

Pricing American Options with Uncertain Volatility through Stochastic Linear Complementarity Models

Guidance

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February 2010

Abstract

In recent years, stemming from the subprime mortgage problem, a serious financial crisis happened and the influence reaches the real economy. Some say that the financial crisis was partly caused by derivatives. A derivative can be defined as a financial instrument whose value depends on the value of underlying assets such as stock, bond, currency and rate of interest. Derivatives may be used for speculation purpose, but they are originally developed to hedge the risk of fluctuation of a commodity or an exchange.

Option is a kind of derivatives; it is the right to buy or sell the underlying assets by a certain date for a certain price. An option which can be exercised at any time during its life is called an American option. There are several ways to compute the price of American options. For example, using finite difference approximation, pricing American options can be formulated as a linear complementarity problem. The prices of American options are dependent on the asset price, the strike price, the expiration date, the risk-free rate and the volatility of the asset price. The Black-Scholes model assumes that these values are constant. However, it is especially difficult to set the volatility as a constant value. In fact, different experts usually make different estimates of the volatility.

In this paper, we consider the problem of pricing American options with uncertain volatility and propose two deterministic formulations based on the expected value method and the expected residual minimization method for a stochastic complementarity problem. We give sufficient conditions that ensure the existence of a solution of those deterministic formulations. Furthermore we show numerical results and discuss the usefulness of the proposed approach.

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1 Introduction

The world economy has fallen into serious recessions such as low stock prices and deflation as symbolized by the Lehman Shock that occurred in September, 2008. It is derivatives consisting of subprime mortgages for low-income persons that are regarded as one reason for the financial crisis. A derivative can be defined as a financial instrument whose value depends on the value of underlying assets such as stock, bond, currency and rate of interest [16]. Derivatives may be used for speculation purpose, which may be one of the causes of the crisis, but derivatives are usually used for hedging the risk of fluctuation of a commodity or an exchange.

Option is a kind of derivatives; it is the right to buy or sell the underlying assets by a certain date for a certain price. A call option is the right to buy an asset for a certain price. A put option is the right to sell an asset for a certain price. Here, the price at which the asset can be bought or sold in an option contract is called the strike price. A European option can be exercised only at the end of its life. An American option can be exercised at any time during its life. Here, the end of a contract is called the expiration date. Using the Black-Scholes model [3], we can compute the prices of European options explicitly under some assumptions. On the other hand, since an American option is permitted to exercise at any time of its life, we have to decide whether or not to exercise it and need to compute its boundary. Hence, pricing American options is more complicated than pricing European options. In particular, we cannot express the prices of American options explicitly and hence we can obtain the prices only by numerical computation. The binomial lattice model, finite difference approximation, and Monte Carlo simulation are used for pricing American options. In the binomial lattice model, we divide the time from now to the expiration date and create a binomial lattice representation of the asset price. Then, by backward induction on the lattice, we compute the prices of American options [10]. In the finite difference approximation method, we approximate the partial differential equation or partial differential inequality that the asset follows, and formulate pricing options as a linear complementarity problem [5, 15]. In Monte Carlo simulation, by sampling random paths of the process of the asset, we calculate the mean of the sample payoff and discount the expected payoff [4, 20].

The prices of European options and American options are dependent on the asset price, the strike price, the expiration date, the risk-free rate and the volatility of the asset price. The Black-Scholes model [3] assumes that these values are constant. Since we know the asset price and the strike price correctly and the contractor can decide the expiration date, these values are absolutely constant. Moreover, we can expect the risk-free rate easily by seeing the interest rate of the bank deposits or the national bonds. However, it is practically difficult to set the volatility as a constant value. That is because each expert has his own view for the volatility. Besides, even if we adopt a historical volatility, it may fluctuate according to the chosen period. In practice, traders work with what are known as implied volatility. The implied volatility is the value calculated backward using the asset price, the strike price, the expiration date, the risk-free rate and the price of option observed in the real market. Traders buy options if the implied volatility is comparatively low and sell options if it is comparatively high.

Recently, there have been many works on pricing options which suppose the volatility is not a constant value in order to remedy the shortcoming of the Black-Scholes model. In most of those works, the volatility of the asset is assumed to be stochastic and its variance is assumed to follow a mean-reverting process that indicates its tendency to return to a long-term average, which is called the stochastic volatility model. The stochastic volatility model [14] gives a closed-form

formula for the prices of the corresponding European options. For American options with varying volatility, their prices are obtained by using Heston model [14] via Monte Carlo simulation [8, 22]. However, the stochastic volatility model assumes that the volatility varies with time. So this model may not suit the situation where the volatility is constant until the expiration time but uncertain at the present time.

In this paper, we assume that the volatility itself follows some probability distribution such as normal distribution and propose the formulation for pricing American options through a stochastic linear complementarity model. The stochastic complementarity problem is the problem whose coefficients are random variables. Since there is in general no solution that satisfies the complementarity conditions for all realizations of the coefficient value simultaneously, some deterministic formulations are constructed. We propose two deterministic formulations for pricing American options with uncertain volatility through the expected value method [13] and the expected residual minimization method [6]. Moreover, by analyzing numerical results based on some criteria, we show the usefulness of the proposed approach.

This paper is organized as follows: In Section 2, we recall the Black-Scholes partial differential equation and formulate pricing American options as the linear complementarity problem. In Section 3, we describe the expected value method and the expected residual minimization method for the stochastic complementarity problem. In Section 4, we propose the formulations for pricing American options with uncertain volatility by means of the the expected value method and the expected residual minimization method. In Section 5, we discuss conditions that ensure the existence of a solution of the proposed formulations. Numerical results are presented and discussed in Section 6. Finally, Section 7 concludes the paper.

2 Pricing American options using linear complementarity models

2.1 Model of asset prices

In this subsection we discuss the model of the behavior of asset prices. The descriptions in this and the next subsections are largely based on [21]. Let S denote the asset price at time t . Consider a small time interval dt , during which S changes to $S + dS$. We can write the corresponding return on the asset as dS/S . The common model decomposes this return into two parts. One is a deterministic return like the return on money invested in a risk-free bank. It gives the contribution

$$\mu dt \tag{2.1}$$

to the return dS/S , where μ is a measure of the average rate of growth of the asset price. In this paper, μ is taken to be a constant. The second part is a random change in the asset price in response to external effects such as unexpected news. It adds the term

$$\sigma dX \tag{2.2}$$

to the return dS/S . Here σ is the standard deviation of returns, called the volatility, and dX is a Wiener process. The Wiener process has the following properties;

- dX has a normal distribution,
- the mean of dX is zero,

- the variance of dX is dt .

Putting (2.1) and (2.2) together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dX.$$

By multiplying both sides of the equation by S , we get the following equation:

$$dS = \mu S dt + \sigma S dX. \quad (2.3)$$

2.2 Black-Scholes partial differential equation

In this subsection, we recall the Black-Scholes partial differential equation which is used for pricing European options. First, we list some assumptions for pricing options considered in this paper.

- The asset price follows the stochastic differential equation (2.3).
- There are no arbitrage possibilities. This means that there is no opportunity to make an instantaneous risk-free profit.
- Trading of the asset can take place continuously.
- Short selling is permitted and the asset is divisible. This means that we may sell assets that we do not own, and we can buy and sell any number (not necessarily an integer) of the asset.

Let $V(S, t)$ denote the option price when the asset price is S and the time is t . The following lemma, called Ito's Lemma, plays an important role in pricing options [3].

Lemma 1. *The function $V(S, t)$ of S and t follows the process*

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

Now, consider the portfolio consisting of one option and a number $-\Delta$ of the asset. Then, the value of the portfolio is given by

$$\Pi = V - \Delta S \quad (2.4)$$

and the jump in the value of this portfolio in one time interval is written as

$$d\Pi = dV - \Delta dS. \quad (2.5)$$

Here Δ is fixed during the time interval. Putting (2.3), (2.5) and Lemma 1 together, we find that Π follows the stochastic differential equation:

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt. \quad (2.6)$$

By choosing

$$\Delta = \frac{\partial V}{\partial S}, \quad (2.7)$$

we can eliminate the random component in (2.6). Note that Δ is the value of $\partial V / \partial S$ at the start of the time interval dt . The increment of the portfolio becomes wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.8)$$

Let the risk-free rate be denoted by r . Then, the return on an amount Π invested in a bank is $r\Pi dt$ in the initial time dt . If the right-hand side of (2.8) is greater than $r\Pi dt$, we could make a guaranteed riskless profit by borrowing an amount Π from a bank and investing in the portfolio. Conversely, if the the right-hand side of (2.8) were less than $r\Pi dt$, we should short the portfolio and invest in the bank. Doing so, we also could make a riskless profit. Therefore, by the assumption of no arbitrage possibilities, we must have

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.9)$$

Substituting (2.4) and (2.7) into (2.9) and dividing both sides by dt , we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (2.10)$$

This is the Black-Scholes partial differential equation.

2.3 Pricing American options

In this subsection we discuss pricing American options. The content of this subsection is based on [15] and [21]. Since we can exercise American options at any time during the life of the option, we have to determine not only option prices but also, for each value of S , whether or not it should be exercised. This is what is known as a free boundary problem. Since it is difficult to deal with free boundary, we reformulate the problem in such a way as to eliminate any explicit dependence on the free boundary. We describe a linear complementarity formulation for American option pricing.

