Master's Thesis

Robust Wardrop Equilibria in the Traffic Assignment Problem with Uncertain Data

Guidance

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February 2011

Abstract

The Traffic Assignment Problem (TAP) is to find the traffic flow satisfying Wardrop's user equilibrium principle, under which each driver selects his/her route with the minimum traffic cost. In order to solve the TAP, the data in the traffic network and each driver's cost function must be evaluated exactly. However, in the real network, those data often involve uncertainties. For such an uncertain network, we consider the robust TAP based on the robust Wardrop equilibrium. Under such an equilibrium, each driver selects his/her route with taking the worst possible case into consideration.

In this paper, we first study the existence condition for the robust Wardrop equilibria. To this end, we reformulate the robust TAP as a nonlinear complementarity problem (NCP), and apply the solvability theorem to such an NCP. Next we formulate the robust TAP with ellipsoidal uncertainty as the Second-Order Cone Complementarity Problem (SOCCP), which can be solved by using an existing algorithm based on the smoothing Newton method. Finally, by means of some numerical experiments, we observe the property of the robust Wardrop equilibria.

Contents

1 Introduction

Since the 1930s, the automobiles have been widely used all over the world because of the economic growth and the technological and scientific development. In order to make the automobile traffic more efficient, we need to design roadway infrastructures such as highways, traffic signals, and toll roads.

When we build new roads or decide new tolls on a traffic network, we need to forecast the traffic flow to estimate the effect due to such decisions. In general, all drivers are supposed to select the route with the minimum cost from the origin to the destination. In other words, the routes with positive traffic flow have the minimum cost, and more costly routes are not used. This flow distribution principle is called *Wardrop's user equilibrium principle* [21]. Also, the problem of finding a flow pattern satisfying Wardrop's user equilibrium principle is called the Traffic Assignment Problem (TAP). The TAP is formulated as mathematical programming problems such as a linear or nonlinear optimization problem, Variational Inequality Problem (VIP), Mixed Complementarity Problem (MCP), and Nonlinear Complementarity Problem (NCP) [1, 2, 5, 12, 18, 19].

In order to formulate the TAP as a mathematical programming problem, it is important to model the cost on each route appropriately. When the route cost function is expressed as the sum of road[1] costs, the route cost function is called *additive* [1, 5, 18, 19]. Otherwise, it is called *non-additive* [2, 12].

In the TAP, we suppose that each user has complete information on the traffic network and can choose a route with minimum cost by using that information. However, in the real traffic network, each user's estimated cost can be often incorrect due to various uncertainties such as weather changeability or traffic accidents. Therefore he/she may choose a route with non-minimal cost, and the flow based on Wardrop's user equilibrium principle does not necessarily express the real network flow.

For the traffic model in which the drivers do not know the complete information on the network, the new concept called the *robust Wardrop equilibrium* [15, 16, 20] attracts much attention recently. In the robust Wardrop equilibrium, we assume that each driver can estimate the "*uncertainty set*" in which the uncertain data of his/her route cost function are contained, and then choose his/her route with taking the value of the *worst (route) cost function* into consideration. In other words, each driver chooses his/her route based on the robust optimization policy [4, 7, 6, 14]. The traffic assignment problem based on the robust Wardrop equilibrium is called a robust TAP, which we will mainly discuss in the paper.

The robust Wardrop equilibrium has been studied by some researchers so far. Ordóñez and Stier-Moses [15, 16] defined the robust Wardrop equilibrium for the restrictive case where each user's cost function can be expressed as the sum of two terms: (1) the term depending on the flow but not involving any uncertainty and (2) the term not depending on the flow but involving some uncertainty. They showed that, when the uncertainty set in each route cost functions is polyhedral, the robust TAP can be formulated as an NCP. On the other hand, Takahashi [20]

[1] The road in a traffic network corresponds to the link in a directed graph. For more detail, see Section 2.

defined the robust Wardrop equilibrium for more general route cost functions without Ordóñez and Stier-Moses' restriction. Moreover, he showed that the robust TAP can be reformulated as a Second-Order Cone Complementarity Problem (SOCCP) [9, 11, 13, 17], when the route cost function is additive, the link cost function is linear and separable[2] , and the uncertain set is ellipsoidal. Also Takahashi showed that the robust TAP can be reformulated as an MCP when the uncertainty set is defined by means of the ∞-norm.

