Abstract

The semi-infinite program (SIP) is normally represented with infinitely many inequality constraints, and has been much studied so far. However, there have been very few studies on the SIP involving second-order cone (SOC) constraints, even though it has important applications such as Chebychev-like approximation and filter design.

In this paper, we focus on the SIP with a convex objective function and infinitely many SOC constraints, called the SISOCP for short. We show that, under a generalized Slater constraint qualification, an optimum of the SISOCP satisfies the KKT conditions that can be represented only with a *finite* subset of the SOC constraints. Next we introduce the regularization and the explicit exchange methods for solving the SISOCP. We first provide an explicit exchange method without a regularization technique, and show that it has global convergence under the strict convexity assumption on the objective function. Then we propose an algorithm combining a regularization method with the explicit exchange method. For the SISOCP, we establish global convergence of the hybrid algorithm without the strict convexity assumption.

Contents

1 Introduction

In this paper, we focus on the following semi-infinite programs with an infinite number of second-order cone constraints (SISOCP):

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & A(t)^\top x - b(t) \in \mathcal{K} \quad (\forall t \in T), \end{array} \tag{1.1}$$

where $T \subseteq \mathbb{R}^l$ is a given compact set, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, $A : \mathbb{R}^l \to \mathbb{R}^{n\times m}$ and $b : \mathbb{R}^l \to \mathbb{R}^m$ are continuous functions, and $\mathcal{K} \subseteq \mathbb{R}^m$ is the Cartesian product of second-order cones (SOCs), that is, $\mathcal{K} := \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \cdots \times \mathcal{K}^{m_k}$ with

$$m = m_1 + m_2 + \cdots + m_k, \quad \mathcal{K}^{m_j} := \left\{ (x_1, \tilde{x}^\top)^\top \in \mathbb{R} \times \mathbb{R}^{m_j - 1} \mid x_1 \geq \|\tilde{x}\| \right\}.$$

Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$, and $\tilde{v}$ denotes $(v_2, v_3, \cdots, v_l)^\top \in \mathbb{R}^{l-1}$ for $v = (v_1, v_2, \cdots, v_l)^\top \in \mathbb{R}^l$. Since $\mathcal{K}^1 = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, we have $\mathcal{K} = \mathbb{R}^k_+$ when $m_1 = m_2 = \cdots = m_k = 1$. For simplicity, we will often write $(x_1, \tilde{x})^\top$ for $(x_1, \tilde{x}^\top)^\top$. Moreover, we always suppose that SISOCP (1.1) has at least one solution.

One of typical applications for SISOCP (1.1) is a Chebychev-like approximation with vector-valued functions. Let $Y \subseteq \mathbb{R}^n$ be a given compact set, and $\Phi : Y \to \mathbb{R}^m$ and $F : \mathbb{R}^l_+ \times Y \to \mathbb{R}^m$ be given functions. Then, how can we determine a parameter $u \in \mathbb{R}^l_+$ such that $\Phi(y) \approx F(u, y)$ for all $y \in Y$? One relevant approach is to solve the following problem:

$$\begin{array}{ll} \underset{u}{\text{Minimize}} & \max_{y \in Y} \|\Phi(y) - F(u, y)\| \\ \text{subject to} & u \in \mathbb{R}^l_+, \end{array} \tag{1.2}$$

which can be rewritten as

$$\begin{array}{ll} \underset{u,r}{\text{Minimize}} & r \\ \text{subject to} & \begin{pmatrix} u \\ r \\ \Phi(y) - F(u, y) \end{pmatrix} \in \mathbb{R}^l_+ \times \mathcal{K}^{m+1} \quad (\forall y \in Y) \end{array}$$

by introducing the auxiliary variable $r \in \mathbb{R}$. If F is affine with respect to u, then the above problem reduces to the SISOCP (1.1) with $\mathcal{K} = (\mathcal{K}^1)^l \times \mathcal{K}^{m+1}$.

When $m = 1$, i.e., $\mathcal{K} = \mathbb{R}_+$, SISOCP (1.1) is the classical semi-infinite program (SIP) [16, 11, 6, 17, 8, 21], which has a wide application in engineering (e.g., the air pollution control, the robot trajectory planning, the stress of materials, etc.[16, 11]). So far, many algorithms have been proposed for solving SIPs, such as the discretization method [6], the local reduction based method [23, 14, 7] and the exchange method [8, 21, 9]. The discretization method solves the relaxed SIP with T replaced by a finite set $T^k \subseteq T$, and the sequence of index sets $\{T^k\}$ is generated so that the distance[1] from T^k to T converges to 0 as k goes to infinity. While this method is comprehensible and easy to implement, the computational cost tends to be huge since the cardinality of T^k grows unboundedly. In the local reduction based method, the infinite number of constraints in the SIP is rewritten as a finite number of constraints with

[1]For two sets $X \subseteq Y$, the distance from X to Y is defined as $\operatorname{dist}(X, Y) := \sup_{y \in Y} \inf_{x \in X} \|x - y\|$.

implicit functions. Although the SIP can be reformulated as a finitely constrained optimization problem by this method, it is not possible in general to evaluate the implicit functions exactly in a direct manner. The exchange method solves a relaxed subproblem with T replaced by a finite subset $T^k \subseteq T$. In this method, T^k is updated so that $T^{k+1} \subseteq T^k \cup \{t_1, t_2, \cdots, t_r\}$ with $\{t_1, t_2, \cdots, t_r\} \subseteq T \setminus T^k$.

Studies on the second-order cones (SOCs) have been advanced significantly in the last decade. One of the most popular problems associated with SOCs is the linear second-order cone program (LSOCP). The primal-dual interior-point method [15, 1] is well known as an effective algorithm for solving LSOCP, and some software packages implementing them [22, 24] have been produced. The nonlinear second-order cone program (NLSOCP) [13, 12, 25] is more complicated and not studied so much as LSOCP. The second-order cone complementarity problem (SOCCP) is another important problem involving SOCs. The Karush-Kuhn-Tucker conditions for LSOCP and NLSOCP are particularly represented as SOCCPs. The smoothing method [10, 4] is one of useful algorithms for solving SOCCP.

Recently, Hayashi and Wu [20] considered the linear semi-infinite program with SOC constraints of the form

$$\begin{aligned} &\text{Minimize} \quad c^\top x \\ &\text{subject to} \quad x \in \mathcal{K}, \;\; a(s)^\top x - b(s) \in \mathbb{R}_+ \; (\forall s \in S) \end{aligned} \tag{1.3}$$

and proposed an explicit exchange method. One may think that SISOCP (1.1) is quite similar to problem (1.3) since both of them combine the SIP with SOCs. However, we note that problem (1.3) includes at most finitely many SOC constraints and cannot be applied to the Chebychev-like approximation problem (1.2). On the other hand, SISOCP (1.1) is formulated with an infinite number of SOC constraints and can be applied to problem (1.2).

The main purpose of the paper is threefold. First, we provide the optimality conditions for SISOCP (1.1). The KKT conditions for SISOCP (1.1) could naturally be described by means of integration and Borel measure since T is infinite. However, we show that, under Slater's constraint qualification, the KKT conditions at the optimum can be represented by using a *finite* number of elements in T. Second, we propose an explicit exchange method for solving the SISOCP (1.1) and show its global convergence under the strict convexity of the objective function. Third, we propose an algorithm that can solve SISOCP (1.1) without the strict convexity. This algorithm is a hybrid of the explicit exchange method and the regularization method, which is known to be effective in handling ill-posed problems. With the help of regularization, local convergence of the algorithm can be established for SISOCP (1.1) without the strict convexity.

This paper is organized as follows. In Section 2, we discuss the optimality conditions for SISOCP (1.1). In Section 3, we introduce the spectral factorization with respect to a SOC. In Section 4, we propose the explicit exchange method for solving SISOCP (1.1). In Section 5, we combine the explicit exchange method with the regularization method, and show that the hybrid algorithm is globally convergent for SISOCP (1.1). In Section 6, we give some numerical results for the hybrid algorithm. In Section 7, we conclude the paper with some remarks.

The notation used in this paper is as follows: For vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the SOC complementarity condition, $x^\top y = 0$, $x \in \mathcal{K}$ and $y \in \mathcal{K}$, is written as $\mathcal{K} \ni x \perp y \in \mathcal{K}$. Occasionally, we denote the inner product of vectors as $\langle x, y \rangle = x^\top y$. For a set $X \subseteq \mathbb{R}^n$ and

$x \in X$, $T_X(x)$ denotes a tangent cone to X at x, i.e., $T_X(x) = \{d \in \mathbb{R}^n \mid \lim_{k\to\infty} a_k(x_k - x) = d,\ \lim_{k\to\infty} x_k = x,\ x_k \in X,\ a_k \geq 0\ (k = 1, 2, \ldots)\}$. For a nonempty cone $C \subseteq \mathbb{R}^n$, C^d denotes the dual cone of C defined as $C^d := \{y \in \mathbb{R}^n \mid y^\top z \geq 0,\ \forall z \in C\}$. In particular, when $C = \mathcal{K}$, $C = C^d$ holds. For a nonempty set $D \subseteq \mathbb{R}^n$, we denote its closure, interior, boundary and convex hull by $\operatorname{cl} D$, $\operatorname{int} D$, $\operatorname{bd} D$ and $\operatorname{co} D$, respectively. Furthermore, for SISOCP (1.1), we denote the feasible set by S.

2 Optimality condition for semi-infinite second-order cone program

In this section, we provide the optimality conditions for SISOCP (1.1). When $m = 1$ and $\mathcal{K} = \mathbb{R}_+$, SISOCP (1.1) reduces to the classical semi-infinite program and the optimality conditions are given as follows [16, Theorem 2].

> Let $\bar{x}$ be an optimum of SISOCP (1.1) with $m = 1$ and $\mathcal{K} = \mathbb{R}_+$. Suppose that the Slater constraint qualification holds for SISOCP (1.1) with $\mathcal{K} = \mathbb{R}_+$, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $A(t)^\top x_0 - b(t) > 0$ $(\forall t \in T)$. Then, there exist p elements $t_1, t_2, \ldots, t_p \in T$ such that $p \leq n$ and
>
> $$\begin{aligned} &\nabla f(\bar{x}) - \sum_{i=1}^{p} \eta_i A(t_i) = 0, \\ &\mathbb{R}_+ \ni \eta_i \perp A(t_i)^\top \bar{x} - b(t_i) \in \mathbb{R}_+ \quad (i = 1, 2, \ldots, p). \end{aligned} \tag{2.1}$$

Does a similar result hold in the general case where $\mathcal{K}$ is the Cartesian product of SOCs? In this section, we define the generalized Slater constraint qualification (GSCQ), and show that the optimality conditions can be represented with finitely many SOC constraints under the GSCQ.