Since a holder of American options may miss the optimal exercise price, there are cases where the portfolio consisting of American options cannot bring as high profit as the money invested in a bank. So (2.9) is modified as

$$r\Pi dt \geq \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Therefore, instead of the Black-Scholes partial differential equation, we obtain the following Black-Scholes partial differential inequality:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0. \quad (2.11)$$

Let $\Lambda(S, t)$ denote the payoff function when the asset price is S and the time is t . Payoff means the amount of money earned by exercising the right of options. For a call option, the payoff function is given by $\Lambda(S, t) = \max(S(t) - E, 0)$, where E is the strike price. For a put option, the payoff function is given by $\Lambda(S, t) = \max(E - S(t), 0)$. If the price of an American option is less than the payoff, then an investor can earn the riskless profit by buying the option and immediately exercising it. Since there are no arbitrage opportunities, we must have

$$V(S, t) \geq \Lambda(S, t). \quad (2.12)$$

In addition, we have two choices for American options; we exercise the right of options or not. If we exercise, the price of an American options is equal to the payoff. If not, American options are

essentially the same as European options. This means that the Black-Scholes partial differential equation is valid. Thus, we obtain

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right) (V(S, t) - \Lambda(S, t)) = 0. \quad (2.13)$$

Putting (2.11), (2.12) and (2.13) together, we conclude that the prices of American options satisfy the partial differential complementarity condition:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &\leq 0 \\ V(S, t) - \Lambda(S, t) &\geq 0 \\ \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right) (V(S, t) - \Lambda(S, t)) &= 0. \end{aligned} \quad (2.14)$$

2.3.1 The linear complementarity formulation

To begin with, we discretize the asset price and time. We divide the time interval $[0, T]$ into L subintervals of equal length and denote

$$t_l = l\delta t, \quad l = 0, 1, 2, \dots, L; \quad \delta t = \frac{T}{L}, \quad (2.15)$$

where T is the expiration date. The range of the asset price is $[0, \infty)$ in principle, but we assume that the actual asset price does not exceed a large positive number S_{\max} . We divide the interval $[0, S_{\max}]$ into N subintervals of equal length and denote

$$S_n = n\delta S, \quad n = 1, 2, \dots, N; \quad \delta S = \frac{S_{\max}}{N}. \quad (2.16)$$

We write the discretized option prices and payoff values as follows:

$$\begin{cases} V_n^l \equiv V(S_n, t_l) \\ \Lambda_n^l \equiv \Lambda(S_n, t_l), \end{cases} \quad 1 \leq n \leq N; \quad 0 \leq l \leq L. \quad (2.17)$$

The partial differential complementarity problem (2.14) is then approximated on a regular grid with step-sizes δt and δS . For the first partial derivative with respect to the time, we use the following forward difference approximation¹:

$$\frac{\partial V}{\partial t} = \frac{V(S, t + \delta t) - V(S, t)}{\delta t} + O(\delta t). \quad (2.18)$$

For the first partial derivative with respect to the asset price, we use the following θ_1 -weighted central difference approximation:

$$\begin{aligned} \frac{\partial V}{\partial S} &= \theta_1 \frac{V(S + \delta S, t) - V(S - \delta S, t)}{2\delta S} \\ &+ (1 - \theta_1) \frac{V(S + \delta S, t + \delta t) - V(S - \delta S, t + \delta t)}{2\delta S} + O(\delta S^2), \end{aligned} \quad (2.19)$$

where $\theta_1 \in [0, 1]$ is a given parameter. When $\theta_1 = 0$, this approximation is called an explicit method. When $\theta_1 = 1$, this approximation is called an implicit method. When $\theta_1 = 1/2$,

¹The function $g(x)$ is written as $O(h(x))$ if $\limsup_{h(x) \rightarrow 0} \frac{|g(x)|}{|h(x)|} < \infty$.

this approximation is called the Crank-Nicolson method. For the second partial derivative with respect to the asset price, we use the following θ_2 -weighted central difference approximation:

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} &= \theta_2 \frac{V(S + \delta S, t) - 2V(S, t) + V(S - \delta S, t)}{(\delta S)^2} \\ &+ (1 - \theta_2) \frac{V(S + \delta S, t + \delta t) - 2V(S, t + \delta t) + V(S - \delta S, t + \delta t)}{(\delta S)^2} + O(\delta S^2), \end{aligned} \quad (2.20)$$

where $\theta_2 \in [0, 1]$ is a given parameter whose role is similar to that of θ_1 .

Using the difference approximations (2.18), (2.19) and (2.20), the left-hand side of the Black-Scholes partial differential inequality can be approximated as follows:

$$\begin{aligned} -\frac{\partial V}{\partial t} - rS \frac{\partial V}{\partial S} + rV - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} &\approx V(S + \delta S, t) \left(-rS\theta_1 \frac{1}{2\delta S} - \frac{1}{2} \sigma^2 S^2 \theta_2 \frac{1}{(\delta S)^2} \right) \\ &+ V(S, t) \left(\frac{1}{\delta t} + r + \sigma^2 S^2 \theta_2 \frac{1}{(\delta S)^2} \right) \\ &+ V(S - \delta S, t) \left(-\frac{1}{2} \sigma^2 S^2 \theta_2 \frac{1}{(\delta S)^2} + rS\theta_1 \frac{1}{2\delta S} \right) \\ &+ V(S + \delta S, t + \delta t) \left(-rS(1 - \theta_1) \frac{1}{2\delta S} - \frac{1}{2} \sigma^2 S^2 (1 - \theta_2) \frac{1}{(\delta S)^2} \right) \\ &+ V(S, t + \delta t) \left(-\frac{1}{\delta t} + \sigma^2 S^2 (1 - \theta_2) \frac{1}{(\delta S)^2} \right) \\ &+ V(S - \delta S, t + \delta t) \left(-\frac{1}{2} \sigma^2 S^2 (1 - \theta_2) \frac{1}{(\delta S)^2} + rS(1 - \theta_1) \frac{1}{2\delta S} \right). \end{aligned}$$

With the above finite difference approximations, the system (2.14) leads to the following finite-dimensional linear complementarity problem:

$$0 \leq (\mathbf{V}^l - \mathbf{\Lambda}^l) \perp (\mathbf{M}\mathbf{V}^l + \mathbf{M}'\mathbf{V}^{l+1}) \geq 0, \quad l = L - 1, L - 2, \dots, 1, 0, \quad (2.21)$$

where \mathbf{V}^l and $\mathbf{\Lambda}^l$ are N -vectors defined by

$$\mathbf{V}^l \equiv \begin{pmatrix} V_1^l \\ \vdots \\ V_N^l \end{pmatrix}, \quad \mathbf{\Lambda}^l \equiv \begin{pmatrix} \Lambda_1^l \\ \vdots \\ \Lambda_N^l \end{pmatrix},$$

\mathbf{M} is the $N \times N$ matrix

$$\mathbf{M} \equiv \begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & a_N & b_N \end{pmatrix}$$

with entries given by

$$\begin{aligned} a_n &= -\frac{1}{2} \sigma^2 n^2 \theta_2 + \frac{rn\theta_1}{2}, & n = 2, \dots, N \\ b_n &= r + \frac{1}{\delta t} + \sigma^2 n^2 \theta_2, & n = 1, \dots, N \\ c_n &= -\frac{rn\theta_1}{2} - \frac{1}{2} \sigma^2 n^2 \theta_2, & n = 1, \dots, N - 1, \end{aligned}$$

and \mathbf{M}' is the $N \times N$ matrix, formed in the same way as \mathbf{M} , with entries given by

$$\begin{aligned} a'_n &= -\frac{1}{2}\sigma^2 n^2(1-\theta_2) + \frac{rn(1-\theta_1)}{2}, & n = 2, \dots, N \\ b'_n &= -\frac{1}{\delta t} + \sigma^2 n^2(1-\theta_2), & n = 1, \dots, N \\ c'_n &= -\frac{rn(1-\theta_1)}{2} - \frac{1}{2}\sigma^2 n^2(1-\theta_2), & n = 1, \dots, N-1. \end{aligned}$$

On the expiration date, we cannot hold American options any more. So we have to exercise the right of options or discard it. This means that, on the expiration date, the price of an American option is equal to the payoff value, that is to say, $\mathbf{V}^L = \mathbf{\Lambda}^L$. Since \mathbf{V}^L is known, we can solve the linear complementarity problems (2.21) for $l = L-1, L-2, \dots, 1, 0$, by proceeding backward in time. Thus, we can obtain a set of discrete option prices at $t = 0$ as $V_n^0, n = 1, \dots, N$.

3 Deterministic formulations for the stochastic complementarity problem

In this section, we consider the general stochastic complementarity problem and describe the expected value method [13] and the expected residual minimization method [6] which give deterministic formulations for the stochastic complementarity problem.

The stochastic complementarity problem in standard form is to find a vector $x \in \mathfrak{R}_+^n$ such that

$$0 \leq x \perp F(x, \omega) \geq 0, \quad \omega \in \Omega, \quad (3.1)$$

where $F : \mathfrak{R}^n \times \Omega \rightarrow \mathfrak{R}^n$ is a vector-valued function, (Ω, \mathcal{F}, P) is a probability space with $\Omega \subseteq \mathfrak{R}^m$, and the perp symbol \perp denotes the orthogonality of two vectors, i.e., $x \perp y$ means $x^T y = 0$. In general, there is no vector $x \in \mathfrak{R}_+^n$ satisfying (3.1) for all $\omega \in \Omega$ simultaneously. Therefore, it is necessary to consider a deterministic formulation for (3.1) which provides an optimal solution of the stochastic complementarity problem in some sense.

3.1 Expected value method

The expected value method [13] considers the deterministic formulation which is to find a vector $x \in \mathfrak{R}_+^n$ such that

$$0 \leq x \perp F_\infty(x) \geq 0, \quad (3.2)$$

where $F_\infty(x) := E[F(x, \omega)]$ is the expectation function of the random function $F(x, \omega)$. Since it is usually difficult to evaluate the expectation function $F_\infty(x)$ exactly, we use a finite number of samples $\{\omega_j, j = 1, \dots, k\}$ and construct an approximating function $F_k(x)$ as

$$F_k(x) := \frac{1}{k} \sum_{j=1}^k F(x, \omega_j).$$

By using the approximating function $F_k(x)$, the complementarity problem (3.2) is rewritten as

$$0 \leq x \perp F_k(x) \geq 0. \quad (3.3)$$

3.2 Expected residual minimization method

We consider a function $\psi : \mathfrak{R}^2 \rightarrow \mathfrak{R}$, called an NCP function, which satisfies

$$\psi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.$$

There are various NCP functions for solving complementarity problems [12]. In this paper we concentrate on two popular NCP functions; the min function

$$\psi(a, b) = \min(a, \nu b) \tag{3.4}$$

and the Fischer-Burmeister (FB) function

$$\psi(a, b) = a + \nu b - \sqrt{a^2 + (\nu b)^2}, \tag{3.5}$$

where ν is a positive parameter. Then, we can easily verify that (3.1) is equivalent to the following equation:

$$\Psi(x, \omega) = 0, \quad \omega \in \Omega, \tag{3.6}$$

where $\Psi : \mathfrak{R}^n \times \Omega \rightarrow \mathfrak{R}^n$ is defined by

$$\Psi(x, \omega) := \begin{pmatrix} \psi(F_1(x, \omega), x_1) \\ \vdots \\ \psi(F_n(x, \omega), x_n) \end{pmatrix}.$$