For the traffic model with uncertain cost functions, Zhang, Chen and Sumalee [23] studied another mathematical approach called a stochastic TAP. They assumed that the uncertain data in the cost functions follow some stochastic distribution, and reformulated the stochastic TAP as a stochastic complementarity problem that can be solved by using the expected residual minimization method. Although Zhang et al. discuss the robustness of the obtained stochastic TAP solution, the meaning of "robust" is essentially different from that in the "robust" TAP model. The robustness in Zhang et al.'s study means that the obtained stochastic TAP solution does not vary so much if the actual value of the stochastic data varies in some degree. On the other hand, the robustness for the robust TAP comes from the "robust optimization," by which each driver chooses his/her route.

In this paper, we consider the robust Wardrop equilibrium in [15, 16, 20] to TAPs with more general uncertainty structures. In [20], Takahashi only considered the case where the link cost functions in traffic network are linear and separable, whereas we study the robust TAP without such a linearity and separability assumption. We also provide the condition for the existence of a robust Wardrop equilibrium, and reformulate the robust TAP as an SOCCP when the uncertainty set is ellipsoidal.

This paper is organized as follows. In Section 2.1, we describe the traffic model and Wardrop's user equilibrium without uncertainty, and formulate the TAP based on the traffic model and Wardrop's user equilibrium. In Section 2.2, we recall background of some equilibrium problems such as SOCCP, MCP, and NCP. In Section 2.3, we formulate the TAP as an NCP and an MCP. Moreover we provide the condition for the existences of a solution for TAP. Section 3 is the main section of this paper. In Section 3.1, we define the robust Wardrop equilibrium, and formulate the robust TAP as an MCP. Furthermore we show the condition for the existence of a solution of the robust TAP. In Section 3.2, we formulate the robust TAP with an ellipsoidal uncertainty set as an SOCCP. In Section 4, we observe the property of equilibria for robust TAPs by means of numerical experiments. In Section 5, we conclude this paper with some remarks.

Throughout the paper, we use the following notations and definitions: $\|\cdot\|$ denotes the 2-norm defined by $\|z\| := \sqrt{z^\top z}$ for a vector z. For a given set S, $|S|$ denotes the cardinality of S. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. For a finite set N and $z = (z_1, z_2, \dots, z_{|N|})$, we write $z = [z_i]_{i\in N}$. We often write $z = (x, y)$ for $[x^\top, y^\top]^\top$. For the vectors a and b of the same dimension, $a \perp b$ means $a^\top b = 0$.

[2] The link cost function is said to be separable if its value depends only on the link flow.

2 Preliminaries

In this section, we recall some fundamental background on the TAP and some related topics. In Subsection 2.1, we give a mathematical expression of the TAP by using Wardrop's user equilibrium principle. In Subsection 2.2, we introduce some classes of complementarity problems, which play an important role in solving TAPs and robust TAPs. In Subsection 2.3, we reformulate the TAP as a complementarity problem, and study the condition under which TAP solutions exist.