This section consists of two subsections. In Subsection 2.1, we define the GSCQ and the generalized Abadie constraint qualifications (GACQ) and show that the GACQ holds under the GSCQ. In Subsection 2.2, we derive the optimality conditions for SISOCP (1.1) by using the results of Subsection 2.1 and Carathéodory's Theorem.

Before going to the subsections, we provide some propositions, which play important roles in proving the propositions and theorems.

Proposition 2.1. [18, Theorem 17.1] *Let $C \subseteq \mathbb{R}^n$ be an arbitrary nonempty cone. Then, we have*

$$C^{dd} = \operatorname{cl} \operatorname{co} C.$$

Particularly, when C is a closed convex cone, we have $C = C^{dd}$.

Proposition 2.2. *Let $D \subseteq \mathbb{R}^n$ be an arbitrary convex set with nonempty interior. Then, we have*

$$x \in \operatorname{int} D,\ y \in \operatorname{cl} D,\ \lambda \in [0, 1) \implies (1 - \lambda)x + \lambda y \in \operatorname{int} D. \tag{2.2}$$

Proof. Choose $x \in \operatorname{int} D$, $y \in \operatorname{cl} D$ and $\lambda \in [0,1)$ arbitrarily. We will show that there exists an $\varepsilon > 0$ such that $(1-\lambda)x + \lambda y + B(0,\varepsilon) \subseteq D$, where $B(0,\varepsilon) := \{x \in \mathbb{R}^n \mid \|x\| \le \varepsilon\}$. From $y \in \operatorname{cl} D$, we have $y \in D + B(0,\varepsilon)$ for any $\varepsilon > 0$. Therefore, by choosing a sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}(1-\lambda)x + \lambda y + B(0,\varepsilon) &\subseteq (1-\lambda)x + \lambda(D + B(0,\varepsilon)) + B(0,\varepsilon)\\ &= (1-\lambda)(x + (1-\lambda)^{-1}(1+\lambda)B(0,\varepsilon)) + \lambda D\\ &\subseteq (1-\lambda)D + \lambda D = D,\end{aligned}$$

where the equalities hold since $\alpha X + \beta X = (\alpha+\beta)X$ for any $\alpha, \beta \ge 0$ and any convex set X, and the last inclusion is due to $x \in \operatorname{int} D$. $\square$

2.1 Generalized Slater and Abadie constraint qualifications

In the case of the convex optimization problem with finitely many inequality constraints, it is known that Abadie's constraint qualification holds under Slater's constraint qualification, and then the KKT conditions serve as a necessary and sufficient condition for the global optimality [2]. In this subsection, we define the generalized Slater and Abadie constraint qualifications (GSCQ and GACQ) for SISOCP (1.1), and show that the GACQ always holds under the GSCQ. Let $\bar{x}$ be an arbitrary feasible solution of SISOCP (1.1), and S be the feasible solution set of SISOCP (1.1), that is,

$$S := \{x \in \mathbb{R}^n \mid A(t)^\top x - b(t) \in \mathcal{K} \ (\forall t \in T)\}.$$

We define the following cones:

$$\begin{aligned}G_t(\bar{x}) &:= \{\alpha(A(t)^\top \bar{x} - b(t)) \mid \alpha \le 0\},\\ \Lambda_t(\bar{x}) &:= \mathcal{K} + G_t(\bar{x}), &(2.3)\\ C_t(\bar{x}) &:= \left\{y \in R^n \mid A(t)^\top y \in \operatorname{cl} \Lambda_t(\bar{x})\right\}, &(2.4)\\ C_S(\bar{x}) &:= \bigcap_{t \in T} C_t(\bar{x}). &(2.5)\end{aligned}$$

We note that the closure of $\Lambda_t(\bar{x})$ is the tangent cone of $\mathcal{K}$ at $A(t)^\top \bar{x} - b(t)$, and the dual cone of $\Lambda_t(\bar{x})$ characterizes the directions satisfying the SOC complementarity conditions for $A(t)^\top \bar{x} - b(t)$, i.e., $\Lambda_t(\bar{x})^d = \{y \in \mathbb{R}^m \mid \mathcal{K} \ni y \perp A(t)^\top \bar{x} - b(t) \in \mathcal{K}\}$. (See Proposition 2.9 below.) Also, $C_S(\bar{x})$ is a generalization of the linearized cone as defined in [5], for the case where $|T| < \infty$ and $\mathcal{K} = \mathbb{R}_+$.

Now, we define GSCQ and GACQ by using the above cones.

Definition 2.3 (GSCQ)**.** *We say that the generalized Slater constraint qualification* (*GSCQ*) *holds for SISOCP* (1.1) *if there exists some $x_0 \in \mathbb{R}^n$ such that*

$$A(t)^\top x_0 - b(t) \in \operatorname{int} \mathcal{K} \quad (\forall t \in T). \tag{2.6}$$

Definition 2.4 (GACQ)**.** *Let S and $\bar{x} \in S$ be the feasible set and a feasible solution of SISOCP* (1.1)*, respectively. Then, we say that the generalized Abadie constraint qualification GACQ holds at $\bar{x} \in S$ if*

$$C_S(\bar{x}) \subseteq T_S(\bar{x}), \tag{2.7}$$

where $C_S(\bar{x})$ is defined by (2.5) *and $T_S(\bar{x})$ is the tangent cone to S at $\bar{x}$.*

Next, we show that the GACQ holds under the GSCQ. To this end, we show the following two lemmas by using the following set:

$$C_S^\circ(\bar{x}) := \bigcap_{t\in T} \left\{ y \in \mathbb{R}^n \mid A(t)^\top y \in \operatorname{int}\mathcal{K} + G_t(\bar{x}) \right\}. \tag{2.8}$$

Notice that $C_S^\circ(\bar{x})$ is not empty for GSCQ.

Lemma 2.5. *Assume that the GSCQ holds for SISOCP* (1.1). *Let $\bar{x}$ be an arbitrary feasible solution of SISOCP* (1.1). *Let $C_S(\bar{x})$ and $C_S^\circ(\bar{x})$ be defined by* (2.5) *and* (2.8), *respectively. Then, $C_S^\circ(\bar{x})$ is nonempty and $C_S(\bar{x}) = \operatorname{cl} C_S^\circ(\bar{x})$.*

Proof. If we have $C_S(\bar{x}) = \operatorname{cl} C_S^\circ(\bar{x})$, then $C_S^\circ(\bar{x})$ must be nonempty since $0 \in C_S(\bar{x})$. So, we only show $C_S(\bar{x}) = \operatorname{cl} C_S^\circ(\bar{x})$. Notice that $C_S(\bar{x}) \supseteq C_S^\circ(\bar{x})$. Then, we have $C_S(\bar{x}) \supseteq \operatorname{cl} C_S^\circ(\bar{x})$, since $C_S(\bar{x})$ is closed. Thus, it suffices to show $C_S(\bar{x}) \subseteq \operatorname{cl} C_S^\circ(\bar{x})$. Let $y \in C_S(\bar{x})$ be chosen arbitrarily. Then, we have to show that there exists some $\{y^k\} \subseteq C_S^\circ(\bar{x})$ such that $y^k \to y$ as $k \to \infty$. By the GSCQ, there is an $x_0 \in \mathbb{R}^n$ such that $A(t)^\top x_0 - b(t) \in \operatorname{int}\mathcal{K}$ for any $t \in T$. Let $y_0 := x_0 - \bar{x}$. Then, we have $A(t)^\top y_0 = (A(t)^\top x_0 - b(t)) - (A(t)^\top \bar{x} - b(t)) \in \operatorname{int}\mathcal{K} + G_t(\bar{x})$. Since $\operatorname{int}\mathcal{K} + G_t(\bar{x})$ is an open convex set, we have

$$A(t)^\top y_0 \in \operatorname{int}\mathcal{K} + G_t(\bar{x}) = \operatorname{int}(\operatorname{int}\mathcal{K} + G_t(\bar{x})).$$

for any $t \in T$. Since $y \in C_S(\bar{x})$ and $\operatorname{cl}\Lambda_t(\bar{x}) = \operatorname{cl}(\mathcal{K} + G_t(\bar{x})) = \operatorname{cl}(\operatorname{int}\mathcal{K} + G_t(\bar{x}))$[2], we have

$$A(t)^\top y \in \operatorname{cl}(\operatorname{int}\mathcal{K} + G_t(\bar{x})).$$

Applying Proposition 2.2 with $D := \operatorname{int}\mathcal{K} + G_t(\bar{x})$, $x := A(t)^\top y_0$, $\lambda := 1 - \eta$ and $y := A(t)^\top y$, we have

$$A(t)^\top((1-\eta)y + \eta y_0) \in \operatorname{int}\mathcal{K} + G_t(\bar{x}) \tag{2.9}$$

for any $t \in T$ and $\eta \in (0, 1]$. Let $\{\eta_k\} \subseteq (0, 1]$ be a sequence such that $\lim_{k\to\infty} \eta_k = 0$ and $\{y^k\}$ be defined by $y^k := (1-\eta_k)y + \eta_k y_0$. Then, (2.9) implies that $A(t)^\top y^k \in \operatorname{int}\mathcal{K} + G_t(\bar{x})$ for any k and $t \in T$. Therefore, $\{y^k\} \subseteq C_S^\circ(\bar{x})$ and $\lim_{k\to\infty} y^k = y$. This completes the proof. □

Lemma 2.6. *Assume that the GSCQ holds for SISOCP* (1.1). *Let $\bar{x}$ be an arbitrary feasible solution of SISOCP* (1.1). *For $y \in \mathbb{R}^n$ and $t \in T$, let $\alpha_y(t) \in \mathbb{R}$ be defined by*

$$\alpha_y(t) := \max_{\alpha\in[0,1]} \{\alpha \mid A(t)^\top(\bar{x} + \alpha y) - b(t) \in \mathcal{K}\}. \tag{2.10}$$

