25] S. Hayashi, N. Yamashita, and M. Fukushima, *A combined smoothing and regularization method for monotone second-order cone complementarity problems*, SIAM Journal on Optimization, 175 (2005), pp. 335–353.

[26] ——, *Robust Nash equilibria and second-order cone complementarity problems*, Journal of Nonlinear and Convex Analysis, 6 (2005), pp. 283–296.

[27] R. Hildebrand, *An LMI description for the cone of Lorentz-positive maps*, Linear and Multilinear Algebra, 55 (2007), pp. 551–573.

[28] ——, *An LMI description for the cone of Lorentz-positive maps II.* preprint, October 2008.

[29] D. S. Huang, F. J. Fabozzi, and M. Fukushima, *Robust portfolio selection with uncertain exit time using worst-case VaR strategy*, Operations Research Letters, 35 (2007), pp. 627–635.

[30] D. S. Huang, S. S. Zhu, F. J. Fabozzi, and M. Fukushima, *Portfolio selection with uncertain exit time: A robust CVaR approach*, Journal of Economic Dynamics and Control, 32 (2008), pp. 594–623.

[31] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, *Applications of second-order*

cone programming, Linear algebra and its applications, 284 (1998), pp. 193–228.

[32] M. LÓPEZ AND G. STILL, *Semi-infinite programming*, European Journal of Operational Research, 180 (2007), pp. 491–518.

[33] R. NISHIMURA, S. HAYASHI, AND M. FUKUSHIMA, *Robust Nash equilibria in N-person non-cooperative games: Uniqueness and reformulation*, Pacific Journal of Optimization, to appear.

[34] R. T. ROCKAFELLAR, *Convex analysis*, Princeton, 1970.

[35] R. H. TÜTÜNCÜ, K. C. TOH, AND M. J. TODD, *Solving semidefinite-quadratic-linear programs using SDPT3*, Mathematical Programming, 95 (2003), pp. 189–217.

[36] L. VANDENBERGHE AND S. BOYD, *Semidefinite programming*, SIAM Review, 38 (1996), pp. 49–95.

[37] N. YAMASHITA AND M. FUKUSHIMA, *A new merit function and a descent method for semidefinite complementarity problems*, in Reformulation – Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, M. Fukushima and L. Qi, eds., Kluwer Academic Publishers, 1999, pp. 405–420.

[38] P. ZHONG AND M. FUKUSHIMA, *Second-order cone programming formulations for robust multiclass classification*, Neural Computation, 19 (2007), pp. 258–282.

[39] S. S. ZHU AND M. FUKUSHIMA, *Worst-case conditional Value-at-Risk with application to robust portfolio management*, Operations Research, to appear.