2.1 Mathematical formulation of traffic assignment problem

In this section, we provide a mathematical formulation of TAP. Consider a directed graph $\mathscr{G} = (\mathscr{N}, \mathscr{L})$ corresponding to the traffic network, where $\mathscr{N}$ and $\mathscr{L}$ denote the node (vertex or point) set and the link (edge or arc) set, respectively. In the real traffic network, the nodes correspond to the origins, the destinations and the intersections, and the links correspond to the roads. W denotes the set which consists of origin-destination pairs (OD pairs). We assume that graph $\mathscr{G}$ is strongly connected, that is, there exists at least one route for every OD pair $w \in W$. Let R_w be the set of all routes between OD pair $w \in W$, and $R := \cup_{w \in W} R_w$. For $r \in R$, $\mathscr{L}_r \subset \mathscr{L}$ denotes the set of all links contained in r. $y_l \in \mathbb{R}$ and $x_r \in \mathbb{R}$ denote the flow of link $l \in \mathscr{L}$ and route $r \in R$, respectively. Let the link and the route flow vectors be denoted as $y := (y_1, y_2, \ldots, y_{|\mathscr{L}|})$ and $x := (x_1, x_2, \ldots, x_{|R|})$, respectively. $f_r : \mathbb{R}^{|R|} \to \mathbb{R}$ denotes the cost function for route $r \in R$ with variable $x \in \mathbb{R}^{|R|}$. $t_l : \mathbb{R}^{|\mathscr{L}|} \to \mathbb{R}$ denotes the cost function for link $l \in \mathscr{L}$ with variable $y \in R^{|\mathscr{L}|}$. For an OD pair $w \in W$, $\lambda_w := \min_{r \in R_w} f_r(x)$ denotes the minimum route cost. $d_w : \mathbb{R}^{|W|} \to \mathbb{R}^{|W|}$ denotes the demand function with variable $\lambda := [\lambda_w]_{w \in W}$.

Next, we describe Wardrop's user equilibrium principle which shows drivers' behavior in the traffic network. A route flow vector $x \in \mathbb{R}^{|R|}$ is called Wardrop's user equilibrium if it satisfies

$$[x_r > 0 \implies f_r(x) \le f_{r'}(x) \quad \forall r' \in R_w] \quad r \in R_w,\ w \in W. \tag{2.1}$$

Wardrop's user equilibrium principle states that each driver in the network selects the route with minimum cost. Conversely, the drivers avoid the routes with non-minimum cost. In other words, under such an equilibrium, the cost of the route with non-zero flow must be less than or equal to other routes for the same OD pair, and conversely, any route with non-minimum cost for an OD pair has no flow.

In addition to Wardrop's user equilibrium principle (2.1), the TAP requires the condition that every route flow is nonnegative and the sum of route flows for each OD pair w is equal to its traffic demand $d_w(\lambda)$, that is,

$$x \ge 0, \quad \sum_{r \in R_w} x_r = d_w(\lambda) \quad (w \in W). \tag{2.2}$$

Combining (2.1) with (2.2), the TAP can be formulated as follows:

$$\begin{aligned} \text{Find} \quad & (x,\lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \le f_r(x) - \lambda_w \perp x_r \ge 0 \quad (r \in R_w,\ w \in W), \\ & \sum_{r \in R_w} x_r = d_w(\lambda) \quad (w \in W), \\ & \lambda_w \ge 0 \quad (w \in W). \end{aligned} \tag{2.3}$$

Furthermore, TAP (2.3) can be rewritten equivalently as follows:

$$\begin{aligned} \text{Find} \quad & (x,\lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \le f(x) - N^\top \lambda \perp x \ge 0, \\ & Nx - d(\lambda) = 0, \quad \lambda \ge 0, \end{aligned} \tag{2.4}$$

where function $f : \mathbb{R}^{|R|} \to \mathbb{R}^{|R|}$ and matrix $N \in \mathbb{R}^{|W| \times |R|}$ are defined by

$$f(x) := [f_r(x)]_{r \in R}, \quad N_{wr} = \begin{cases} 1 & r \in R_w \\ 0 & r \notin R_w \end{cases}, \tag{2.5}$$

respectively.

2.2 Complementarity problems

In this subsection, we introduce some classes of complementarity problems [10]. The complementarity problem is a kind of equilibrium problem, and has been studied extensively so far since it is mathematically tractable and can be solved efficiently by existing algorithms such as the smoothing Newton method. In the subsequent sections, we reformulate robust TAPs as complementarity problems.