Then, for any $y \in C_S^\circ(\bar{x})$, we have

$$\inf_{t\in T} \alpha_y(t) > 0.$$

Proof. Let $y \in C_S^\circ(\bar{x})$ and $t \in T$ be chosen arbitrarily. First note that $\alpha_y(t) \geq 0$, since $\bar{x}$ is feasible to SISOCP (1.1). Then, we first prove $\alpha_y(t) > 0$. To this end, it suffices to show the existence of $\alpha \in (0, 1]$ such that

$$A(t)^\top(\bar{x} + \alpha y) - b(t) \in \operatorname{int}\mathcal{K}. \tag{2.11}$$

[2] This equality can be obtained easily from the fact that $\operatorname{cl}(\operatorname{int}\mathcal{K}) = \mathcal{K}$

Since $y \in C_S^\circ(\bar{x})$, we have $A(t)^\top y \in \operatorname{int}\mathcal{K} + G_t(\bar{x})$, which together with the definition of $G_t(\bar{x})$ implies the existence of some $\beta \geq 0$ such that

$$\beta\left(A(t)^\top \bar{x} - b(t)\right) + A(t)^\top y \in \operatorname{int}\mathcal{K}. \tag{2.12}$$

When $\beta = 0$, (2.12) reduces to $A(t)^\top y \in \operatorname{int}\mathcal{K}$, which together with $A(t)^\top \bar{x} - b(t) \in \mathcal{K}$ and Proposition 2.2 implies $\frac{1}{2}A(t)^\top y + \frac{1}{2}\left(A(t)^\top \bar{x} - b(t)\right) = \frac{1}{2}\left(A(t)^\top(\bar{x}+y) - b(t)\right) \in \operatorname{int}\mathcal{K}$, and hence, $A(t)^\top(\bar{x}+y) - b(t) \in \operatorname{int}\mathcal{K}$. We thus have (2.11) with $\alpha = 1$. When $\beta > 0$, by multiplying (2.12) by β^{-1}, we have $A(t)^\top(\bar{x} + \beta^{-1}y) - b(t) \in \operatorname{int}\mathcal{K}$. Due to Proposition 2.2, we have $A(t)^\top(\bar{x} + sy) - b(t) \in \operatorname{int}\mathcal{K}$ for any $s \in (0, \beta^{-1}]$, which implies $A(t)^\top(\bar{x} + \min(\beta^{-1}, 1)y) - b(t) \in \operatorname{int}\mathcal{K}$. Hence, we also have (2.11).

In what follows, we show $\inf_{t\in T} \alpha_y(t) > 0$. Suppose to the contrary that there exists a sequence $\{t^k\} \subseteq T$ such that $\alpha_y(t^k) \to 0$ as $k \to \infty$. Let t^* be an arbitrary accumulation point of $\{t^k\}$. Then, by taking an appropriate subsequence, we have

$$\lim_{k\to\infty} t^k = t^*, \qquad \lim_{k\to\infty} \alpha_y(t^k) = 0. \tag{2.13}$$

From (2.11), there exists an $\bar{\alpha} > 0$ such that

$$A(t^*)^\top(\bar{x} + \bar{\alpha}y) - b(t^*) \in \operatorname{int}\mathcal{K}. \tag{2.14}$$

Hence, by the continuity of functions A and b, we have

$$A(t^k)^\top(\bar{x} + \bar{\alpha}y) - b(t^k) \in \operatorname{int}\mathcal{K} \tag{2.15}$$

for all k sufficiently large. From (2.15) and (2.10), we have $0 < \bar{\alpha} \leq \alpha_y(t^k)$, which together with (2.13) implies $\bar{\alpha} = 0$. However, this contradicts $\bar{\alpha} > 0$. Hence, we have $\inf_{t\in T} \alpha_y(t) > 0$. □

Now, we show the main theorem of this section, which claims that the GSCQ implies the GACQ for SISOCP (1.1).

Theorem 2.7. *Let $\bar{x}$ be an arbitrary feasible solution of SISOCP* (1.1)*. Assume that the GSCQ holds. Then, the GACQ holds at $\bar{x}$.*

Proof. Let $C_S^\circ(\bar{x})$ be defined by (2.8). Then we have $\operatorname{cl} C_S^\circ(\bar{x}) = C_S(\bar{x})$ from Lemma 2.5. Therefore, due to the closedness of $T_S(\bar{x})$, we only have to show

$$C_S^\circ(\bar{x}) \subseteq T_S(\bar{x}).$$

Let $y \in C_S^\circ(\bar{x})$ be chosen arbitrarily and $\alpha_y := \inf_{t\in T} \alpha_y(t)$, where $\alpha_y(t)$ is given by (2.10). Then, we have

$$A(t)^\top(\bar{x} + \beta y) - b(t) \in \mathcal{K} \tag{2.16}$$

for any $\beta \in [0, \alpha_y]$ and $t \in T$, since $A(t)^\top \bar{x} - b(t) \in \mathcal{K}$ and $\mathcal{K}$ is convex.

By Lemma 2.6, we have $\alpha_y > 0$. Hence, we can choose $\{b_k\} \subseteq (0, \alpha_y]$ such that $\lim_{k\to\infty} b_k = 0$. By (2.16), we have

$$A(t)^\top(\bar{x} + b_k y) - b(t) \in \mathcal{K} \qquad (\forall t \in T),$$

which implies $\bar{x} + b_k y \in S$ for all k. Now, recall that the definition of $T_S(\bar{x})$ is given by

$$T_S(\bar{x}) := \left\{ y \in \mathbb{R}^n \mid \lim_{k\to\infty} a_k(x_k - \bar{x}) = y, \lim_{k\to\infty} x_k = \bar{x},\ x_k \in S,\ a_k \geq 0\ (k = 1, 2, \dots) \right\}. \tag{2.17}$$

Thus, by setting $x_k := \bar{x} + b_k y$ and $a_k := 1/b_k$, we have $y \in T_S(\bar{x})$. The proof is completed. □

2.2 The KKT conditions for SISOCP

As we have shown in the previous subsection, the GACQ holds under the GSCQ. In this subsection, by using this result, we show that the optimality condition for SISOCP (1.1) can be represented as the KKT conditions with finitely many SOC constraints. It is well known that the following Carathéodory's Theorem plays a significant role in deriving the optimality condition for the ordinary SIP with inequality constraints. The theorem is also important in deriving the optimality conditions for SISOCP (1.1).

Lemma 2.8. (Carathéodory's Theorem [18, Theorem 17.1]) *Let $D \subseteq \mathbb{R}^n$ be an arbitrary nonempty set, and $\operatorname{co} D$ be the convex hull of D. Then, for any $x \in \operatorname{co} D$, there exist p elements $s_1, s_2, \dots, s_p \in D$ and p positive numbers $\lambda_1, \lambda_2, \dots, \lambda_p > 0$ such that $p \le n+1$, $\sum_{i=1}^p \lambda_i = 1$, and $x = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_p s_p$.*

The SOC complementarity condition that appears in the KKT conditions is written as $\mathcal{K} \ni y(t) \perp A(t)^\top x - b(t) \in \mathcal{K}$ with a Lagrange multiplier vector $y(t)$. The next proposition claims that the dual cone of $\Lambda_t(\bar{x})$ defined by (2.3) characterizes the Lagrange multiplier $y(t)$.

Proposition 2.9. *Let $t \in T$ be chosen arbitrarily, and $\bar{x}$ be an arbitrary feasible solution of SISOCP (1.1). Let $\Lambda_t(\bar{x})$ be defined by (2.3). Then, we have*

$$\begin{aligned} \Lambda_t(\bar{x}) &= \operatorname{co}(\mathcal{K} \cup G_t(\bar{x})), \\ \Lambda_t(\bar{x})^d &= \mathcal{K} \cap G_t(\bar{x})^d \\ &= \{y \in \mathbb{R}^m \mid \mathcal{K} \ni y \perp A(t)^\top \bar{x} - b(t) \in \mathcal{K}\}. \end{aligned}$$

Proof. First, we show $\Lambda_t(\bar{x}) = \mathcal{K} + G_t(\bar{x}) = \operatorname{co}(\mathcal{K} \cup G_t(\bar{x}))$. Since $0 \in G_t(\bar{x})$ and $0 \in \mathcal{K}$, we have $\mathcal{K} + G_t(\bar{x}) \supseteq \mathcal{K}$ and $\mathcal{K} + G_t(\bar{x}) \supseteq G_t(\bar{x})$, that is, $\mathcal{K} + G_t(\bar{x}) \supseteq \mathcal{K} \cup G_t(\bar{x})$. Noticing the convexity of $\mathcal{K} + G_t(\bar{x})$, we have $\mathcal{K} + G_t(\bar{x}) \supseteq \operatorname{co}(\mathcal{K} \cup G_t(\bar{x}))$. Conversely, we show $\mathcal{K} + G_t(\bar{x}) \subseteq \operatorname{co}(\mathcal{K} \cup G_t(\bar{x}))$. Choose $y \in \mathcal{K} + G_t(\bar{x})$ arbitrarily. Then, there exist some $k \in \mathcal{K}$ and $g \in G_t(\bar{x})$ such that $y = k + g$. Since $\mathcal{K}$ and $G_t(\bar{x})$ are cones, we have $2k \in \mathcal{K}$ and $2g \in G_t(\bar{x})$, and hence $y = (2k + 2g)/2 \in \operatorname{co}(\mathcal{K} \cup G_t(\bar{x}))$. This shows $\mathcal{K} + G_t(\bar{x}) \subseteq \operatorname{co}(\mathcal{K} \cup G_t(\bar{x}))$.

We can readily show $\Lambda_t(\bar{x})^d = \mathcal{K} \cap G_t(\bar{x})^d$ since $\Lambda_t(\bar{x})^d = (\operatorname{co}(\mathcal{K} \cup G_t(\bar{x})))^d = (\mathcal{K} \cup G_t(\bar{x}))^d = \mathcal{K}^d \cap G_t(\bar{x})^d = \mathcal{K} \cap G_t(\bar{x})^d$, where the second equality follows since $(\operatorname{co} C)^d = C^d$ for any cone C, the third equality holds since $(C_1 \cup C_2)^d = C_1^d \cap C_2^d$ for any cones C_1 and C_2, and the last equality holds since $\mathcal{K}$ is self-dual.