As mentioned above, there is usually no $x \in \mathfrak{R}_+^n$ satisfying (3.6) for all $\omega \in \Omega$ simultaneously. In [6], the expected residual minimization method is proposed to give the following deterministic formulation for the stochastic complementarity problem:

$$\begin{aligned} \min_x \quad & \mathbb{E} [\|\Psi(x, \omega)\|^2] \\ \text{s.t.} \quad & x \in \mathfrak{R}_+^n, \end{aligned} \tag{3.7}$$

where $\|\cdot\|$ denotes the Euclidean norm. Like the expected value method, it is usually difficult to evaluate the expectation $\mathbb{E} [\|\Psi(x, \omega)\|^2]$ exactly. So we use a finite number of samples $\{\omega_j, j = 1, \dots, k\}$ and construct an approximating function of $\mathbb{E} [\|\Psi(x, \omega)\|^2]$ as

$$f_k(x) := \frac{1}{k} \sum_{j=1}^k \|\Psi(x, \omega_j)\|^2.$$

By using the approximating function, problem (3.7) is rewritten as

$$\begin{aligned} \min_x \quad & f_k(x) \\ \text{s.t.} \quad & x \in \mathfrak{R}_+^n. \end{aligned} \tag{3.8}$$

This approach may be regarded as an extension of the least-squares method for an overdetermined system of equations.

We note that, if Ω has only one realization, then we get the same solution by using the expected value method (3.2) and the expected residual minimization method (3.7) as long as the original complementarity problem has a solution, and the solubility of the expected residual minimization method (3.7) does not depend on the choice of NCP functions. However, as shown below, we usually get different solutions by using the expected value method and the expected residual minimization method if Ω has more than one realization. Moreover, the solubility of the expected residual minimization method (3.7) is dependent on the choice of NCP functions.

Example 1. [6] Let $x \in \mathfrak{R}^1$, $\omega \in \mathfrak{R}^1$, $\Omega = \{\omega_1, \omega_2\} = \{0, 1\}$, $P\{\omega = \omega_1\} = P\{\omega = \omega_2\} = 1/2$, $F(x, \omega) = (1 - \omega)\omega x + 1 - 2\omega$. Then we have $F(x, \omega_1) = 1$, $F(x, \omega_2) = -1 \quad \forall x$. If we adopt the expected value formulation, we solve the following linear complementarity problem:

$$0 \leq x \perp 0 \cdot x + 0 \geq 0.$$

The solution set of this problem consists of all x satisfying $x \geq 0$.

Note that $E [\|\Psi(x, \omega)\|^2]$ can be written as

$$E [\|\Psi(x, \omega)\|^2] = \frac{1}{2} \sum_{j=1}^2 \|\Psi(x, \omega_j)\|^2.$$

If we adopt the expected residual minimization formulation defined by the min function and set $\nu = 1$ in (3.4), then the objective function of problem (3.7) is given by

$$\frac{1}{2} \left[(\min(1, x))^2 + (\min(-1, x))^2 \right] = \begin{cases} x^2 & x < -1 \\ \frac{1}{2}(x^2 + 1) & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}.$$

The expected residual minimization formulation (3.7) seeks solutions in the nonnegative orthant. So this problem has the unique solution $x^* = 0$. If we adopt the expected residual minimization formulation defined by the FB function and set $\nu = 1$ in (3.5), the objective function of problem (3.7) is given by

$$\frac{1}{2} \left[\left(1 + x - \sqrt{1 + x^2}\right)^2 + \left(-1 + x - \sqrt{1 + x^2}\right)^2 \right].$$

Since this function is monotonically decreasing on $[0, \infty)$, problem (3.7) does not have a solution.

As shown in Example 1, a solution of the stochastic complementarity problem depends on the choice of deterministic formulations. Besides, there are cases where the solution set is empty or there are many solutions. In Section 5, we discuss conditions that ensure the existence of a solution in deterministic formulations for the stochastic linear complementarity problem derived from the model for pricing American options.

4 Pricing American options with uncertain volatility

In this section, we present two deterministic formulations for pricing American options with uncertain volatility, which are based on the expected value method and the expected residual minimization method for the stochastic complementarity problem discussed in Section 3. Since the entries of the matrices \mathbf{M} and \mathbf{M}' defined in Subsection 2.3 are dependent on the volatility σ , we write $\mathbf{M}(\sigma)$ and $\mathbf{M}'(\sigma)$.

If we regard the volatility σ as a random variable, pricing American options with uncertain volatility is formulated as the following stochastic linear complementarity problem:

$$0 \leq (\mathbf{V}^l - \mathbf{\Lambda}^l) \perp \left(\mathbf{M}(\sigma)\mathbf{V}^l + \mathbf{M}'(\sigma)\mathbf{V}^{l+1} \right) \geq 0, \quad l = L - 1, L - 2, \dots, 1, 0. \quad (4.1)$$

As mentioned in Section 3, there are usually no $V^l, l = 0, 1, \dots, L - 1$ satisfying (4.1) for all σ simultaneously. So we apply the expected value method and the expected residual minimization method to the stochastic linear complementarity problem (4.1).

First, we give the formulation based on the expected value method. In the expected value method, we substitute the expected values $E[\mathbf{M}(\sigma)]$ and $E[\mathbf{M}'(\sigma)]$ for $\mathbf{M}(\sigma)$ and $\mathbf{M}'(\sigma)$, respectively. Then we have the following linear complementarity problem:

$$0 \leq (\mathbf{V}^l - \mathbf{\Lambda}^l) \perp \left(E[\mathbf{M}(\sigma)] \mathbf{V}^l + E[\mathbf{M}'(\sigma)] \mathbf{V}^{l+1} \right) \geq 0, \quad l = L-1, L-2, \dots, 1, 0. \quad (4.2)$$

Using discrete samples $\{\sigma_j, j = 1, \dots, k\}$, the expected values $E[\mathbf{M}(\sigma)]$ and $E[\mathbf{M}'(\sigma)]$ can be approximated by $\frac{1}{k} \sum_{j=1}^k \mathbf{M}(\sigma_j)$ and $\frac{1}{k} \sum_{j=1}^k \mathbf{M}'(\sigma_j)$, respectively. So (4.2) can be rewritten as

$$0 \leq (\mathbf{V}^l - \mathbf{\Lambda}^l) \perp \left(\frac{1}{k} \sum_{j=1}^k \mathbf{M}(\sigma_j) \mathbf{V}^l + \frac{1}{k} \sum_{j=1}^k \mathbf{M}'(\sigma_j) \mathbf{V}^{l+1} \right) \geq 0, \quad l = L-1, L-2, \dots, 1, 0. \quad (4.3)$$

Like pricing American options with constant volatility, the price of an option on the expiration date is equal to the payoff value, that is $\mathbf{V}^L = \mathbf{\Lambda}^L$. By solving (4.3) backward in time, we can obtain a set of discrete option prices at $t = 0$ as $V_n^0, n = 1, \dots, N$.

Next, we give the formulation based on the expected residual minimization method. Using the equality $\mathbf{V}^L = \mathbf{\Lambda}^L$, the stochastic linear complementarity problem (4.1) can be rewritten as the following stochastic linear complementarity problem:

$$0 \leq \begin{pmatrix} \mathbf{V}^0 - \mathbf{\Lambda}^0 \\ \mathbf{V}^1 - \mathbf{\Lambda}^1 \\ \vdots \\ \mathbf{V}^{L-2} - \mathbf{\Lambda}^{L-2} \\ \mathbf{V}^{L-1} - \mathbf{\Lambda}^{L-1} \end{pmatrix} \perp \begin{pmatrix} \mathbf{M}(\sigma) & \mathbf{M}'(\sigma) & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{M}(\sigma) & \mathbf{M}'(\sigma) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mathbf{M}(\sigma) & \mathbf{M}'(\sigma) \\ 0 & 0 & \cdots & 0 & 0 & \mathbf{M}(\sigma) \end{pmatrix} \begin{pmatrix} \mathbf{V}^0 \\ \mathbf{V}^1 \\ \vdots \\ \mathbf{V}^{L-2} \\ \mathbf{V}^{L-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbf{M}'(\sigma) \mathbf{\Lambda}^L \end{pmatrix} \geq 0, \quad (4.4)$$

where $\mathbf{V}^l, l = 0, 1, \dots, L-1$ are the variables. We define \mathbf{V} and $\Psi(\mathbf{V}, \sigma)$ as

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^0 \\ \mathbf{V}^1 \\ \vdots \\ \mathbf{V}^{L-2} \\ \mathbf{V}^{L-1} \end{pmatrix}, \quad \Psi(\mathbf{V}, \sigma) = \begin{pmatrix} \psi(V_1^0 - \Lambda_1^0, (\mathbf{M}(\sigma) \mathbf{V}^0 + \mathbf{M}'(\sigma) \mathbf{V}^1)_1) \\ \vdots \\ \psi(V_N^0 - \Lambda_N^0, (\mathbf{M}(\sigma) \mathbf{V}^0 + \mathbf{M}'(\sigma) \mathbf{V}^1)_N) \\ \psi(V_1^1 - \Lambda_1^1, (\mathbf{M}(\sigma) \mathbf{V}^1 + \mathbf{M}'(\sigma) \mathbf{V}^2)_1) \\ \vdots \\ \psi(V_N^1 - \Lambda_N^1, (\mathbf{M}(\sigma) \mathbf{V}^1 + \mathbf{M}'(\sigma) \mathbf{V}^2)_N) \\ \vdots \\ \psi(V_1^{L-1} - \Lambda_1^{L-1}, (\mathbf{M}(\sigma) \mathbf{V}^{L-1} + \mathbf{M}'(\sigma) \mathbf{V}^L)_1) \\ \vdots \\ \psi(V_N^{L-1} - \Lambda_N^{L-1}, (\mathbf{M}(\sigma) \mathbf{V}^{L-1} + \mathbf{M}'(\sigma) \mathbf{V}^L)_N) \end{pmatrix},$$

where ψ is an NCP function and $(\mathbf{M}(\sigma) \mathbf{V}^l + \mathbf{M}'(\sigma) \mathbf{V}^{l+1})_n$ denotes the n th component of the vector $\mathbf{M}(\sigma) \mathbf{V}^l + \mathbf{M}'(\sigma) \mathbf{V}^{l+1}$.