For given functions $h : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n \times \mathbb{R}^\nu$, NCP and MCP can be formulated as

$$\begin{aligned} \text{Find} \quad & x \in \mathbb{R}^n \\ \text{such that} \quad & 0 \le x \perp h(x) \ge 0, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \text{Find} \quad & (x, y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \\ \text{such that} \quad & 0 \le x \perp y \ge 0,\ F(x, y, \zeta) = 0, \end{aligned} \tag{2.7}$$

respectively. Notice that MCP contains NCP as a subclass since NCP (2.6) reduces to MCP (2.7) by setting $F(x, y, \zeta) := y - h(x)$.

The second-order cone complementarity problem (SOCCP) [9, 11, 13, 17] is a more general class of complementarity problems written as follows:

$$\begin{aligned} \text{Find} \quad & (x, y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \\ \text{such that} \quad & K \ni x \perp y \in K, \quad F(x, y, \zeta) = 0, \end{aligned} \tag{2.8}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n \times \mathbb{R}^\nu$ is a given function, and K is the Cartesian product of several second-order cones, that is, $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$, and the n_i-dimensional second-order cone $K^{n_i} \subset \mathbb{R}^{n_i}$ is defined as

$$K^{n_i} = \left\{ (z_1, z_2^\top)^\top \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \,|\, z_1 \ge \|z_2\| \right\}.$$

Notice that SOCCP contains MCP as a subclass since K coincides with the nonnegative orthant when $n_1 = n_2 = \cdots = n_m = 1$. In this paper, we formulate the robust TAP as an SOCCP of the form

$$\begin{array}{rl} \text{Find} & \zeta \in \mathbb{R}^{\nu} \\ \text{such that} & K \ni G(\zeta) \perp H(\zeta) \in K, \ C\zeta = h, \end{array} \tag{2.9}$$

where $G : \mathbb{R}^{\nu} \to \mathbb{R}^n$ and $H : \mathbb{R}^{\nu} \to \mathbb{R}^n$ are given functions, and $C \in \mathbb{R}^{\nu\times\nu}$ and $h \in \mathbb{R}^{\nu}$ are given constants. We can easily see that SOCCP(2.9) can be rewritten as SOCCP(2.8) by letting $x := G(\zeta)$, $y := H(\zeta)$, and

$$F(x, y, \zeta) := \begin{bmatrix} x - G(\zeta) \\ y - H(\zeta) \\ C\zeta - d \end{bmatrix}.$$

2.3 Complementarity reformulation of TAP and existence of solution

In this section,we show some relation between TAP(2.4) and NCP(2.6) or MCP(2.7), and discuss the existence of a TAP solution. In order to formulate the TAP as an NCP,we make the following assumption.

Assumption 2.1. *In TAP* (2.4)*,the following conditions hold:*

(a) $f(x) \geq 0$ *and* $d(\lambda) \geq 0$ *for any* $(x, \lambda) \in \mathbb{R}_+^{|R|} \times \mathbb{R}_+^{|W|}$,

(b) *For all* $r \in R$, $f_r(x)x_r = 0$ *implies* $x_r = 0$.

Notice that (b) automatically holds if $f(x) > 0$ for any $x \in \mathbb{R}_+^{|R|}$. Under this assumption, TAP (2.4) can be rewritten in the form of NCP (2.6).

Theorem 2.1. [10, Proposition 1.4.6] *Suppose that TAP* (2.4) *satisfies Assumption* 2.1*. Then,the TAP can be reformulated as the following NCP equivalently:*

$$\begin{array}{rl} \text{Find} & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} & 0 \leq \begin{bmatrix} f(x) - N^{\top}\lambda \\ Nx - d(\lambda) \end{bmatrix} \perp \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0. \end{array} \tag{2.10}$$

By using the above NCP reformulation, we can derive a sufficient condition under which there exists at least one solution of TAP (2.4).

Assumption 2.2. *In TAP* (2.4)*, functions* f *and* d *are continuous. Moreover, there exists* $M > 0$ *such that* $d_w(\lambda) \leq M$ *for any* $w \in W$ *and* $\lambda \in \mathbb{R}^{|W|}$.