Finally we show $\mathcal{K} \cap G_t(\bar{x})^d = \{y \in \mathbb{R}^m \mid \mathcal{K} \ni y \perp A(t)^\top \bar{x} - b(t) \in \mathcal{K}\}$. Note that $\{y \in \mathbb{R}^m \mid \mathcal{K} \ni y \perp A(t)^\top \bar{x} - b(t) \in \mathcal{K}\} = \mathcal{K} \cap (A(t)^\top \bar{x} - b(t))^\perp$, where $v^\perp$ denotes the hyper plane orthogonal to vector v. Since it is not difficult to see $G_t(\bar{x})^d \supseteq (A(t)^\top \bar{x} - b(t))^\perp$, we have $\mathcal{K} \cap G_t(\bar{x})^d \supseteq \mathcal{K} \cap (A(t)^\top \bar{x} - b(t))^\perp$. Therefore, the proof will be complete if we show the converse inclusion. Choose $z \in \mathcal{K} \cap G_t(\bar{x})^d$ arbitrarily. Since $z \in G_t(\bar{x})^d$, we have $z^\top (A(t)^\top \bar{x} - b(t)) \le 0$. On the other hand, $z \in \mathcal{K}$ and $A(t)^\top \bar{x} - b(t) \in \mathcal{K}$ imply $z^\top (A(t)^\top \bar{x} - b(t)) \ge 0$. Hence, $z^\top (A(t)^\top \bar{x} - b(t)) = 0$, i.e., $z \in \mathcal{K} \cap (A(t)^\top \bar{x} - b(t))^\perp$. This completes the proof. □

Now, in order to obtain the optimality condition for SISOCP (1.1), we introduce the following cones:

$$H_t(\bar{x}) := \{z \in \mathbb{R}^n \mid z = A(t)\lambda,\ \lambda \in \Lambda_t(\bar{x})^d\}, \tag{2.18}$$

$$H(\bar{x}) := \bigcup_{t \in T} H_t(\bar{x}), \tag{2.19}$$

where $t \in T$ and $\bar{x}$ is a feasible solution. Note that $H_t(\bar{x})$ is a convex cone but may not be closed, and $H_t(\bar{x})$ is a cone but may not be closed or convex.

The next proposition shows the relation between $H(\bar{x})$ and $C_S(\bar{x})$.

Proposition 2.10. *Let $\bar{x} \in S$ be an arbitrary feasible solution of SISOCP* (1.1)*. Let $C_S(\bar{x})$ and $H(\bar{x})$ be defined by* (2.5) *and* (2.19)*, respectively. Then, we have*

$$C_S(\bar{x})^d \subseteq \operatorname{cl} \operatorname{co} H(\bar{x}).$$

Proof. It suffices to prove $C_S(\bar{x}) \supseteq H(\bar{x})^d$. Choose $y \in H(\bar{x})^d$, $t \in T$ and $\lambda \in \Lambda_t(\bar{x})^d$ arbitrarily. Since $y \in H(\bar{x})^d$ and $A(t)\lambda \in H_t(\bar{x}) \subseteq H(\bar{x})$, we have $\langle A(t)^\top y, \lambda \rangle = \langle y, A(t)\lambda \rangle \geq 0$. Note that $t \in T$ and $\lambda \in \Lambda_t(\bar{x})^d$ were chosen arbitrarily. Therefore, we have $A(t)^\top y \in \Lambda_t(\bar{x})^{dd} = \operatorname{cl} \operatorname{co} \Lambda_t(\bar{x}) = \operatorname{cl} \Lambda_t(\bar{x})$ for any $t \in T$, which implies $y \in C_S(\bar{x})$. □

The following lemma is also important for the proof of the subsequent theorem.

Lemma 2.11. *Assume that the GSCQ holds for SISOCP* (1.1)*. Let x_0 be an arbitrary point satisfying* (2.6) *and $z \in \mathcal{K}$ be an arbitrary vector. Then, there exists some $\varepsilon > 0$ such that*

$$(A(t)^\top x_0 - b(t))^\top z \geq \varepsilon \|z\| \tag{2.20}$$

for any $t \in T$.

Proof. For simplicity, let $y(t) := A(t)^\top x_0 - b(t)$. When $z = 0$, inequality (2.20) holds obviously for any $t \in T$. So we only consider the case where $z \neq 0$. Let

$$\delta(t) := \frac{y(t)^\top z}{\|z\|}. \tag{2.21}$$

To show (2.20), it suffices to prove $\inf_{t\in T} \delta(t) > 0$. Suppose that $\inf_{t\in T} \delta(t) \leq 0$ for contradiction. Then, we must have $\inf_{t\in T} \delta(t) = 0$ since $y(t) \in \operatorname{int} \mathcal{K}$ and $z \in \mathcal{K}$ implies $\delta(t) \geq 0$. Due to the compactness of T, there exist some subsequence $\{t^k\} \subseteq T$ and $t^* \in T$ such that $\lim_{k\to\infty} \delta(t^k) = 0$ and $\lim_{k\to\infty} t^k = t^*$. Moreover, the continuity of $y(t)$ yields $\lim_{k\to\infty} y(t^k) = y(t^*)$. Then, by (2.21), we obtain $y(t^*)^\top z = 0$. However, this contradicts $0 \neq z \in \mathcal{K}$ and $y(t^*) \in \operatorname{int} \mathcal{K}$. Therefore, we have $\inf_{t\in T} \delta(t) > 0$. □

Now, we are in the position to show the theorem on the optimality condition for SISOCP (1.1).

Theorem 2.12 (Optimality condition)**.** *Assume that the GSCQ holds for SISOCP* (1.1)*. Let x^* be an arbitrary optimizer of SISOCP* (1.1)*. Then, there exist $t_1, t_2, \ldots, t_p \in T$ and $y_1, y_2, \ldots, y_p \in \mathbb{R}^m$ such that $p \leq n + 1$ and*

$$\nabla f(x^*) - \sum_{i=1}^{p} A(t_i) y_i = 0, \tag{2.22}$$

$$\mathcal{K} \ni y_i \perp A(t_i)^\top x^* - b(t_i) \in \mathcal{K} \ \ (i = 1, 2, \ldots, p). \tag{2.23}$$

Proof. From $x^* \in \operatorname{argmin}_{x\in S} f(x)$ and [19, Theorem 3.6], we have $\nabla f(x^*) \in T_S(x^*)^d$. Also we have $T_S(x^*)^d \subseteq C_S(x^*)^d \subseteq \operatorname{cl}\operatorname{co} H(x^*)$, where the first inclusion holds since $C_S(x^*) \subseteq T_S(x^*)$ from Theorem 2.7, and the second inclusion follows from Proposition 2.10. Therefore, we have

$$\nabla f(x^*) \in \operatorname{cl}\operatorname{co} H(x^*),$$

which indicates the existence of a sequence $\{z^k\} \subseteq \operatorname{co} H(x^*)$ such that

$$\lim_{k\to\infty} z^k = \nabla f(x^*).$$

By Lemma 2.8, (2.18) and (2.19), there exist $n+1$ nonnegative scalars[3] $\alpha_1^k, \alpha_2^k, \ldots, \alpha_{n+1}^k \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i^k = 1$ and

$$z^k = \sum_{i=1}^{n+1} A(t_i^k)\alpha_i^k \lambda_i^k, \quad \lambda_i^k \in \Lambda_{t_i^k}(x^*)^d. \tag{2.24}$$

Denote $y_i^k := \alpha_i^k \lambda_i^k \in \Lambda_{t_i^k}(x^*)^d$ for each i in (2.24).

In what follows, we show that the sequence $\{y_i^k\}$ is bounded and any accumulation point satisfies (2.22) and (2.23). From the GSCQ, there exists an $x_0 \in \mathbb{R}^n$ such that $A(t_i^k)^\top x_0 - b(t_i^k) \in \operatorname{int}\mathcal{K}$ for each i. By $y_i^k \in \Lambda_{t_i^k}(x^*)^d \subseteq \mathcal{K}$ and Lemma 2.11, there exists $\varepsilon > 0$ such that

$$\langle y_i^k,\ A(t_i^k)^\top x_0 - b(t_i^k)\rangle \geq \varepsilon \|y_i^k\| \tag{2.25}$$

for each i. Since $y_i^k \in \Lambda_{t_i^k}(x^*)^d \subseteq G_{t_i^k}(x^*)^d$ from Proposition 2.9, we have

$$\langle y_i^k, A(t_i^k)^\top x^* - b(t_i^k)\rangle \leq 0. \tag{2.26}$$

It, then, follows from (2.26) and (2.25) that

$$\langle y_i^k, A(t_i^k)^\top (x_0 - x^*)\rangle \geq \varepsilon \|y_i^k\|. \tag{2.27}$$

From (2.24), (2.27) and $y_i^k = \alpha_i^k \lambda_i^k$, we have $(z^k)^\top (x_0 - x^*) = \sum_{i=1}^{n+1} \langle y_i^k, A(t_i^k)^\top (x_0 - x^*)\rangle \geq \sum_{i=1}^{n+1} \varepsilon \|y_i^k\|$. Moreover, since $\{z^k\}$ is convergent, there exists $M > 0$ such that $(z^k)^\top (x_0 - x^*) \leq M$ for all k. Therefore, we have

$$M \geq \varepsilon \sum_{i=1}^{n+1} \|y_i^k\|.$$

which implies the boundedness of $\{y_i^k\}$. ここで $\{y_i^k\}$ の有界性について述べていますが, これは ε が y_i^k に独立であるから言えることです. もし, 独立でないならば, その時点で証明が壊れてしまうように思いますが, どうでしょうか？Now, let y_i and t_i be arbitrary accumulation points of $\{y_i^k\}$ and $\{t_i^k\}$, respectively. Then there exist subsequences such that $z^k \to \nabla f(x^*)$, $t_i^k \to t_i$ and $y_i^k \to y_i$ for each $i = 1, 2, \ldots, n+1$. From (2.24) with $y_i^k = \alpha_i^k \lambda_i^k$ and the continuity of function A, we obtain $\nabla f(x^*) = \sum_{i=1}^{n+1} A(t_i) y_i$. Hence, we have (2.22). From $y_i^k \in \Lambda_{t_i^k}(x^*)^d$ and Proposition 2.9, it follows that $\mathcal{K} \ni y_i^k \perp A(t_i^k)^\top x^* - b(t_i^k) \in \mathcal{K}$ for each k. Since $\mathcal{K}$ is closed, we have $y_i \in \mathcal{K}$ and $A(t_i)^\top x^* - b(t_i) \in \mathcal{K}$. Moreover, we have $\langle y_i, A(t_i)^\top x^* - b(t_i)\rangle = 0$, since the function defined by $\theta(y,t) := \langle y, A(t)^\top x^* - b(t)\rangle$ is continuous at any $y \in \mathbb{R}^m$ and $t \in T$. Therefore, (2.23) is obtained. □

Note that the results obtained so far in this section do not rely on any specific property of SOCs from Jordan algebra. Therefore, Theorem 2.12 can be extended directly to the case where $\mathcal{K}$ in SISOCP (1.1) is replaced by a general closed convex cone with nonempty interior.