Using the expected residual minimization method, pricing American options with uncertain volatility is formulated as the following optimization problem:

$$\begin{aligned} \min_{\mathbf{V}} \quad & \mathbb{E} [\|\Psi(\mathbf{V}, \sigma)\|^2] \\ \text{s.t.} \quad & \mathbf{V}^l \geq \Lambda^l, \quad l = 0, 1, \dots, L-1, \\ & \mathbf{V}^L = \Lambda^L. \end{aligned} \tag{4.5}$$

Adopting the min function as the NCP function ψ , (4.5) can be rewritten as

$$\begin{aligned} \min_{\mathbf{V}} \quad & \mathbb{E} \left[\sum_{l=0}^{L-1} \sum_{n=1}^N \left\{ \min \left(V_n^l - \Lambda_n^l, \nu \left(\mathbf{M}(\sigma) \mathbf{V}^l + \mathbf{M}'(\sigma) \mathbf{V}^{l+1} \right)_n \right) \right\}^2 \right] \\ \text{s.t.} \quad & \mathbf{V}^l \geq \Lambda^l, \quad l = 0, 1, \dots, L-1, \\ & \mathbf{V}^L = \Lambda^L. \end{aligned} \tag{4.6}$$

Moreover, by using discrete samples $\{\sigma_j, j = 1, \dots, k\}$, (4.6) can be approximated as follows:

$$\begin{aligned} \min_{\mathbf{V}} \quad & \frac{1}{k} \sum_{j=1}^k \sum_{l=0}^{L-1} \sum_{n=1}^N \left\{ \min \left(V_n^l - \Lambda_n^l, \nu \left(\mathbf{M}(\sigma_j) \mathbf{V}^l + \mathbf{M}'(\sigma_j) \mathbf{V}^{l+1} \right)_n \right) \right\}^2 \\ \text{s.t.} \quad & \mathbf{V}^l \geq \Lambda^l, \quad l = 0, 1, \dots, L-1, \\ & \mathbf{V}^L = \Lambda^L. \end{aligned} \tag{4.7}$$

5 Choice of step-size parameter and existence of a solution

In this section, we give conditions that ensure the existence of a solution in the formulation by the expected value method (4.3) and the formulation by the expected residual minimization method (4.3) for pricing American options with uncertain volatility. Recall that we can take the step-size parameter δt arbitrarily for a certain positive integer L satisfying (2.15). So we mainly examine conditions for the parameter δt that ensure the existence of a solution.

5.1 Existence of a solution in the expected value method

We denote the discrete samples of σ as $\{\sigma_j, j = 1, \dots, k\}$. Then, the coefficient matrix of the linear complementarity problem in the expected value method (4.3) is written as

$$\tilde{\mathbf{M}} \equiv \begin{pmatrix} \tilde{b}_1 & \tilde{c}_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \tilde{a}_{N-1} & \tilde{b}_{N-1} & \tilde{c}_{N-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{a}_N & \tilde{b}_N \end{pmatrix}$$

with entries given by

$$\begin{aligned}\tilde{a}_n &= -\frac{n^2\theta_2}{2k} \sum_{j=1}^k \sigma_j^2 + \frac{rn\theta_1}{2}, \quad n = 2, \dots, N \\ \tilde{b}_n &= r + \frac{1}{\delta t} + \frac{n^2\theta_2}{k} \sum_{j=1}^k \sigma_j^2, \quad n = 1, \dots, N \\ \tilde{c}_n &= -\frac{rn\theta_1}{2} - \frac{n^2\theta_2}{2k} \sum_{j=1}^k \sigma_j^2, \quad n = 1, \dots, N-1.\end{aligned}$$

For a square matrix $A \in \mathfrak{R}^{n \times n}$, the following results are known [19].

Lemma 2. *If a square matrix A is a strictly row diagonally dominant matrix with positive diagonal elements, then A is a P-matrix.*

Recall that $A = (a_{ij})$ is said to be strictly row diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \dots, n.$$

A square matrix is said to be a P-matrix if all its principal minors are positive. About a P-matrix, the following results are known [9].

Lemma 3. *Let $A \in \mathfrak{R}^{n \times n}$. Then the following statements are equivalent:*

- (a) A is a P-matrix.
- (b) Matrix A reverses the sign of no vector, i.e.,

$$x_i(Ax)_i \leq 0, \quad \forall i \quad \Rightarrow \quad x = 0.$$

- (c) the linear complementarity problem

$$0 \leq x \perp Ax + q \geq 0$$

has a unique solution for any vector $q \in \mathfrak{R}^n$.

Concerning the choice of δt , we can establish the following proposition.

Proposition 1. *If we choose δt such that*

$$\frac{1}{\delta t} > \frac{kr^2\theta_1^2}{4\theta_2 \sum_{j=1}^k \sigma_j^2} - r, \quad (5.1)$$

then the linear complementarity problem (4.3) in the expected value method has a unique solution.

Proof. Clearly, all diagonal elements of $\tilde{\mathbf{M}}$ are positive. We will prove that $\tilde{\mathbf{M}}$ is a strictly row diagonally dominant matrix. Note that $\tilde{\mathbf{M}}$ is a strictly row diagonally dominant if and only if

$$\begin{aligned}|\tilde{b}_1| &> |\tilde{c}_1|, \\ |\tilde{b}_n| &> |\tilde{a}_n| + |\tilde{c}_n|, \quad n = 2, \dots, N-1, \\ |\tilde{b}_N| &> |\tilde{a}_N|.\end{aligned} \quad (5.2)$$

Since $\tilde{b}_n, n = 1, \dots, N$ are positive and $\tilde{c}_n, n = 1, \dots, N - 1$ are negative, we can write

$$\begin{aligned} |\tilde{b}_1| - |\tilde{c}_1| &= r + \frac{1}{\delta t} + \frac{\theta_2}{k} \sum_{j=1}^k \sigma_j^2 - \frac{r\theta_1}{2} - \frac{\theta_2}{2k} \sum_{j=1}^k \sigma_j^2, \\ |\tilde{b}_n| - |\tilde{a}_n| - |\tilde{c}_n| &= r + \frac{1}{\delta t} + \frac{n^2\theta_2}{k} \sum_{j=1}^k \sigma_j^2 - \left| -\frac{n^2\theta_2}{2k} \sum_{j=1}^k \sigma_j^2 + \frac{rn\theta_1}{2} \right| \\ &\quad - \frac{rn\theta_1}{2} - \frac{n^2\theta_2}{2k} \sum_{j=1}^k \sigma_j^2, \quad n = 2, \dots, N - 1, \\ |\tilde{b}_N| - |\tilde{a}_N| &= r + \frac{1}{\delta t} + \frac{N^2\theta_2}{k} \sum_{j=1}^k \sigma_j^2 - \left| -\frac{N^2\theta_2}{2k} \sum_{j=1}^k \sigma_j^2 + \frac{rN\theta_1}{2} \right|. \end{aligned}$$

We only consider the cases of $n = 2, \dots, N - 1$, because the cases $n = 1$ and $n = N$ can be treated similarly. First, suppose $a_n \geq 0$. Then we can write

$$|\tilde{b}_n| - |\tilde{a}_n| - |\tilde{c}_n| = r + \frac{1}{\delta t} + \frac{n^2\theta_2}{k} \sum_{j=1}^k \sigma_j^2 - rn\theta_1, \quad n = 2, \dots, N - 1. \quad (5.3)$$

Note that the right-hand of (5.3) can be rewritten as

$$\frac{\theta_2}{k} \sum_{j=1}^k \sigma_j^2 \left(n - \frac{kr\theta_1}{2\theta_2 \sum_{j=1}^k \sigma_j^2} \right)^2 + \frac{1}{\delta t} - \frac{kr^2\theta_1^2}{4\theta_2 \sum_{j=1}^k \sigma_j^2} + r, \quad n = 2, \dots, N - 1. \quad (5.4)$$

Hence if δt satisfies (5.1), we have (5.2).

Next, suppose $a_n < 0$. Then we can write

$$|\tilde{b}_n| - |\tilde{a}_n| - |\tilde{c}_n| = r + \frac{1}{\delta t}, \quad n = 2, \dots, N - 1.$$

Since $r \geq 0$ and $\delta t > 0$, we have (5.2).

Therefore, if δt is chosen to satisfy (5.1), then $\tilde{\mathbf{M}}$ is a strictly row diagonally dominant matrix. By Lemma 2, this implies that $\tilde{\mathbf{M}}$ is a P-matrix. Then the assertion of the proposition follows from Lemma 3. \blacksquare