Theorem 2.2. [10, Proposition 2.2.14] *Suppose that Assumptions* 2.1 *and* 2.2 *hold. Then, TAP* (2.4) *has at least one solution.*

We have shown that TAP (2.4) reduces to an NCP under Assumption 2.1. On the other hand, it also reduced to an MCP under another assumption. In the subsequent numerical experiments, in order to some (robust) TAPs, we apply a smoothing Newton algorithm to this MCP.

Assumption 2.3. *In TAP (2.4), It follows $f(x) \geq 0$ and $d(\lambda) > 0$ for any $(x, \lambda) \in \mathbb{R}_+^{|R|} \times \mathbb{R}_+^{|W|}$.*

Theorem 2.3. *Suppose that TAP* (2.4) *satisfies Assumption* 2.3. *Then, the TAP can be reformulated as the following MCP equivalently:*

$$
\begin{array}{rl}
\text{Find} & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\
\text{such that} & 0 \leq f(x) - N^\top \lambda \perp x \geq 0, \\
& Nx = d(\lambda).
\end{array}
\tag{2.11}
$$

Proof. For any solution (x, λ) of MCP (2.11), it suffices to show $\lambda \geq 0$. Let (x, λ) be an arbitrary solution of MCP (2.11), and fix $w \in W$ arbitrarily. Then, by Assumption 2.3 and the equality in MCP (2.11), we have $0 < d_w(\lambda) = \sum_{r \in R_w} x_r$. Hence, there exists an $\bar{r} \in R_w$ such that $x_{\bar{r}} > 0$, which together with the complementarity condition $0 \leq f_{\bar{r}}(x) - \lambda_w \perp x_{\bar{r}} \geq 0$ implies $f_{\bar{r}}(x) - \lambda_w = 0$. We thus have $\lambda_w = f_{\bar{r}}(x) \geq 0$ by Assumption 2.3. Since $w \in W$ was chosen arbitrarily, we have $\lambda \geq 0$. □

3 Traffic assignment problem based on robust Wardrop's user equilibrium

In this section, we define the robust TAP and discuss the existence of its solution. We also formulate the robust TAP with special uncertainty structure as an SOCCP.

3.1 Robust traffic assignment problem and existence of solutions

In this subsection, we provide a mathematical expression of the robust TAP, and study the existence of a robust Wardrop equilibrium.

Consider the following situation. The cost function $f_r^{\hat{u}^r}$ for route $r \in R$ contains uncertain data $\hat{u}^r$. Even though the users cannot estimate the value of $\hat{u}^r$ accurately, they know that it belongs to a certain compact set U_r. In such a situation, we assume that each user with OD pair w chooses a route with minimum worst cost, i.e., a route r such that $r = \operatorname{argmin}_{r \in R_w} \tilde{f}_r(x)$, where

$$
\tilde{f}_r(x) := \max \left\{ f_r^{\hat{u}^r}(x) \mid \hat{u}^r \in U_r \right\}
\tag{3.1}
$$

is called the *worst cost function.* Moreover, a Wardrop equilibrium with respect to the worst cost $\tilde{f}_r(x)$ is called a *robust Wardrop equlibrium.*

Definition 3.1. Let the worst cost function $\tilde{f}_r$ be defined by (3.1). Then, a route flow vector $x \in \mathbb{R}^{|R|}$ satisfying

$$
[x_r > 0 \Longrightarrow \tilde{f}_r(x) \leq \tilde{f}_{r'}(x) \quad \forall r' \in R_w] \, (r \in R_w, \, w \in W),
\tag{3.2}
$$

is called a robust Wardrop equlibrium. Moreover, the problem of finding the robust Wardrop equilibrium is called a robust TAP, i.e., it is to find $(x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|}$ such that

$$
\begin{array}{c}
0 \leq \tilde{f}_r(x) - \lambda_w \perp x_r \geq 0 \quad (r \in R_w, \; w \in W), \\
\displaystyle\sum_{r \in R} x_r = d_w(\lambda), \; \lambda_w \geq 0 \quad (w \in W).
\end{array}
\tag{3.3}
$$

By using the complementarity reformulation technique in the previous section, we can also show conditions under which a robust Wardrop equilibrium exists. In what follows, we denote $\tilde{f}(x) := [\tilde{f}_r(x)]_{r=1}^{|R|} \in \mathbb{R}_+^{|R|}$.