[3] If we have $p < n+1$ scalars, then we can set $\alpha_{p+1}^k = \alpha_{p+2}^k = \cdots = \alpha_{n+1}^k = 0$ without loss of generality.

3 Spectral factorization with respect to second-order cone

In this section, we introduce some properties on the spectral factorization with respect to second-order cones. The spectral factorization is a fundamental concept in Jordan algebra [3].

First, we consider the case where $\mathcal{K} = \mathcal{K}^m$. For any $z = (z_1, \tilde{z})^\top \in \mathbb{R} \times \mathbb{R}^{m-1}$ $(m \geq 2)$, its spectral factorization with respect to $\mathcal{K}^m$ is written as

$$z = \lambda^1(z)u_1(z) + \lambda^2(z)u_2(z),$$

where λ^1, $\lambda^2 \in \mathbb{R}$ are the spectral values of z defined by

$$\lambda^i(z) = z_1 + (-1)^i \|\tilde{z}\| \quad (i = 1, 2), \tag{3.1}$$

and $u_1(z), u_2(z) \in \mathbb{R}^m$ are the spectral vectors of z given by

$$u_i(z) = \begin{cases} \dfrac{1}{2}\begin{pmatrix} 1 \\ (-1)^i \frac{\tilde{z}}{\|\tilde{z}\|} \end{pmatrix} & (\|\tilde{z}\| \neq 0), \\ \dfrac{1}{2}\begin{pmatrix} 1 \\ (-1)^i \|w\| \end{pmatrix} & (\|\tilde{z}\| = 0), \end{cases} \tag{3.2}$$

with $w \in \mathbb{R}^{m-1}$ such that $\|w\| = 1$. Obviously, $\lambda^1(z) \leq \lambda^2(z)$ holds. For any $z \in \mathbb{R}^m$, it follows that

$$\begin{aligned} \lambda^1(z) \geq 0 &\iff z \in \mathcal{K}^m, \\ \lambda^1(z) > 0 &\iff z \in \operatorname{int} \mathcal{K}^m, \\ \lambda^1(z) = 0 &\iff z \in \operatorname{bd} \mathcal{K}^m. \end{aligned} \tag{3.3}$$

We next consider the case of $\mathcal{K} = \mathcal{K}^{m_1} \times \mathcal{K}^{m_2} \times \cdots \times \mathcal{K}^{m_k}$. For any $z \in \mathbb{R}^m$, define the subvector $z_j \in \mathbb{R}^{m_j}$ for each $j = 1, 2, \ldots, k$ as

$$z = (z_1, z_2, \ldots, z_k)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k}. \tag{3.4}$$

Then, using the spectral factorizations of z_j for each j, we have

$$z = \begin{pmatrix} \lambda^1(z_1)u_1(z_1) + \lambda^2(z_1)u_2(z_1) \\ \vdots \\ \lambda^1(z_k)u_1(z_k) + \lambda^2(z_k)u_2(z_k) \end{pmatrix}, \tag{3.5}$$

where $\lambda^1(z_j)$ and $\lambda^2(z_j)$ are spectral values of z_j with respect to $\mathcal{K}^{m_j}$ defined by (3.1), and $u_1(z_j)$ and $u_2(z_j)$ are spectral vectors of z_j with respect to $\mathcal{K}^{m_j}$.

In addition, define a function $\lambda_{\mathcal{K}}^{\min} : \mathbb{R}^m \to \mathbb{R}$ by

$$\lambda_{\mathcal{K}}^{\min}(z) := \min\{\lambda^1(z_1), \lambda^1(z_2), \ldots, \lambda^1(z_k)\}. \tag{3.6}$$

From (3.3), we have

$$\lambda_{\mathcal{K}}^{\min}(z) \geq 0 \iff z \in \mathcal{K}. \tag{3.7}$$

Now, we define the projection of $z \in \mathbb{R}^m$ onto $\mathcal{K}$ as

$$P_{\mathcal{K}}(z) := \mathrm{argmin}_{w\in\mathcal{K}}\|z - w\|,$$

and the vector $e \in \mathbb{R}^m$ as

$$e := (e_1, e_2, \dots, e_k)^\top, \tag{3.8}$$

where

$$e_j := (1, 0, \dots, 0)^\top \in \mathbb{R}^{m_j}.$$

The vector defined above is the central axis of cone $\mathcal{K}$, and also plays a role of the identity element in Jordan algebra [3]. The followings two propositions are associated with e, which will be needed in the convergence analyses of the algorithms proposed in this paper.

Proposition 3.1. *Let $z = (z_1, z_2, \dots, z_k)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k}$ be an arbitrary vector. Then, we have*

$$P_{\mathcal{K}}(z) = \begin{pmatrix} \max(\lambda^1(z_1), 0)u_1(z_1) + \max(\lambda^2(z_1), 0)u_2(z_1) \\ \vdots \\ \max(\lambda^1(z_k), 0)u_1(z_k) + \max(\lambda^2(z_k), 0)u_2(z_k) \end{pmatrix}. \tag{3.9}$$

Proof. We have $P_{\mathcal{K}^{m_j}}(z_j) = \max(\lambda^1(z_j), 0)u_1(z_j)+\max(\lambda^2(z_j), 0)u_2(z_j)$ from [4, Proposition 3.3]. Since $P_{\mathcal{K}}(z) = P_{\mathcal{K}^{m_1}}(z_1) \times P_{\mathcal{K}^{m_2}}(z_2) \times \cdots \times P_{\mathcal{K}^{m_k}}(z_k)$, we have (3.9). □

Proposition 3.2. *Let $z = (z_1, z_2, \dots, z_k)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k}$ be an arbitrary vector and e be defined by* (3.8)*. Then, we have*

$$e = \begin{pmatrix} u_1(z_1) + u_2(z_1) \\ \vdots \\ u_1(z_k) + u_2(z_k) \end{pmatrix}. \tag{3.10}$$

Proof. By the definition of spectral vectors $u_1(z_j)$ and $u_2(z_j)$, we immediately have $e_j = u_1(z_j)+u_2(z_j)$. Hence, by (3.8), we obtain (3.10). □

4 Explicit exchange method for SISOCP

In this section, we propose an explicit exchange method for solving SISOCP (1.1). Moreover, we show that the algorithm has a global convergence property under mild assumptions.

4.1 Algorithm

The algorithm proposed in this section requires solving second-order cone programs (SOCP) with finitely many constraints as subproblems. Let SOCP (T') denote the relaxed problem of SISOCP (1.1) with T replaced by a finite subset $T' := \{t_1, t_2, \dots, t_p\} \subseteq T$. Then, the SOCP$(T')$ can be formulated as follows:

$$\text{SOCP}(T') \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & A(t_j)^\top x - b(t_j) \in \mathcal{K} \quad (j = 1, 2, \dots, p). \end{array}$$

We suppose that the subproblem SOCP (T') can be solved by any suitable existing algorithm. Let $\bar{x}$ be an optimal solution of SOCP(T'). Then, $\bar{x}$ satisfies the following KKT conditions under some constraint qualification [15, 1]:

$$\begin{aligned} &\nabla f(\bar{x}) - \sum_{t_j \in T'} A(t_j) y_{t_j} = 0, \\ &\mathcal{K} \ni y_{t_j} \perp A(t_j)^\top \bar{x} - b(t_j) \in \mathcal{K} \ \ (j = 1, 2, \ldots, p), \end{aligned} \tag{4.1}$$

where y_{t_j} is a Lagrange multiplier vector corresponding to the constraint $A(t_j)^\top \bar{x} - b(t_j) \in \mathcal{K}$ for each j.

Now, we propose the following algorithm.

Algorithm 1 (Explicit exchange method)

Step 0. Choose a positive sequence $\{\gamma_k\} \subseteq \mathbb{R}_{++}$ such that $\lim_{k\to\infty} \gamma_k = 0$. Choose a finite subset $E^0 := \{t_1^0, \ldots, t_p^0\} \subseteq T$ and solve SOCP(E^0) to obtain an optimal solution x^0. Set $k := 0$.

Step 1. Set $r := 0$, $T_0 := E^k$ and $v^0 := x^k$. Do the following (a)–(c):

(a) Find a $t_{\mathrm{new}}^r \in T$ such that

$$A(t_{\mathrm{new}}^r)^\top v^r - b(t_{\mathrm{new}}^r) \notin -\gamma_k e + \mathcal{K}. \tag{4.2}$$

If such a t_{new}^r does not exist, i.e.,

$$A(t)^\top v^r - b(t) \in -\gamma_k e + \mathcal{K} \tag{4.3}$$

for any $t \in T$, then set $x^{k+1} := v^r$, $E^{k+1} := T_r$, and go to Step 2. Otherwise, let

$$\overline{T}_{r+1} := T_r \cup \{t_{\mathrm{new}}^r\},$$

and go to (b).

(b) Solve SOCP$(\overline{T}_{r+1})$ to obtain an optimum v_{r+1} and Lagrange multipliers y_t^{r+1}, for $t \in \overline{T}_{r+1}$.

(c) Let $T_{r+1} := \{t \in \overline{T}_{r+1} \mid y_t^{r+1} \neq 0\}$. Set $r := r + 1$ and return to (a).

Step 2. If γ_k is sufficiently small, terminate. Otherwise, set $k := k + 1$ and return to Step 1.