5.2 Existence of a solution in the expected residual minimization method

Next, we examine conditions that ensure the existence of a solution in the expected residual minimization method. We denote the discrete samples of σ as $\{\sigma_j, j = 1, \dots, k\}$. For each σ_j , the coefficient matrix (4.4) is written as

$$\mathbf{G}(\sigma_j) = \begin{pmatrix} \hat{\mathbf{M}}(\sigma_j) & \hat{\mathbf{M}}'(\sigma_j) & 0 & 0 & \cdots & 0 \\ 0 & \hat{\mathbf{M}}(\sigma_j) & \hat{\mathbf{M}}'(\sigma_j) & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \hat{\mathbf{M}}(\sigma_j) & \hat{\mathbf{M}}'(\sigma_j) \\ 0 & 0 & \cdots & 0 & 0 & \hat{\mathbf{M}}(\sigma_j) \end{pmatrix}, \quad (5.5)$$

where $\hat{\mathbf{M}}(\sigma_j)$ is the $N \times N$ matrix

$$\hat{\mathbf{M}}(\sigma_j) = \begin{pmatrix} \hat{b}_1 & \hat{c}_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hat{a}_2 & \hat{b}_2 & \hat{c}_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{a}_3 & \hat{b}_3 & \hat{c}_3 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \hat{a}_{N-1} & \hat{b}_{N-1} & \hat{c}_{N-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \hat{a}_N & \hat{b}_N \end{pmatrix}$$

with entries given by

$$\begin{aligned} \hat{a}_n &= -\frac{1}{2}\sigma_j^2 n^2 \theta_2 + \frac{rn\theta_1}{2}, \quad n = 2, \dots, N \\ \hat{b}_n &= r + \frac{1}{\delta t} + \sigma_j^2 n^2 \theta_2, \quad n = 1, \dots, N \\ \hat{c}_n &= -\frac{rn\theta_1}{2} - \frac{1}{2}\sigma_j^2 n^2 \theta_2, \quad n = 1, \dots, N-1, \end{aligned}$$

and $\hat{\mathbf{M}}'(\sigma_j)$ is the $N \times N$ matrix, formed in the same way as $\hat{\mathbf{M}}(\sigma_j)$, with entries given by

$$\begin{aligned} \hat{a}'_n &= -\frac{1}{2}\sigma_j^2 n^2 (1 - \theta_2) + \frac{rn(1 - \theta_1)}{2}, \quad n = 2, \dots, N \\ \hat{b}'_n &= -\frac{1}{\delta t} + \sigma_j^2 n^2 (1 - \theta_2), \quad n = 1, \dots, N \\ \hat{c}'_n &= -\frac{rn(1 - \theta_1)}{2} - \frac{1}{2}\sigma_j^2 n^2 (1 - \theta_2), \quad n = 1, \dots, N-1. \end{aligned}$$

Recall that a square matrix H is called an R_0 matrix if

$$x^T H x = 0, \quad H x \geq 0, \quad x \geq 0 \quad \Rightarrow \quad x = 0.$$

In particular, any P-matrix is an R_0 matrix [9]. The following existence result has been established for the expected residual minimization method [6].

Lemma 4. *If $\mathbf{G}(\sigma_j)$ is an R_0 matrix for some $j \in \{1, \dots, k\}$, then the solution set of the optimization problem (4.7) is nonempty and bounded.*

Considering the choice of the parameter δt , we have the following proposition.

Proposition 2. *If we choose δt such that*

$$\frac{1}{\delta t} > \frac{r^2 \theta_1^2}{4\theta_2 \sigma_j^2} - r, \quad (5.6)$$

for some $j \in \{1, \dots, k\}$, then the solution set of the optimization problem (4.7) in the expected residual minimization method is nonempty and bounded.

Proof. In a similar manner to the proof of Proposition 1, we can verify that all diagonal elements of $\hat{\mathbf{M}}(\sigma_j)$ are positive and $\hat{\mathbf{M}}(\sigma_j)$ is a strictly row diagonally dominant matrix, whenever δt satisfies (2). Therefore, $\hat{\mathbf{M}}(\sigma_j)$ is a P-matrix. Below we will show that $\mathbf{G}(\sigma_j) \in \mathfrak{R}^{N^2 \times N^2}$ is a P-matrix. From Lemma 3, $\mathbf{G}(\sigma_j)$ is a P-matrix if and only if, for any $x \in \mathfrak{R}^{N^2}$,

$$x_i (\mathbf{G}(\sigma_j)x)_i \leq 0, \quad \forall i \quad \Rightarrow \quad x = 0. \quad (5.7)$$

Let us denote

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{pmatrix},$$

where $x^p \in \mathbb{R}^N, p = 1, 2, \dots, N$. Then, we can write

$$\mathbf{G}(\sigma_j)x = \begin{pmatrix} \hat{\mathbf{M}}(\sigma_j)x^1 + \hat{\mathbf{M}}'(\sigma_j)x^2 \\ \hat{\mathbf{M}}(\sigma_j)x^2 + \hat{\mathbf{M}}'(\sigma_j)x^3 \\ \vdots \\ \hat{\mathbf{M}}(\sigma_j)x^{N-1} + \hat{\mathbf{M}}'(\sigma_j)x^N \\ \hat{\mathbf{M}}(\sigma_j)x^N \end{pmatrix}. \quad (5.8)$$

Assume

$$x_i (\mathbf{G}(\sigma_j)x)_i \leq 0, \quad \forall i. \quad (5.9)$$

First, we show $x^N = 0$. By Lemma 3, since $\hat{\mathbf{M}}(\sigma_j)$ is a P-matrix, we have

$$y_i (\hat{\mathbf{M}}(\sigma_j)y)_i \leq 0, \quad \forall i \Rightarrow y = 0 \quad (5.10)$$

for any $y \in \mathbb{R}^N$. It then follows from (5.8), (5.9) and (5.10) that $x^N = 0$.

Next, notice that the $(N-1)$ th block of the vector $\mathbf{G}(\sigma_j)x$ equals $\hat{\mathbf{M}}(\sigma_j)x^{N-1}$ since $x^N = 0$. Hence, by the same reasoning as above, we have $x^{N-1} = 0$. Repeating similar arguments, we deduce $x^{N-2} = x^{N-3} = \dots = x^1 = 0$, implying (5.7) hold. Thus, $\mathbf{G}(\sigma_j)$ is a P-matrix. Since every P-matrix is an R_0 matrix [9], it follows from Lemma 4 that the solution set of the optimization problem (4.7) is nonempty and bounded. \blacksquare

From Proposition 1 and Proposition 2, if we choose the step-size parameter δt small enough to satisfy the conditions (5.1) and (5.6), respectively, then we can ensure that the linear complementarity problem (4.3) in the expected value method and the optimization problem (4.7) in the expected residual minimization method have a solution. However, when δt is small, the size of problem (4.3) or (4.7) becomes large, which may make the problem more expensive computationally.

6 Numerical experiments

In this section, we describe numerical experiments. All computations were carried out using Matlab on a PC. We use put options whose underlying asset is S&P100. S&P100 is a market value weighted index consisting of 100 leading United States stocks. First, we state how to set the parameters of the stochastic linear complementarity problem (4.1). Next, we describe some criteria used to compare the results. Finally, we show and discuss the computational results.

6.1 Parameter setting

In this subsection, we describe how to set the parameters to derive the stochastic complementarity problem (4.1).

We set the parameters in the finite difference approximation as $\theta_1 = 1/2$ and $\theta_2 = 1/2$. We set $L = 4$ to divide the time interval $[0, T]$, where T is the expiration date. From (2.15), the length of each subinterval is $\delta t = T/4$. We assume that the underlying asset does not exceed a large positive number $S_{\max} = 900$. Then the interval $[0, S_{\max}]$ is divided into N subintervals of equal length, where we set $N = 30$. From (2.16), the length of each subinterval is $\delta S = S_{\max}/N = 30$. The payoff $\Lambda(S, t)$ is discretized for the asset price and the time. Since we consider put options, the elements of $\Lambda^l, l = 0, 1, \dots, L$ can be written as

$$\Lambda_n^l = \begin{cases} E - n\delta S & 1 \leq n \leq E/\delta S \\ 0 & E/\delta S < n \leq N \end{cases} ; \quad 0 \leq l \leq L. \quad (6.1)$$

We obtained the data listed in Appendix from the Wall Street Journal's homepage². Here, we use the interest rate of 6 month U.S. government bond obtained from the same page as the risk-free rate r in the list. From the data, we give the risk-free rate r and, noting that the expiration date is the third Friday of the expiration month, calculate the expiration date T . Then, using the strike price E , we calculate the payoff $\Lambda^l, l = 0, 1, \dots, L$ from (6.1).

For example, for the option whose expiration month is January, 2010 and whose strike price is 360 on November 30, 2009, the risk-free rate is $r = 0.00242$ and the expiration date is $T = 46/365$, since there are 46 days from November 30 to January 15 (the third Friday of January, 2010). Since the strike price is $E = 360$ and we set $\delta S = 30, N = 30$ and $L = 4$, we have $E/\delta S = 360/30 = 12$ and, from (6.1), we have

$$\Lambda_n^l = \begin{cases} 360 - 30n & 1 \leq n \leq 12 \\ 0 & 12 < n \leq 30 \end{cases} ; \quad 0 \leq l \leq 4.$$

Now we describe how to estimate the volatility of the rate of return of S&P100. From the Yahoo! finance homepage³, we obtain the historical data S_0, \dots, S_{180} , where S_t is the asset price observed t days ago. From the historical data, we calculate the continuously compounded rates of return of S&P100 as

$$u_t = \ln \left(\frac{S_{t-1}}{S_t} \right), \quad t = 1, \dots, 180.$$

Using the continuously compounded rates of return u_1, \dots, u_{180} , we obtain the average of the continuously compounded rate of return in the most recent 60 days, the average rate of the return in the next 60 days, and the average rate of the return in the remaining 60 days, denoted as \bar{u}^1, \bar{u}^2 and \bar{u}^3 , respectively, by the following formulas:

$$\bar{u}^1 = \frac{1}{60} \sum_{t=1}^{60} u_t, \quad \bar{u}^2 = \frac{1}{60} \sum_{t=61}^{120} u_t, \quad \bar{u}^3 = \frac{1}{60} \sum_{t=121}^{180} u_t.$$

Similarly, we compute the volatilities of the continuously compounded rate of return in the above-mentioned three periods by

$$\sigma_1 = \sqrt{\frac{250}{59} \sum_{t=1}^{60} (u_t - \bar{u}^1)^2}, \quad \sigma_2 = \sqrt{\frac{250}{59} \sum_{t=61}^{120} (u_t - \bar{u}^2)^2}, \quad \sigma_3 = \sqrt{\frac{250}{59} \sum_{t=121}^{180} (u_t - \bar{u}^3)^2}. \quad (6.2)$$

²<http://asia.wsj.com/home-page>

³<http://finance.yahoo.com>

We regard these values as realizations of the volatility and set $P\{\sigma = \sigma_j\} = 1/3, j = 1, 2, 3$. Since there are 250 business days in a year, the right-hand side of (6.2) contains the factor $\sqrt{250}$ to convert the day rate of the volatility into the annual rate of the volatility. Since we adopt the unbiased variance, the right-hand side of (6.2) contains the factor $1/\sqrt{59}$ rather than $1/\sqrt{60}$.