Assumption 3.1. *For the robust TAP* (3.3)*, the following four conditions hold:*

(a) *For any $x \in \mathbb{R}_+^{|R|}$, there exists $\hat{u}^r \in U_r$ such that $f_r^{\hat{u}^r}(x) > 0$ for each $r \in R$.*

(b) *For each $r \in R$, the function $h_r : \mathbb{R}_+^{|R|} \times U_r \to \mathbb{R}_+$ defined by $h_r(x, \hat{u}^r) := f_r^{\hat{u}^r}(x)$ is continuous on $\mathbb{R}_+^{|R|} \times U_r$.*

(c) *$d(\lambda) > 0$ for any $\lambda \in \mathbb{R}_+^{|W|}$.*

(d) *For each $w \in W$, function $d_w(\lambda)$ is continuous and bounded above on $\mathbb{R}^{|W|}$.*

Theorem 3.1. *Suppose that Assumption* 3.1 *holds. Then the robust TAP* (3.3) *is equilivalent to the following MCP:*

$$
\begin{array}{rl}
\text{Find} & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\
\text{such that} & 0 \le \tilde{f}(x) - N^\top \lambda \perp x \ge 0, \\
& Nx - d(\lambda) = 0.
\end{array}
\tag{3.4}
$$

Proof. From (b) together with [3, Theorem 1.4.6], we can see that the worst cost function $\tilde{f}_r$ is continuous on $\mathbb{R}_+^{|R|}$. Moreover, by (a) and the compactness of U_r, we have $\tilde{f}(x) > 0$ for any $x \in \mathbb{R}_+$. Therefore, Assumptions 2.1 – 2.3 hold for $f(x) := \tilde{f}(x)$. (Notice that Assumption 2.1 (b) holds since $\tilde{f}(x) > 0$.) Thus, by Theorems 2.2 and 2.3 with $f(x) = \tilde{f}(x)$, we obtain the theorem. □

3.2 SOCCP reformulation for robust TAP with ellipsoidal uncertainty sets

In the previous subsection, we have defined the robust TAP and showed the condition for the existence of a solution. In this section, we show that the robust TAP can be reformulated as an SOCCP when the uncertainty sets are described by means of the Euclidean norm.

3.2.1 Robust TAP with general link cost function

In what follows, we assume that each link cost function is expressed as

$$
t_l^{\hat{u}_l}(y) = t_l(y) + \hat{u}_l \Delta t_l(y),
\tag{3.5}
$$

where $t_l : \mathbb{R}^{|\mathscr{L}|} \to \mathbb{R}$ and $\Delta t_l : \mathbb{R}^{|\mathscr{L}|} \to \mathbb{R}$ are given functions , and $\hat{u}_l \in \mathbb{R}$ denotes the uncertainty parameter. Moreover, we suppose that the uncertain route cost function $f_r^{\hat{u}^r}(x)$ is additive, i.e.,

$$
f_r^{\hat{u}^r}(x) = \sum_{l \in \mathscr{L}_r} t_l^{\hat{u}_l}(y),
\tag{3.6}
$$

where the uncertainty parameter satisfies $\hat{u}^r = [\hat{u}_l]_{l \in \mathscr{L}} \in \mathbb{R}^{|\mathscr{L}|}$. Now, let $M \in \mathbb{R}^{|\mathscr{L}| \times |R|}$ be the link-route incidence matrix with the (l, r) entry

$$
M_{lr} := \begin{cases} 1 & (l \in \mathscr{L}_r) \\ 0 & (l \notin \mathscr{L}_r). \end{cases}
$$

Then we have $y = Mx$, which together with (3.5) and (3.6) yields

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathscr{L}_r} t_l(Mx) + \hat{u}_l \Delta t_l(Mx). \tag{3.7}$$

Furthermore, we make the following assumption on the uncertainty set U_r.