In Step 1-(a), we can verify (4.2) by checking whether $\lambda_{\mathcal{K}}^{\min}(A(t_{\mathrm{new}}^r)^\top v^r - b(t_{\mathrm{new}}^r) + \gamma_k e)$ is negative, where $\lambda_{\mathcal{K}}^{\min}$ is defined by (3.6). On the other hand, to verify (4.3), we have to solve $\min_{t\in T} \lambda_{\mathcal{K}}^{\min}(A(t)^\top v^r - b(t) + \gamma_k e)$ and check the nonnegativity of the optimal value. In Step 1-(b), SOCP$(\overline{T}_{r+1})$ can be solved by applying an existing method such as the primal-dual interior point method, the regularized smoothing method, and so on [15, 1, 13, 10, 4]. In Step 1-(c), SOCP(T_{r+1}) is obtained from SOCP$(\overline{T}_{r+1})$ by removing only the constraints with zero Lagrange multipliers, then the optimal values of those two problems are equal. In addition, the feasible region of SOCP$(\overline{T}_{r+1})$ is contained in that of SOCP(T_r). Therefore, we have

$$V_P(T_0) \leq V_P(\overline{T}_1) = V_P(T_1) \leq \cdots \leq V_P(T_r) \leq V_P(\overline{T}_{r+1}) = V_P(T_{r+1}) \leq \cdots \leq V_P(T) < +\infty, \tag{4.4}$$

where $V_P(T')$ denotes the optimal value of SOCP(T').

4.2 Global convergence under strict convexity assumption

In the previous subsection, we proposed the explicit exchange method for solving SISOCP (1.1). In this subsection, we show that the algorithm generates a sequence converging to the optimal solution under the following assumption.

Assumption A. i) Function f is strictly convex over the feasible region of SISOCP (1.1). ii) In Step 1-(b) of Algorithm 1, SOCP($\overline{T}_{r+1}$) is solvable for each r. iii) A sequence generated $\{v^r\}$ in every Step 1 of Algorithm 1 is bounded.

Under this assumption, we have the following proposition, which shows that the distance between v^{r+1} and v^r does not tend to 0 during the inner iterations in Step 1.

Proposition 4.1. *Let that Assumption A hold. Then, in every Step 1 of Algorithm 1, there exists a positive number $N > 0$ such that*

$$\|v^{r+1} - v^r\| \geq N\gamma_k$$

for any $r \geq 0$.

Proof. Denote $z(v,t) := A(t)^\top v - b(t)$ for simplicity. From the continuity of the matrix norm $\|A(t)\| := \max_{\|w\|=1} \|A(t)^\top w\|$ with respect to t and the compactness of T, there exists a sufficiently large $M > 0$ such that $\|A(t)\| \leq M$ for any $t \in T$. Hence, we have

$$\|z(v^{r+1},t) - z(v^r,t)\| = A(t)^\top(v^{r+1} - v^r) \leq M\|v^{r+1} - v^r\| \tag{4.5}$$

for any $t \in T$.

We next show that $\{\|z(v^{r+1},t) - z(v^r,t)\|\}$ is bounded below by some positive number. Let $z := z(v^r, t^r_{\text{new}})$ be decomposed as in (3.4) and denote $\lambda^i_j := \lambda^i(z_j)$ and $u^i_j := u_i(z_j)$ for each $(i,j) \in \{1,2\} \times \{1,2,\dots,k\}$, where $\lambda^i(z_j)$ and $u_i(z_j)$ are the spectral values and vectors of z_j, respectively. Then, by (3.5) and Proposition 3.2, we have

$$z(v^r, t^r_{\text{new}}) + \gamma_k e = \begin{pmatrix} (\gamma_k + \lambda^1_1)u^1_1 + (\gamma_k + \lambda^2_1)u^2_1 \\ \vdots \\ (\gamma_k + \lambda^1_k)u^1_k + (\gamma_k + \lambda^2_k)u^2_k \end{pmatrix}.$$

Since $z(v^r, t^r_{\text{new}}) + \gamma_k e \notin \mathcal{K}$ from (4.2), it must hold that $\gamma_k + \lambda^1_j < 0$ for some j. Therefore, by choosing $l \in \{1,2,\dots,k\}$ such that $\lambda^1_l = \min_{1\leq j\leq k} \lambda^1_j$, we have

$$\lambda^1_l < -\gamma_k < 0. \tag{4.6}$$

Now, denote $(\alpha)_+ := \max(\alpha, 0)$ for $\alpha \in \mathbb{R}$. Then, by Proposition 3.1, we have

$$P_{\mathcal{K}}(z(v^{r+1}, t^r_{\text{new}})) = \begin{pmatrix} (\lambda^1_1)_+ u^1_1 + (\lambda^2_1)_+ u^2_1 \\ \vdots \\ (\lambda^1_k)_+ u^1_k + (\lambda^2_k)_+ u^2_k \end{pmatrix}. \tag{4.7}$$

Hence,

$$\begin{aligned}
\|z(v^r, t^r_{\text{new}}) - z(v^{r+1}, t^r_{\text{new}})\|^2 &\geq \|z(v^r, t^r_{\text{new}}) - P_{\mathcal{K}}(z(v^r, t^r_{\text{new}}))\|^2 \\
&= \left\| \begin{pmatrix} (\lambda^1_1 - (\lambda^1_1)_+)u^1_1 + (\lambda^2_1 - (\lambda^2_1)_+)u^2_1 \\ \vdots \\ (\lambda^1_k - (\lambda^1_k)_+)u^1_k + (\lambda^2_k - (\lambda^2_k)_+)u^2_k \end{pmatrix} \right\|^2 \\
&\geq \|(\lambda^1_l - (\lambda^1_l)_+)u^1_l + (\lambda^2_l - (\lambda^2_l)_+)u^2_l\|^2 \\
&\geq (\lambda^1_l)^2 \|u^1_l\|^2 \qquad (4.8) \\
&\geq \gamma_k^2/2,
\end{aligned}$$

where the first inequality follows since v^{r+1} solves SOCP$(\overline{T}_{r+1})$ and hence $z(v^{r+1}, t^r_{\text{new}}) \in \mathcal{K}$, the third inequality follows from $(u^1_l)^\top u^2_l = 0$, and the last inequality is due to (4.6). Then, we have

$$\|z(v^{r+1}, t^r_{\text{new}}) - z(v^r, t^r_{\text{new}})\| \geq \frac{\gamma_k}{\sqrt{2}}.$$

This together with (4.5) yields

$$\|v^{r+1} - v^r\| \geq \frac{\gamma_k}{\sqrt{2}M}.$$

By letting $N := 1/\sqrt{2}M$, the proof is completed. □

Theorem 4.2. *Let that Assumption A hold. Then, the inner iterations in every Step 1 of Algorithm 1 terminate finitely.*

Proof. Suppose for the sake of contradiction that, at some outer iteration k, the inner iterations in every Step 1 does not terminate finitely and an infinite sequence $\{v^r\}$ is generated. For simplicity, denote $g^r_t := A(t)^\top v^r - b(t)$. Since v^r solves SOCP$(\overline{T}_r)$, it satisfies the following KKT conditions:

$$\nabla f(v^r) - \sum_{t \in \overline{T}_r} A(t) y^r_t = 0, \qquad (4.9)$$

$$\mathcal{K} \ni y^r_t \perp g^r_t \in \mathcal{K} \ \ (t \in \overline{T}_r), \qquad (4.10)$$

where y^r_t is the Lagrange multiplier vector corresponding to g^r_t for each $t \in \overline{T}_r$. From (4.4), we have $f(v^1) \leq f(v^2) \leq \cdots \leq f(\bar{x}) < +\infty$, where $\bar{x}$ is an optimal solution of the original problem SISOCP (1.1), This implies

$$\lim_{r \to \infty} \big(f(v^{r+1}) - f(v^r)\big) = 0. \qquad (4.11)$$

Let $F_r := f(v^{r+1}) - f(v^r) - \nabla f(v^r)^\top (v^{r+1} - v^r)$. Then, we have

$$
\begin{aligned}
f(v^{r+1}) - f(v^r) &= \big(f(v^{r+1}) - f(v^r) - \nabla f(v^r)^\top (v^{r+1} - v^r)\big) + \nabla f(v^r)^\top (v^{r+1} - v^r) \\
&= F_r + \left(\sum_{t\in\overline{T}_r} A(t) y_t^r\right)^\top (v^{r+1} - v^r) \qquad (4.12) \\
&= F_r + \sum_{t\in\overline{T}_r} (y_t^r)^\top A(t)^\top (v^{r+1} - v^r) \\
&= F_r + \sum_{t\in\overline{T}_r} (y_t^r)^\top (g_t^{r+1} - g_t^r) \\
&= F_r + \sum_{t\in\overline{T}_r} (y_t^r)^\top g_t^{r+1} - \sum_{t\in\overline{T}_r} (g_t^r)^\top y_t^r \\
&= F_r + \sum_{t\in\overline{T}_r} (y_t^r)^\top g_t^{r+1}, \qquad (4.13)
\end{aligned}
$$

where (4.12) and (4.13) follow from (4.9) and (4.10), respectively. Since function f is convex, we have $F_r \geq 0$ for any $r \geq 1$. In addition, since $y_t^r \in \mathcal{K}$ and $g_t^{r+1} \in \mathcal{K}$ for any $t \in \overline{T}_r$, we have $\sum_{t\in\overline{T}_r} (y_t^r)^\top g_t^{r+1} \geq 0$. Therefore, from (4.11) and (4.13), we have

$$\lim_{r\to\infty} F_r = \lim_{r\to\infty} \big(f(v^{r+1}) - f(v^r) - \nabla f(v^r)^\top (v^{r+1} - v^r)\big) = 0. \qquad (4.14)$$

From Assumption A iii), there exist subsequences $\{v^{r_j}\}$ and $\{v^{r_j+1}\}$ such that $v^{r_j} \to v^*$ and $v^{r_j+1} \to v^{**}$ as j goes to infinity. This together with (4.14) implies

$$\lim_{r\to\infty} \big(f(v^{r+1}) - f(v^r) - \nabla f(v^r)^\top (v^{r+1} - v^r)\big) = f(v^{**}) - f(v^*) - \nabla f(v^*)^\top (v^{**} - v^*) = 0.$$

From the strict convexity of f, we have $\|v^{**} - v^*\| = 0$, which contradicts Proposition 4.1. Hence, the inner iterations in Step 1 must terminate finitely. □

The next theorem shows the global convergence of Algorithm 1 under the strict convexity of the objective function. We omit the proof since it is analogous to [20, Theorem 3.4].