We use these parameter values to calculate the coefficients in the stochastic linear complementarity problem (4.1). Then, we obtain the deterministic formulations based on the expected value method (4.3) and the expected residual minimization method (4.7). We let \mathbf{V}_{EV} and \mathbf{V}_{ERM} denote the solutions obtained by the expected value method and the expected residual minimization method, respectively. In the expected residual minimization method, we adopt the min function (3.4) and the FB function (3.5) as an NCP function and set the parameter ν in (3.4) and (3.5) as $\nu = 0.1, 1, 10$. We use the PATH solver [11] to solve the linear complementarity problem (4.3) and use the fmincon solver in the Matlab Toolbox to solve the optimization problem (4.7).

6.2 Criteria for comparing solutions

In this subsection, we describe two criteria used to compare solutions obtained by different formulations. As a standard for comparison, we use the simple average of the solutions \mathbf{V}_j obtained by solving the linear complementarity problems (4.1) for $\sigma_j, j = 1, \dots, k$:

$$\mathbf{V}_{\text{avg}} = \frac{1}{k} \sum_{j=1}^k \mathbf{V}_j. \quad (6.3)$$

6.2.1 Estimation error

One criterion is to analyze the prices obtained from each method against the prices observed in the real market.

First, we describe how the prices of options corresponding to the current asset price can be estimated from the solutions \mathbf{V}_{EV} , \mathbf{V}_{ERM} and \mathbf{V}_{avg} . Note that the vector \mathbf{V} can be written as

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^0 \\ \mathbf{V}^1 \\ \vdots \\ \mathbf{V}^l \\ \vdots \\ \mathbf{V}^{L-2} \\ \mathbf{V}^{L-1} \end{pmatrix} \quad \text{with} \quad \mathbf{V}^l = \begin{pmatrix} V_1^l \\ \vdots \\ V_n^l \\ \vdots \\ V_N^l \end{pmatrix}, \quad l = 0, 1, \dots, L-1.$$

From (2.17), V_n^l is the price corresponding to time t_l and asset price S_n , where t_l and S_n are given by (2.15) and (2.16), respectively. Recall that we want to obtain the prices of options corresponding to the asset price ‘‘at present’’, i.e., $t = 0$. We can get the prices of options at $t = 0$ by taking the first block component of the solution \mathbf{V} , i.e., \mathbf{V}^0 .

Notice that the solution \mathbf{V} gives us only the prices corresponding to N asset prices $S_n, n = 1, \dots, N$. We want to obtain the prices of options corresponding to the given asset price at $t = 0$. This can be obtained by the following procedure: Suppose that the given asset price lies in the interval $[n\delta S, (n+1)\delta S]$ for some integer $n > 0$, i.e., it is represented as $n\delta S + a$, where a is a positive number such that $0 \leq a < \delta S$. Then we can obtain the corresponding price of

options by interpolating the value of the n th element of \mathbf{V}^0 and the $(n + 1)$ th element of \mathbf{V}^0 . Specifically, we calculate the prices of options by the formula:

$$\left(1 - \frac{a}{\delta S}\right) V_n^0 + \frac{a}{\delta S} V_{n+1}^0.$$

For example, look at the data on November 30, 2009. The current asset price is \$511. If we set $\delta S = 30$, then 511 is in between 510 ($=17\delta S$) and 540 ($=18\delta S$). So we use the values of V_{17}^0 and V_{18}^0 in the solution \mathbf{V} . Since we can write $511=17\delta S+1$, we obtain the corresponding price of options by computing

$$\left(1 - \frac{1}{30}\right) V_{17}^0 + \frac{1}{30} V_{18}^0.$$

Let Z be the number of options used for numerical experiments. Then, for the z th option ($z = 1, 2, \dots, Z$), its prices corresponding to the given asset price at $t = 0$ obtained by the expected value method, the expected residual minimization method, and the simple average method as V_{EV}^z , V_{ERM}^z and V_{avg}^z , respectively. We get the prices of options observed in the real market from the data listed in the appendix and denote it as V_{mkt}^z . For example, the price of the option whose expiration date is January, 2010 and strike price is \$360 on November 30, 2009 is \$0.35.

Using these values, we calculate the root mean squared error rate (RMSER)

$$\text{RMSER}_i = \sqrt{\frac{1}{Z} \sum_{z=1}^Z \left(\frac{V_i^z - V_{\text{mkt}}^z}{V_{\text{mkt}}^z} \right)^2}, \quad i = \text{avg, EV, ERM}$$

in order to compare the preciseness of estimation. We also calculate the mean error rate (MER)

$$\text{MER}_i = \frac{1}{Z} \sum_{z=1}^Z \left(\frac{V_i^z - V_{\text{mkt}}^z}{V_{\text{mkt}}^z} \right), \quad i = \text{avg, EV, ERM}$$

in order to compare the bias of the price obtained by each method relative to the price observed in the real market.

6.2.2 Measures of feasibility and optimality

Another criterion is to compare the solutions obtained by each method in terms of some measures of feasibility and optimality, as was done in [7]. For any $\omega \in \Omega$, the complementarity problem

$$0 \leq x \perp F(x, \omega) \geq 0 \tag{6.4}$$

is equivalent to the optimization problem

$$\begin{aligned} \min \quad & x^T F(x, \omega) \\ \text{s.t.} \quad & x \geq 0, \quad F(x, \omega) \geq 0 \end{aligned} \tag{6.5}$$

in the sense that an optimal solution of (6.5) with zero objective value coincides with a solution of (6.4). We denote the discrete samples of ω as $\{\omega_j, j = 1, \dots, k\}$ and denote the solution obtained by the expected value method (3.3) or the expected residual minimization method (3.8) as x^* .

Using the ideas from the literature on stochastic programming [2, 17, 18], we evaluate the violation of the nonnegativity condition $F(x, \omega) \geq 0$ in problem (6.5) by

$$\gamma_{\text{feas}}(x^*, \omega) = \|\min(0, F(x^*, \omega))\|, \quad (6.6)$$

and evaluate the loss in the objective function of (6.5) by

$$\gamma_{\text{opt}}(x^*, \omega) = x^{*T} \max(0, F(x^*, \omega)). \quad (6.7)$$

Here $\min(0, F(x^*, \omega))$ and $\max(0, F(x^*, \omega))$ denote the vectors with components $\min(0, F_i(x^*, \omega))$ and $\max(0, F_i(x^*, \omega))$, respectively, where $F_i(x^*, \omega)$ is the i th element of $F(x^*, \omega)$. We apply these measures to the stochastic linear complementarity problem (4.4) derived from the model for pricing American options with uncertain volatility. As in Subsection 6.1, we denote the solution from the expected value method (4.3), the expected residual minimization method, and the simple average method as \mathbf{V}_{EV} , \mathbf{V}_{ERM} and \mathbf{V}_{avg} , respectively. Then, (6.6) and (6.7) are written as

$$\gamma_{\text{feas}}(\mathbf{V}_i, \sigma) = \sqrt{\sum_{l=0}^{L-1} \left\| \min\left(0, \mathbf{M}(\sigma)\mathbf{V}_i^l + \mathbf{M}'(\sigma)\mathbf{V}_i^{l+1}\right) \right\|^2}, \quad i = \text{avg, EV, ERM},$$

$$\gamma_{\text{opt}}(\mathbf{V}_i, \sigma) = \sum_{l=0}^{L-1} \left(\mathbf{V}_i^l - \mathbf{\Lambda}^l\right)^T \max\left(0, \mathbf{M}(\sigma)\mathbf{V}_i^l + \mathbf{M}'(\sigma)\mathbf{V}_i^{l+1}\right), \quad i = \text{avg, EV, ERM},$$

respectively. We denote the discrete samples of σ as $\{\sigma_j, j = 1, \dots, k\}$. For the z th option and $\sigma = \sigma_j$, we denote the values of $\gamma_{\text{feas}}(\mathbf{V}_i, \sigma)$ and $\gamma_{\text{opt}}(\mathbf{V}_i, \sigma)$ as $\gamma_{\text{feas}}(\mathbf{V}_i^z, \sigma_j)$ and $\gamma_{\text{opt}}(\mathbf{V}_i^z, \sigma_j)$, respectively. We calculate the average values of $\gamma_{\text{feas}}(\mathbf{V}_i^z, \sigma_j)$ and $\gamma_{\text{opt}}(\mathbf{V}_i^z, \sigma_j)$ for $\sigma_j, j = 1, \dots, k$ and $z = 1, \dots, Z$ by the following formulas:

$$\Gamma_i^{\text{feas}} = \frac{1}{kZ} \sum_{z=1}^Z \sum_{j=1}^k \gamma_{\text{feas}}(\mathbf{V}_i^z, \sigma_j), \quad i = \text{avg, EV, ERM},$$

$$\Gamma_i^{\text{opt}} = \frac{1}{kZ} \sum_{z=1}^Z \sum_{j=1}^k \gamma_{\text{opt}}(\mathbf{V}_i^z, \sigma_j), \quad i = \text{avg, EV, ERM},$$

where Z is the total number of options used for numerical experiments. We use Γ_i^{feas} and Γ_i^{opt} to compare solutions obtained by the different methods.

6.3 Numerical results

In this subsection, we show the numerical results. First, we show the values of RMSER and MER in Table 2 and Table 3, respectively. In both tables, we classify options into 4 categories according to their moneyness [1] which is the asset price S divided by the strike price E , as shown in Tables 1. The numbers of options used in the numerical experiments are 88 for DOTM, 148 for OTM, 66 for ATM, and 47 for ITM.

In terms of RMSER, the most precise estimate of the prices of options observed in the real market is given by the expected residual minimization method using the FB function (3.5) with parameter $\nu = 1$. If we focus on ATM and ITM, the expected residual minimization method using the FB function (3.5) with parameter $\nu = 0.1$ estimates most precisely the prices of options

moneyiness	category
$1.2 < S/E$	deep-out-of-the-money(DOTM)
$1.04 < S/E \leq 1.2$	out-of-the-money(OTM)
$0.98 \leq S/E \leq 1.04$	at-the-money(ATM)
$S/E < 0.98$	in-the-money(ITM)

Table 1: The classification according to moneyiness

observed in the real market. Regarding the positive parameter ν , the best choice to estimate the prices observed in the real market is $\nu = 1$ for both the min function and the FB function. If we set $\nu = 10$, the values of RMSER become large, that is to say, the method fails to estimate the prices observed in the real market. So we may conclude that $\nu = 1$ or even a smaller value is an appropriate choice. In terms of MER, all the prices V_{avg} , V_{EV} and V_{ERM} , except those obtained by the expected residual minimization method using the min function with parameter $\nu = 10$ and the FB function with parameter $\nu = 10$, tend to be much lower than the prices observed in the real market.