Assumption 3.2. *Uncertainty set U_r is ellipsoidal for each $r \in R$, i.e.,*

$$U_r := \left\{ \hat{u}^r \in \mathbb{R}^{|\mathscr{L}|} \;\middle|\; \hat{u}^r = \bar{u}^r + D_r \hat{v}^r,\ \|\hat{v}^r\| \leq \delta_r, \right\},$$

where $\bar{u}^r$ is a given vector, $D_r \in \mathbb{R}^{|\mathscr{L}| \times |\mathscr{L}|}$ is a given symmetric positive definite matrix, and δ_r is a given positive scalar.

Under Assumption 3.2, we can represent the worst cost function $\tilde{f}_r$ explicitly. To this end, we need the following lemma.

Lemma 3.1. *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ be arbitrary vectors, $C \in \mathbb{R}^{m \times n}$ be an arbitrary matrix, and $\delta > 0$ be any positive scalar. Let $P \subset \mathbb{R}^m$ be defined by*

$$P := \left\{ p \in \mathbb{R}^m \,\middle|\, p = b + Cq, \|q\| \leq \delta \right\}.$$

Then we have

$$\max_{p \in \mathbb{R}^m} \left\{ a^\top p \,\middle|\, p \in P \right\} = a^\top b + \delta \|C^\top a\|. \tag{3.8}$$

Proof. Since (3.8) is evident for $a = 0$, we assume $a \neq 0$. Let $q \in \mathbb{R}^n$ be an arbitrary vector with $\|q\| \leq \delta$, and let $p := b + Cq$. Then, by Cauchy's inequality, we have

$$a^\top p = a^\top b + a^\top Cq \leq a^\top b + \|C^\top a\| \|q\| \leq a^\top b + \delta \|C^\top a\|. \tag{3.9}$$

Moreover, the above inequalities hold as equalities when we choose $q := \delta C^\top a / \|C^\top a\|$. □

Applying Lemma 3.1 to the uncertain route cost $f_r^{\hat{u}^r}$ with (3.7) under Assumption 3.2, we readily obtain

$$\tilde{f}_r(x) = \sum_{l \in \mathscr{L}_r} t_l(Mx) + \bar{u}_l^r \Delta t_l(Mx) + \delta_r \|D_r \mathrm{diag}(M_r) \Delta t(Mx)\|, \tag{3.10}$$

where $\mathrm{diag}(M_r) \in \mathbb{R}^{|\mathscr{L}| \times |\mathscr{L}|}$ is the diagonal matrix whose diagonal components are given by M_{lr} $(l \in \mathscr{L})$.

By Theorem 3.1, the robust TAP with $\tilde{f}_r(x)$ defined by (3.10) reduces to MCP (3.4) under Assumption 3.1. However, since $\tilde{f}$ is nondifferentiable, it is difficult to apply existing algorithms to MCP(3.4) directly. To avoid this difficulty, we reformulate the robust TAP as an SOCCP that contains differentiable functions only.

Let $g_r(x) := \sum_{l \in \mathscr{L}_r} t_l(Mx) + \bar{u}_l^r \Delta t_l(Mx)$ and $g(x) := [g_r(x)]_{r=1}^{|R|} \in \mathbb{R}^{|R|}$. Then the worst cost function (3.10) can be expressed explicitly as

$$\tilde{f}(x) = g(x) + \begin{bmatrix} \delta_1 \|D_1 \mathrm{diag}(M_1) \Delta t(Mx)\| \\ \delta_2 \|D_2 \mathrm{diag}(M_2) \Delta t(Mx)\| \\ \vdots \\ \delta_{|R|} \|D_{|R|} \mathrm{diag}(M_{|R|}) \Delta t(Mx)\| \end{bmatrix}.$$