Theorem 4.3. *Let Assumption A hold. Let x^* be the optimum[4] of SISOCP* (1.1) *, and $\{x^k\}$ be the sequence generated by Algorithm 1. Then, it follows that*

$$\lim_{k\to\infty} x^k = x^*.$$

5 Regularized explicit exchange method for SISOCP

In the previous section, we proposed the explicit exchange method for SISOCP (1.1) and analyzed the convergence property. However, to ensure the global convergence, the strict convexity of the objective function was required (Assumption A). In this section, we propose a regularized explicit exchange method, and establish global convergence of the method without assuming the strict convexity.

[4] From the strictly convexity of f, SISOCP (1.1) has a unique solution.

5.1 Algorithm

Let ε be a positive number, and $T' := \{t_1, t_2, \dots, t_p\}$ be a finite subset of T. The regularized explicit exchange method solves the following SOCP, denoted SOCP(ε, T'), in each iteration.

$$\text{SOCP}(\varepsilon, T') \qquad \begin{array}{ll} \text{Minimize} & \frac{1}{2}\varepsilon\|x\|^2 + f(x) \\ \text{subject to} & A(t_j)^\top x - b(t_j) \in \mathcal{K} \quad (j = 1, 2, \dots, p). \end{array} \tag{5.1}$$

When the function f is convex, $\frac{1}{2}\varepsilon\|x\|^2 + f(x)$ is strongly convex. Then, if we solve SOCP$(\varepsilon_k, \overline{T}_{r+1})$ with $\varepsilon_k > 0$ instead of SOCP$(\overline{T}_{r+1})$ in Step 1-(b) of Algorithm 1, it is ensured by Theorem 4.2 that the inner iterations terminate[5] finitely. Moreover, by choosing positive sequences $\{\varepsilon_k\}$ and $\{\gamma_k\}$ both converging to 0, the generated sequence is expected to converge to a solution of SISOCP (1.1). Now we propose the following algorithm for SISOCP (1.1).

Algorithm 2 (Regularized Explicit Exchange Method)

Step 0. Choose positive sequences $\{\gamma_k\} \subseteq \mathbb{R}_{++}$ and $\{\varepsilon_k\} \subseteq \mathbb{R}_{++}$ such that $\lim_{k\to\infty} \gamma_k = 0$, $\lim_{k\to\infty} \varepsilon_k = 0$ and $\gamma_k = O(\varepsilon_k)$. Choose a finite subset $E^0 := \{t_1^0, \dots, t_l^0\} \subseteq T$. Set $k := 0$.

Step 1. Set $r := 0$ and $T_0 := E^k$. Solve SOCP(ε_k, T_0) and let v^0 be an optimum. Do the following (a)–(c):

(a) Find $t_{\text{new}}^r \in T$ such that

$$A(t_{\text{new}}^r)^\top v^r - b(t_{\text{new}}^r) \notin -\gamma_k e + \mathcal{K}. \tag{5.2}$$

If such a t_{new}^r does not exist, i.e.,

$$A(t)^\top v^r - b(t) \in -\gamma_k e + \mathcal{K} \tag{5.3}$$

for any $t \in T$, then set $x^{k+1} := v^r$ and $E^{k+1} := T_r$, and go to Step 2. Otherwise, let

$$\overline{T}_{r+1} := T_r \cup \{t_{\text{new}}^r\},$$

and go to (b).

(b) Solve SOCP$(\varepsilon_k, \overline{T}_{r+1})$ to obtain an optimum v^{r+1} and Lagrange multipliers y_t^{r+1}, for $t \in \overline{T}_{r+1}$.

(c) Let $T_{r+1} := \{t \in \overline{T}_{r+1} \mid y_t^{r+1} \neq 0\}$. Set $r := r + 1$ and return to (a).

Step 2. If γ_k and ε_k are sufficiently small, then terminate. Otherwise, set $k := k + 1$ and return to Step 1.

[5] We can verify Assumption A as follows. Assumption A i) is obvious from the definition of strict and strong convexity. Assumption A ii) holds, since $\operatorname{argmin}\{g(x) \mid x \in X\}$ is nonempty if g is strongly convex and X is closed. Assumption A iii) also holds, since the sequence $\{v^r\}$ generated in Step 1 is contained in the set $L := \{x \mid f(x) \leq V_P(T)\}$ from (4.4), and L is bounded due to the strong convexity of f.

5.2 Global convergence without strict convexity assumption

In this section, we show the global convergence of Algorithm 2 for SISOCP(1.1) without assuming strict convexity. Indeed, we only need the following assumption.

Assumption B. The GSCQ holds for SISOCP (1.1).

Under this assumption, Algorithm 2 has the global convergence property in the sense that the distance between x^k and the solution set of SISOCP (1.1) tends to 0. The following lemma shows the boundedness of $\{x^k\}$, from which we can derive the global convergence of Algorithm 2.

Lemma 5.1. *Let Assumption B hold. Then, a sequence $\{x^k\}$ generated by Algorithm 2 is bounded.*

Proof. Let $\bar{x}$ be an arbitrary solution of SISOCP (1.1). Since the GSCQ holds, by Theorem 2.12, there exist $t_1, t_2, \ldots, t_p \in T$ and $p \leq n+1$ such that

$$\nabla f(\bar{x}) - \sum_{i=1}^{p} A(t_i) y_i = 0, \tag{5.4}$$

$$\mathcal{K} \ni y_i \perp A(t_i)^\top \bar{x} - b(t_i) \in \mathcal{K} \quad (i = 1, 2, \ldots, p). \tag{5.5}$$

Let $\{x^k\}$ be a sequence generated by Algorithm 2. Since x^k solves SOCP$(\varepsilon_{k-1}, \overline{E}_k)$. and $\bar{x}$ is feasible for SOCP $(\varepsilon_{k-1}, \overline{E}_k)$, we have

$$\frac{1}{2}\varepsilon_{k-1}\|x^k\|^2 + f(x^k) \leq \frac{1}{2}\varepsilon_{k-1}\|\bar{x}\|^2 + f(\bar{x}). \tag{5.6}$$

Multiplying both sides of (5.6) by $2/\varepsilon_{k-1}$, we have

$$\begin{aligned}
\|x^k\|^2 &\leq \|\bar{x}\|^2 - \frac{2}{\varepsilon_{k-1}}(f(x^k) - f(\bar{x})) \\
&\leq \|\bar{x}\|^2 - \frac{2}{\varepsilon_{k-1}}\nabla f(\bar{x})^\top (x^k - \bar{x}) \\
&= \|\bar{x}\|^2 - \frac{2}{\varepsilon_{k-1}}\left(\sum_{i=1}^{p} A(t_i) y_i\right)^\top (x^k - \bar{x}),
\end{aligned} \tag{5.7}$$

where the second inequality holds since f is convex, and the equality follows from (5.4). Moreover, the last term of (5.7) satisfies the following inequalities:

$$\begin{aligned}
&-\left(\sum_{i=1}^{p} A(t_i) y_i\right)^\top (x^k - \bar{x}) \\
&\quad = -\sum_{i=1}^{p} y_i^\top (A(t_i)^\top x^k - b(t_i) + \gamma_{k-1} e) + \sum_{i=1}^{p} {y_i}^\top (\gamma_{k-1} e) + \sum_{i=1}^{p} {y_i}^\top (A(t_i)^\top \bar{x} - b(t_i)) \\
&\quad \leq \sum_{i=1}^{p} {y_i}^\top (\gamma_{k-1} e) \leq p\mu \|e\| \gamma_{k-1},
\end{aligned} \tag{5.8}$$

where $\mu := \max\{\|y_1\|, \|y_2\|, \ldots, \|y_p\|\}$, and the first inequality follows since $y_i \in \mathcal{K}$ and $A(t_i)^\top x^k - b(t_i) + \gamma_{k-1}e \in \mathcal{K}$ from (5.3), and $y_i{}^\top(A(t_i)^\top \bar{x} - b(t_i)) = 0$ from (5.5). Then, by substituting (5.8) into (5.7), we have

$$\|x^k\|^2 \leq \|\bar{x}\|^2 + 2p\mu\|e\|\gamma_{k-1}/\varepsilon_{k-1}.$$

Since $\gamma_{k-1} = O(\varepsilon_{k-1})$ by the choice in Step 0, there exists some $M > 0$ such that $\gamma_{k-1}/\varepsilon_{k-1} \leq M$ for all k. Therefore, $\{x^k\}$ is bounded. □

Theorem 5.2. *Let Assumption B hold. Let $\{x^k\}$ be a sequence generated by Algorithm 2. Then, $\{x^k\}$ is bounded, and every accumulation point of $\{x^k\}$ solves SISOCP (1.1).*

Proof. Since the boundedness of $\{x^k\}$ follows from Lemma 5.1, we only show the second statement. Let x^* be an arbitrary accumulation point of $\{x^k\}$. Then, taking a subsequence if necessary, we have

$$x^k \to x^*, \qquad \varepsilon_{k-1} \to 0, \qquad \gamma_{k-1} \to 0 \qquad (k \to \infty).$$

First, we show that x^* is feasible to SISOCP (1.1). Since x^k is determined as v^r satisfying (5.3) with γ_k replaced by γ_{k-1}, $A(t)^\top x^k - b(t) + \gamma_{k-1}e \in \mathcal{K}$ holds for any $t \in T$. Noticing that $\mathcal{K}$ is closed, we have $\lim_{k\to\infty} A(t)^\top x^k - b(t) + \gamma_{k-1}e = A(t)^\top x^* - b(t) \in \mathcal{K}$ for any $t \in T$. Hence, x^* is feasible to SISOCP (1.1).