We show the values of Γ^{feas} defined in Subsection 6.2.2 in Table 4. Table 5 shows the values Γ^{opt} divided by 100. The solution \mathbf{V}_{ERM} by the expected residual minimization method has smaller Γ^{feas} values and larger Γ^{opt} values than the other solutions. Recall that the inequality (2.11) is derived from the no arbitrage assumption and Γ^{feas} represents the violation of this inequality. Thus, the expected residual minimization method (4.7) produces a solution which tends to satisfy no arbitrage assumption, which is one of the most important assumptions in the theory of options. Regarding the positive parameter ν in (3.4) and (3.5), the larger ν we set, the smaller Γ^{feas} values the solution has, that is to say, the stronger tendency to satisfy the no arbitrage assumption the solution has. However, the formulation with a large ν may yield a solution with a large Γ^{opt} value.

	avg	EV	ERM					
			min			FB		
			$\nu=0.1$	$\nu=1$	$\nu=10$	$\nu=0.1$	$\nu=1$	$\nu=10$
DOTM	0.95	0.97	0.97	0.84	13.25	0.94	0.78	18.99
OTM	0.65	0.67	0.65	0.47	2.54	0.56	0.46	8.87
ATM	0.18	0.17	0.16	0.22	0.34	0.16	0.21	0.40
ITM	0.07	0.07	0.07	0.08	0.11	0.06	0.08	0.38
total	0.64	0.66	0.65	0.53	6.86	0.60	0.50	11.15

Table 2: Comparison of RMSER

	avg	EV	ERM					
			min			FB		
			$\nu=0.1$	$\nu=1$	$\nu=10$	$\nu=0.1$	$\nu=1$	$\nu=10$
DOTM	-0.95	-0.97	-0.97	-0.82	3.48	-0.94	-0.76	6.17
OTM	-0.67	-0.68	-0.66	-0.41	0.51	-0.55	-0.41	1.40
ATM	-0.18	-0.17	-0.13	0.08	0.15	-0.04	0.07	0.21
ITM	-0.04	-0.04	-0.03	0.03	0.04	-0.01	0.02	0.10
total	-0.56	-0.57	-0.56	-0.36	1.13	-0.48	-0.35	2.20

Table 3: Comparison of MER

	avg	EV	ERM					
			min			FB		
			$\nu=0.1$	$\nu=1$	$\nu=10$	$\nu=0.1$	$\nu=1$	$\nu=10$
DOTM	24.15	22.63	19.99	2.17	0.11	11.22	1.00	0.07
OTM	27.63	25.94	22.32	2.54	0.11	12.42	1.09	0.08
ATM	32.40	30.31	25.71	2.49	0.08	13.98	1.16	0.07
ITM	31.66	29.68	25.65	3.43	0.06	14.55	1.31	0.07
total	28.20	26.44	22.82	2.56	0.10	12.70	1.11	0.07

Table 4: Comparison of Γ^{feas}

	avg	EV	ERM					
			min			FB		
			$\nu=0.1$	$\nu=1$	$\nu=10$	$\nu=0.1$	$\nu=1$	$\nu=10$
DOTM	2.39	2.85	3.45	11.50	267.84	5.84	10.95	170.07
OTM	3.07	3.63	4.47	14.76	344.31	7.45	13.00	392.87
ATM	3.86	4.57	5.76	18.29	116.77	9.61	16.28	111.62
ITM	4.26	5.02	6.17	21.92	49.42	10.10	18.79	346.91
total	3.21	3.80	4.69	15.57	242.29	7.81	13.88	277.31

Table 5: Comparison of Γ^{opt}

7 Conclusion

In this paper, we have proposed two deterministic formulations for pricing American options with uncertain volatility based on the expected value method and the expected residual minimization method for stochastic linear complementarity problems. We have shown sufficient conditions that guarantee the formulations by the expected value method and the expected residual minimization method to have solutions. The numerical results indicate that the expected residual minimization method yield solutions that tend to satisfy the no arbitrage assumption than the expected value method.

Acknowledgements

First of all, I would like to express sincere appreciation to Professor Masao Fukushima. His invaluable suggestions for my research and gentle behavior in daily life always encourage me. I would like to tender my acknowledgements to Associate Professor Nobuo Yamashita for his precise advice and great help in program coding. I also would like to express my thanks to Assistant Professor Shunsuke Hayashi for his appropriate comments and hard work. I am deeply indebted to Professor Jun Sekine. He kindly gave me a clear explanation about option pricing theory. I am very grateful to all members of Fukushima Laboratory for their warm friendship during the course. Finally, I really thank my family Tatsuo Hamatani, Yukiko Hamatani and Yasuhiro Hamatani for their constant support.

References

- [1] G. BAKSHI, C. CAO, AND Z. CHEN, *Empirical performance of alternative option pricing models*, The Journal of Finance, 52 (1997), pp. 2003–2049.
- [2] J. R. BIRGE AND F. LOUVEAUX, *Introduction to Stochastic Programming*, Springer-Verlag, New York, 1997.
- [3] F. BLACK AND M. SHOLES, *The pricing of options and corporate liabilities*, Journal of Political Economy, 4 (1973), pp. 637–654.
- [4] P. BOYLE AND P. PHELIM, *Options: A Monte Carlo approach*, Journal of Finance Economics, 4 (1977), pp. 323–338.
- [5] M. J. BRENNAN AND E. S. SCHWARTZ, *The valuation of American put option*, The Journal of Finance, 32 (1977), pp. 449–462.
- [6] X. CHEN AND M. FUKUSHIMA, *Expected residual minimization method for stochastic linear complementarity problems*, Mathematics of Operations Research, 30 (2004), pp. 1022–1038.
- [7] X. CHEN, C. ZHANG, AND M. FUKUSHIMA, *Robust solution of monotone stochastic linear complementarity problems*, Mathematical Programming, 117 (2009), pp. 51–80.
- [8] N. CLARK AND K. PARROTT, *Multigrid for American option pricing with stochastic volatility*, Applied Mathematical Finance, 6 (1999), pp. 177–195.
- [9] R. W. COTTLE, J. S. PANG, AND R. E. STONE, *The Linear Complementarity Problem*, Academic Press, San Diego, 1992.

- [10] J. C. COX, S. A. ROSS, AND M. RUBINSTEIN, *Option pricing : A simplified approach*, Journal of Finance Economics, 7 (1979), pp. 229–268.
- [11] S. P. DIRKSE AND M. C. FERRIS, *The PATH solver: A non-monotone stabilization for mixed complementarity problems*, Optimization Methods and Software, 5 (1995), pp. 123–156.
- [12] F. FACCHINEI AND J. S. PANG, *Finite-Dimensional Variational Inequalities and Complementarity Problems , I and II*, Springer-Verlag, New York, 2003.
- [13] G. GURKAN, A. Y. OZGE, AND S. M. ROBINSON, *Sample-path solution of stochastic variational inequalities*, Mathematical Programming, 84 (1999), pp. 313–333.
- [14] S. L. HESTON, *A closed form solution for options with stochastic volatility with applications to bond and currency options*, Reviews of Financial Studies, 6 (1993), pp. 327–343.
- [15] J. HUANG AND J. S. PANG, *Option pricing and linear complementarity*, The Journal of Computational Finance, 2 (1998), pp. 31–60.
- [16] J. C. HULL, *Options, Futures, and Other Derivatives*, Prentice Hall, Upper Saddle River, 2000.
- [17] P. KALL AND S. W. WALLACE, *Stochastic Programming*, John Wiley & Sons, 1994.
- [18] K. MARTI, *Stochastic Optimization Methods*, Springer-Verlag, Berlin, 2005.
- [19] O. SVENSSON AND S. VOROBYOV, *A subexponential algorithm for a subclass of P-matrix generalized linear complementarity problems*, tech. rep., DIMACS, 2005.
- [20] J. A. TELLY, *Valuing American options in path simulation model*, Transactions of Society of Actuaries, XLV (1994), pp. 499–520.
- [21] P. WILMOTT, J. N. DEWYNNE, AND S. HOWISON, *The Mathematics of Financial Derivatives - A Student Introduction.*, Cambridge University Press., Cambridge, 1995.
- [22] R. ZVAN, P. A. FORSYTH, AND K. R. VETZAL, *Penalty methods for American options with stochastic volatility*, Journal of Computational and Applied Mathematics, 91 (1998), pp. 119–218.

A Data for numerical experiments

Nov 30, 2009 Asset price : 511
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Jan	360	0.35
Jan	390	0.55
Jan	420	1.33
Jan	450	2.8
Jan	510	17.1
Jan	540	38.2
Feb	360	0.9
Feb	450	6.4
Mar	420	5.6
Mar	480	15.28

Dec 8, 2009 Asset price : 507
Risk-free rate r : 0.147 %

Expiration	Strike	Price
Jan	390	0.4
Jan	420	0.9
Jan	450	2.06
Jan	480	5.5
Feb	300	0.15
Feb	390	1.5
Feb	420	2.85
Feb	450	5.5
Feb	480	10.9
Feb	510	21.64
Mar	360	1.45
Mar	480	15.1

Dec 11, 2009 Asset price : 514
Risk-free rate r : 0.157 %

Expiration	Strike	Price
Jan	420	0.6
Jan	450	1.31
Jan	480	3.26
Feb	480	8.4
Mar	420	3.73
Mar	480	12.33

Dec 15, 2009 Asset price : 512
Risk-free rate r : 0.164 %

Expiration	Strike	Price
Feb	300	0.05
Feb	390	0.9
Feb	420	1.75
Feb	510	16.45
Mar	360	0.85
Mar	420	3.3
Mar	480	11.5

Dec 16, 2009 Asset price : 506
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Jan	420	0.55
Jan	450	1.4
Jan	480	3.8
Jan	510	12.2
Feb	420	1.75
Feb	450	4.11
Feb	480	9
Feb	510	19.52
Mar	420	3.8

Dec 18, 2009 Asset price : 509
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Jan	360	0.1
Jan	420	0.53
Jan	450	1.2
Jan	510	11
Feb	480	8.5
Mar	420	3.7
Mar	540	41.8