We next show that x^* is optimal to SISOCP (1.1). Let $\bar{x}$ be an arbitrary optimum of SISOCP (1.1). Since x^* is feasible to SISOCP (1.1), we have $f(x^*) \geq f(\bar{x})$. On the other hand, $\bar{x}$ is feasible to SOCP(ε_{k-1}, E_k) since it solves SISOCP (1.1) whose feasible region is contained in that of SOCP(ε_{k-1}, E_k). Hence, we have

$$\frac{1}{2}\varepsilon_{k-1}\|x^k\|^2 + f(x^k) \leq \frac{1}{2}\varepsilon_{k-1}\|\bar{x}\|^2 + f(\bar{x}). \tag{5.9}$$

Due to the continuity of f, by letting $k \to \infty$ in (5.9), we have $f(x^*) \leq f(\bar{x})$. Therefore, we obtain $f(x^*) = f(\bar{x})$. x^* solves the SISOCP(1.1). □

6 Numerical experiments

In this section, we report some numerical results with Algorithm 2. The program was coded in Matlab 2008a and run on a machine with an Intel®Core2 Duo E6850 3.00GHz CPU and 4GB RAM. In the experiments, we solve the following SISOCP with a linear objective function:

$$\begin{aligned} &\text{Minimize} \quad c^\top x \\ &\text{subject to} \quad A(t)^\top x - b(t) \in \mathcal{K} \quad (\forall t \in T) \end{aligned} \tag{6.1}$$

with the index set $T = [-1, 1]$, $c \in \mathbb{R}^{15}$, $A_{ij}(t) := \alpha^0_{ij} + \alpha^1_{ij}t + \alpha^2_{ij}t^2 + \alpha^3_{ij}t^3$ ($i = 1, 2, \ldots, 15,\ j = 1, 2, \ldots, 30$) and $b_j(t) := \beta^0_j + \beta^1_j t + \beta^2_j t^2 + \beta^3_j t^3$ ($j = 1, 2, \ldots, 30$). The second-order cone $\mathcal{K} \subseteq \mathbb{R}^{30}$ is chosen to be one of the following four cases: (i) $\mathcal{K} = \mathcal{K}^{30}$, (ii) $\mathcal{K} = \mathcal{K}^{10} \times \mathcal{K}^{20}$, (iii) $\mathcal{K} = (\mathcal{K}^{10})^3 = \mathcal{K}^{10} \times \mathcal{K}^{10} \times \mathcal{K}^{10}$, (iv) $\mathcal{K} = (\mathcal{K}^5)^6 = \mathcal{K}^5 \times \mathcal{K}^5 \times \mathcal{K}^5 \times \mathcal{K}^5 \times \mathcal{K}^5 \times \mathcal{K}^5$. In (6.1), all components of $c \in \mathbb{R}^{15}$ are chosen randomly from $[-2, 2]$. β^0_j ($j = 1, 2, \ldots, 30$) are determined so that $(\beta_1^0, \beta_2^0, \ldots, \beta_{30}^0)^\top = e \in \mathbb{R}^{30}$, where e is defined by (3.8) according to the Cartesian structure of $\mathcal{K}$. In addition, $\alpha_{ij}^k, \beta_i^l \, (i = 1, 2, \ldots, 15, \ j = 1, 2, \ldots, 30, \ k = 0, 1, 2, 3, \ l = 1, 2, 3)$ are chosen randomly from $[-2, 2]$ so that the origin is an interior feasible point of (6.1). By using $A(t)$ and $b(t)$ generated in this way, (6.1) satisfies the GSCQ.

In Step 0 of Algorithm 2, we set $\{\varepsilon_k\}$ and $\{\gamma_k\}$ such that $\varepsilon_k = \gamma_k = 2^{-k}$ for each k. Moreover, we choose 10 points $t_1^0, t_2^0, \ldots, t_{10}^0 \in T$ randomly, and set $E^0 := \{t_1^0, t_2^0, \ldots, t_{10}^0\}$. In Step 1-(a), to find t_{new}^r satisfying (5.2), we first check the values of $\lambda_{\mathcal{K}}^{\min}\left(A(t)^\top v^r - b(t) + \gamma_k e\right)$ at $t = -1.0\ , -0.9,\ -0.8, \ldots,\ 0.9,\ 1.0$. If we fail to find t_{new}^r among them, then we solve

$$\min_{t \in T} \lambda_{\mathcal{K}}^{\min}(A(t)^\top v^r - b(t) + \gamma_k e)$$

and check whether or not its optimal value is nonnegative. To this end, due to (3.7), it suffices to solve the following problems for each $s = 1, 2, \ldots, k$ and check if the smallest optimal value among them is nonnegative:

$$\begin{array}{ll} \text{Minimize} & \lambda_s^1(\left[A(t)^\top v^r - b(t) + \gamma_k e\right]_s) \\ \text{subject to} & t \in T, \end{array} \tag{6.2}$$

where λ_s^1 is the function defined by (3.1), and $\left[A(t)^\top v^r - b(t) + \gamma_k e\right]_s$ is the sth block component of vector $A(t)^\top v^r - b(t) + \gamma_k e$ as decomposed in (3.4). In Step 1-(b), we solve SOCP$(\varepsilon_k, \overline{T}_{r+1})$ by the smoothing method [4, 10]. In Step 2, we terminate the algorithm if $\max(\varepsilon_k, \gamma_k) \leq 10^{-6}$, which means that we always stop the algorithm when $k = 20$ since $\varepsilon_{20} = \gamma_{20} = 2^{-20} < 10^{-6}$.

The obtained results are shown in Table 1, Table 2, Table 3 and Table 4, in which cpu(s), t_{add} and E_{fin} denote the CPU time in seconds, the cumulative number of times t_{new}^r is added to T_r in Step 1, and the value of E^k at the termination of the algorithm, respectively.

From the tables, we can observe that the computational cost tends to be higher as the number of SOCs in $\mathcal{K}$ gets larger. For example, in the case of $\mathcal{K} = \mathcal{K}^{30}$ (one SOC in $\mathcal{K}$), the cpu time is only 5 seconds or so, whereas it becomes around 20 seconds when $\mathcal{K} = (\mathcal{K}^5)^6$ (six SOCs in $\mathcal{K}$). Presumably, such a result is due to the fact that, when we fail to find t_{new}^r in Step 1-(a), problem (6.2) has to be solved as many times as the number of SOCs in $\mathcal{K}$. We also note that both $t = -1$ and 1 (the extreme points of T) belong to E_{fin} for all the test problems. This implies that the constraints $A(t)^\top x - b(t) \in \mathcal{K}$ with $t = -1$ and 1 are both active at the optimal solutions. For the problems with $E_{\mathrm{fin}} = \{-1, 1\}$, the values of cpu(s) and t_{add} seem relatively small. In fact, for such problems, the active sets at the optima can often be identified in a small number of iterations. On the other hand, if E_{fin} has elements other than -1 or 1, then the values of cpu(s) and t_{add} tend to be larger. Especially, problems A4, B3, C2 and D9 yield the largest values among the test problems with $\mathcal{K} = \mathcal{K}^{30}$, $\mathcal{K} = \mathcal{K}^{10} \times \mathcal{K}^{20}$, $\mathcal{K} = (\mathcal{K}^{10})^3$, and $\mathcal{K} = (\mathcal{K}^5)^6$, respectively. Indeed, for those four problems, E_{fin} has the *third* element that could not be identified at an early stage of the iterations.

7 Concluding remarks

For the semi-infinite program with a convex objective function and infinitely many second-order cone constraints (SISOCP), we have shown that the KKT conditions can be represented with finitely many SOC constraints, as long as the generalized Slater constraint qualification

Table 1: Results for $\mathcal{K} = \mathcal{K}^{30}$

problem	cpu(s)	t_{add}	E_{fin}
A1	4.78	3	$\{-1, 1\}$
A2	5.27	3	$\{-1, 1\}$
A3	5.70	3	$\{-1, 1\}$
A4	8.39	7	$\{-1, 0.14, 1\}$
A5	3.94	3	$\{-1, 1\}$
A6	4.51	6	$\{-1, 1\}$
A7	4.99	3	$\{-1, 1\}$
A8	4.70	3	$\{-1, 1\}$
A9	4.52	4	$\{-1, 1\}$
A10	4.75	3	$\{-1, 1\}$

Table 2: Results for $\mathcal{K} = \mathcal{K}^{10} \times \mathcal{K}^{20}$

problem	cpu(s)	t_{add}	E_{fin}
B1	9.79	7	$\{-1, 1\}$
B2	6.31	6	$\{-1, 1\}$
B3	10.02	15	$\{-1, 0.3, 1\}$
B4	6.90	3	$\{-1, -0.24, 1\}$
B5	6.63	6	$\{-1, 0.4, 1\}$
B6	8.23	8	$\{-1, 1\}$
B7	6.54	4	$\{-1, -0.14, 1\}$
B8	7.32	6	$\{-1, 1\}$
B9	6.78	2	$\{-1, 1\}$
B10	6.29	5	$\{-1, 1\}$

Table 3: Results for $\mathcal{K} = (\mathcal{K}^{10})^3$

problem	cpu(s)	t_{add}	E_{fin}
C1	7.85	2	$\{-1, 1\}$
C2	15.69	7	$\{-1, 0.16, 1\}$
C3	9.72	7	$\{-1, -0.38, 1\}$
C4	10.16	3	$\{-1, 1\}$
C5	7.31	2	$\{-1, 1\}$
C6	7.76	5	$\{-1, 1\}$
C7	7.73	5	$\{-1, 1\}$
C8	8.97	4	$\{-1, 1\}$
C9	10.78	5	$\{-1, 1\}$
C10	7.23	2	$\{-1, 1\}$

Table 4: Results for $\mathcal{K} = (\mathcal{K}^5)^6$

problem	cpu(s)	t_{add}	E_{fin}
D1	27.00	7	$\{-1, -0.58, 1\}$
D2	20.10	14	$\{-1, 0.36, 1\}$
D3	17.42	6	$\{-1, 0.90, 1\}$
D4	26.16	6	$\{-1, -0.56, 1\}$
D5	20.52	6	$\{-1, 0.07, 1\}$
D6	27.57	13	$\{-1, 0.33, 1\}$
D7	25.96	8	$\{-1, -0.73, 1\}$
D8	22.64	4	$\{-1, 1\}$
D9	65.15	26	$\{-1, -0.63, -0.41, -0.23, 0.25, 1\}$
D10	20.12	8	$\{-1, 1\}$

(GSCQ) holds. Furthermore, we have proposed the regularized explicit exchange method for solving SISOCP, and established its global convergence under the GSCQ. Finally, we have conducted numerical experiments with the proposed algorithm and made some observations about its behavior.

The proposed algorithm in this paper is based on the exchange method. But for the ordinary SIP, there have been developed many methods other than the exchange method. It is an interesting future subject to extend those methods to the SISOCP.

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