Dec 21, 2009 Asset price : 514
Risk-free rate r : 0.165 %

Expiration	Strike	Price
Jan	450	0.6
Feb	450	2.8
Feb	480	6.1
Feb	510	14.39
Mar	480	10
Mar	510	18.7
Mar	540	35.9

Dec 22, 2009 Asset price : 515
Risk-free rate r : 0.172 %

Expiration	Strike	Price
Jan	480	1.4
Jan	510	6.5
Jan	420	1.05
Feb	480	5.8
Feb	510	13.3
Mar	480	9
Mar	510	18.4

Dec 23, 2009 Asset price : 515
Risk-free rate r : 0.172 %

Expiration	Strike	Price
Jan	390	0.1
Jan	450	0.5
Jan	510	6
Feb	390	0.45
Feb	450	2.25
Feb	480	5.4
Feb	510	13.17
Mar	420	2.25
Mar	540	34.9

Dec 24, 2009 Asset price : 518
Risk-free rate r : 0.168 %

Expiration	Strike	Price
Jan	420	0.2
Jan	450	0.45
Jan	480	1.14
Jan	510	5
Feb	420	0.89
Feb	480	4.8
Mar	420	2
Mar	450	4
Mar	540	32.9

Jan 4, 2010 Asset price : 523
Risk-free rate r : 0.172 %

Expiration	Strike	Price
Feb	420	0.7
Feb	450	1.34
Feb	480	3.25
Feb	510	8.8
Feb	570	50
Mar	420	1.55
Mar	480	6.5
Mar	510	13.6
Mar	540	28.6
Apr	480	9.53

Jan 5, 2010 Asset price : 525
Risk-free rate r : 0.165 %

Expiration	Strike	Price
Feb	450	1.15
Feb	480	2.7
Feb	510	8.4
Feb	540	23.9
Mar	420	1.35
Mar	450	3.1
Mar	510	13.25
Apr	300	0.25
Apr	480	9.45

Jan 6, 2010 Asset price : 524
Risk-free rate r : 0.157 %

Expiration	Strike	Price
Feb	390	0.25
Feb	450	0.95
Feb	480	2.47
Feb	510	7
Mar	420	1.2
Mar	450	2.7
Mar	480	5.6
Apr	420	2.45
Apr	480	8.5

Jan 7, 2010 Asset price : 526
Risk-free rate r : 0.162 %

Expiration	Strike	Price
Feb	390	0.2
Feb	480	2.18
Feb	510	6.5
Mar	390	0.6
Mar	420	1.2
Apr	420	2.1

Jan 8, 2010 Asset price : 528
Risk-free rate r : 0.149 %

Expiration	Strike	Price
Feb	450	0.65
Feb	480	2
Feb	510	5.7
Mar	450	2.1
Mar	480	4.3
Apr	360	0.55
Apr	420	2.11

Jan 11, 2010 Asset price : 529
Risk-free rate r : 0.13 %

Expiration	Strike	Price
Feb	420	0.31
Feb	450	0.65
Feb	480	1.55
Feb	510	5.45
Feb	570	43.8
Mar	360	0.15
Mar	420	0.9
Mar	450	1.8
Mar	480	4.2
Mar	510	9.82
Mar	570	43.8
Apr	360	0.5

Jan 12, 2010 Asset price : 524
Risk-free rate r : 0.137 %

Expiration	Strike	Price
Feb	420	0.35
Feb	450	0.9
Feb	480	2.2
Feb	510	7.2
Mar	480	5.4
Mar	510	11.68

Jan 13, 2010 Asset price : 528
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Feb	450	0.75
Feb	480	1.72
Feb	510	5
Feb	540	17.4
Mar	360	0.35
Mar	420	1.2
Mar	540	23.2

Jan 14, 2010 Asset price : 530
Risk-free rate r : 0.137 %

Expiration	Strike	Price
Feb	450	0.7
Feb	480	1.4
Feb	510	4.6
Feb	540	16.4
Mar	330	0.15
Mar	360	0.2
Mar	420	0.9
Mar	450	1.85
Mar	480	3.83
Mar	510	8.4
Mar	540	21
Apr	480	6.2

Jan 15, 2010 Asset price : 524
Risk-free rate r : 0.149 %

Expiration	Strike	Price
Feb	420	0.4
Feb	450	0.75
Feb	480	1.9
Feb	510	5.7
Feb	540	20.5
Mar	420	0.8
Mar	450	2.3
Mar	480	4.7
Mar	510	10.5
Mar	540	26.05
Apr	480	7.5
Apr	540	29.2

Jan 19, 2010 Asset price : 530
Risk-free rate r : 0.137 %

Expiration	Strike	Price
Feb	420	0.21
Feb	450	0.45
Feb	480	1.15
Feb	510	3.7
Mar	420	1
Mar	480	3.3
Mar	510	8.1
Mar	540	19.9
Apr	510	11.91
May	300	0.25
May	420	3.4

Jan 20, 2010 Asset price : 525
Risk-free rate r : 0.14 %

Expiration	Strike	Price
Feb	300	0.05
Feb	450	0.55
Feb	480	1.45
Feb	510	4.8
Feb	540	19.6
Feb	600	80.34
Mar	390	0.55
Mar	420	0.98
Mar	450	1.85
Mar	480	4.5
Mar	510	9.7
Mar	540	25.28

Jan 21, 2010 Asset price : 514
Risk-free rate r : 0.13 %

Expiration	Strike	Price
Feb	360	0.08
Feb	450	1.25
Feb	480	3.1
Feb	510	9.6
Mar	300	0.1
Mar	420	1.17
Mar	450	2.9
Mar	480	6.5
Mar	510	15.4
Mar	540	29.5
Apr	360	0.7
Apr	390	1.3
Apr	420	2.6
Apr	450	4.1
Apr	480	6.62
May	480	13.2

Jan 22, 2010 Asset price : 502
Risk-free rate r : 0.124 %

Expiration	Strike	Price
Feb	390	0.4
Feb	420	1.4
Feb	450	2.8
Feb	480	6.2
Feb	510	15.9
Feb	540	38
Mar	300	0.1
Mar	360	0.85
Mar	390	1.6
Mar	420	3
Mar	450	5.6
Mar	480	10.3
Mar	510	21.9
Mar	540	39.4
Mar	570	60.37
Apr	420	4.6
Apr	450	7.11
Apr	480	13.1
Apr	510	25.7
Apr	540	38.3

Jan 25, 2010 Asset price : 505
Risk-free rate r : 0.134 %

Expiration	Strike	Price
Feb	420	0.95
Feb	450	1.9
Feb	480	4.8
Feb	510	14
Feb	540	34.86
Mar	330	0.15
Mar	420	2.5
Mar	450	4.47
Mar	480	9.2
Mar	510	19.6
Mar	540	40.2
Apr	420	3.7
Apr	480	11.6

Jan 26, 2010 Asset price : 503
Risk-free rate r : 0.142 %

Expiration	Strike	Price
Feb	420	1.05
Feb	450	1.35
Feb	480	4.7
Mar	420	1.7
Mar	450	4.2
Mar	480	9.16
Mar	510	20.9
Mar	540	42
Mar	570	67.29
Apr	360	0.9
Apr	450	6.7
Apr	480	13
Apr	510	24
Apr	540	44.1

Jan 27, 2010 Asset price : 506
Risk-free rate r : 0.157 %

Expiration	Strike	Price
Feb	420	0.35
Feb	450	1
Feb	480	3.4
Feb	510	12.6
Feb	540	39.4
Mar	330	0.2
Mar	420	1.9
Mar	450	4.1
Mar	480	7.8
Mar	510	19.3
Mar	540	41.7
Apr	480	11.25

Jan 28, 2010 Asset price : 500
Risk-free rate r : 0.145 %

Expiration	Strike	Price
Feb	420	0.55
Feb	450	1.3
Feb	480	4.5
Feb	510	16
Mar	360	0.3
Mar	420	1.8
Mar	450	4.6
Mar	480	9.2
Mar	510	20.6
Mar	540	42
Apr	420	3.2
Apr	480	13.5
Apr	510	23
May	480	17.6

Jan 29, 2010 Asset price : 495
Risk-free rate r : 0.142 %

Expiration	Strike	Price
Feb	420	0.25
Feb	450	1.6
Feb	480	5.2
Feb	510	20.4
Feb	540	42.8
Mar	360	0.3
Mar	420	1.5
Mar	450	3.3
Mar	480	10.9
Mar	510	25.6
Mar	540	41
Apr	420	3.3
Apr	480	14.6
Apr	510	27.1
Apr	540	49.5

Feb 01, 2010 Asset price : 501
Risk-free rate r : 0.157 %

Expiration	Strike	Price
Mar	420	1.35
Mar	450	3.1
Mar	480	8.1
Mar	510	19.8
Apr	420	3.2
Apr	510	24
May	420	4.7
May	480	16.5

Feb 02, 2010 Asset price : 508
Risk-free rate r : 0.165 %

Expiration	Strike	Price
Mar	300	0.05
Mar	420	1.15
Mar	450	2.14
Mar	480	5.8
Mar	510	15.09
Mar	540	37.6
Apr	480	9.24
Apr	510	19.5
Apr	540	39.8

Feb 03, 2010 Asset price : 505
Risk-free rate r : 0.165 %

Expiration	Strike	Price
Mar	420	0.75
Mar	450	2.1
Mar	480	6
Mar	510	16.02
Apr	360	0.55
Apr	450	4.7
Apr	510	20.5
May	480	14

Feb 04, 2010 Asset price : 490
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Mar	360	0.35
Mar	420	1.84
Mar	450	4.83
Mar	480	11.8
Mar	510	27
Mar	540	49.4
Mar	570	80.7
Apr	420	3.3
Apr	450	7.33
Apr	480	15.8
Apr	510	29.2
Apr	540	51.6
May	480	20.5

Feb 05, 2010 Asset price : 491
Risk-free rate r : 0.152 %

Expiration	Strike	Price
Mar	360	0.35
Mar	420	3.1
Mar	450	6.7
Mar	480	11.4
Mar	510	26.8
Mar	570	89.2
Apr	420	4.3
Apr	450	10.55
Apr	480	17.2
May	420	6.6
May	480	21.6