Moreover, by using an auxiliary variable $s := [s_r]_{r=1}^{|R|} \in \mathbb{R}^{|R|}$, MCP(3.4) can be rewritten as the following problem:

$$
\begin{array}{rc}
\text{Find} & (x, \lambda, s) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \times \mathbb{R}^{|R|} \\
\text{such that} & 0 \leq g(x) + s - N^\top \lambda \perp x \geq 0, \\
& s_r = \delta_r \| D_r \text{diag}(M_r) \Delta t(Mx) \| \ (r \in R), \\
& Nx = d(\lambda).
\end{array}
\tag{3.11}
$$

Furthermore, we can reformulate (3.11) as an SOCCP by the following lemma.

Lemma 3.2. *Let $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^{k-1}$ be an arbitrary vector with $k \geq 2$. Then, $\xi_1 = \|\xi_2\|$ if and only if there exists a vector $v \in \mathbb{R}^{k-1}$ such that*

$$
K^k \ni \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ v \end{bmatrix} \in K^k. \tag{3.12}
$$

Proof. First we show the "only if" part. Suppose that $\xi_1 = \|\xi_2\|$ holds.When $\xi_1 = \|\xi_2\| = 0$, it is obvious that there exists a v satisfying (3.12). So, we assume $\xi_1 = \|\xi_2\| > 0$. Let $v := -\xi_2/\|\xi_2\|$. Then we have $(1, v) = (1, -\xi_2/\|\xi_2\|) \in K^k$ and $(\xi_1, \xi_2)^\top (1, v) = \xi_1 - \|\xi_2\| = 0$. We thus have (3.12).

Next we show the "if" part. Let $v \in \mathbb{R}^{k-1}$ and $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^{k-1}$ be arbitrary vectors satisfying (3.12). Then the following three formulas hold:

$$
0 = \xi_1 + v^\top \xi_2, \tag{3.13}
$$

$$
1 \geq \|v\|, \tag{3.14}
$$

$$
\xi_1 \geq \|\xi_2\|. \tag{3.15}
$$

By Cauchy's inequality, we have

$$
0 = \xi_1 + v^\top \xi_2 \geq \|v\| \|\xi_2\| + v^\top \xi_2 \geq 0,
$$

which implies that all the inequalities hold as equalities, and hence $\xi_1 = \|v\| \|\zeta_2\|$. Substituting (3.14) to this equality, we obtain

$$
\xi_1 = \|v\| \|\xi_2\| \leq \|\xi_2\|.
$$

On the other hand, we also have (3.15). Hence, $\xi_1 = \|\xi_2\|$. $\square$

By Lemma 3.2, problem (3.11) can be reformulated as the following SOCCP:

$$
\begin{array}{rc}
\text{Find} & (x, \lambda, s, v) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \times \mathbb{R}^{|R|} \times \mathbb{R}^{|\mathscr{L}||R|} \\
\text{such that} & 0 \leq g(x) + s - N^\top \lambda \perp x \geq 0, \\
K^{|\mathscr{L}|+1} \ni & \begin{bmatrix} s_r \\ \delta_r D_r \text{diag}(M_r) \Delta t(Mx) \end{bmatrix} \perp \begin{bmatrix} 1 \\ v^r \end{bmatrix} \in K^{|\mathscr{L}|+1} \quad (r \in R), \\
& Nx = d(\lambda).
\end{array}
\tag{3.16}
$$

Moreover, if d is a constant function, i.e., $d(\lambda) = d$ for any $\lambda \in \mathbb{R}^{|W|}$, then SOCCP (3.16) can be rewritten in the form of SOCCP (2.9) with $K := (K^1)^{|R|} \times (K^{|\mathscr{L}|+1})^{|R|}$, $\zeta := (x, \lambda, s, v) \in \mathbb{R}^{|R|+|W|+|R|(|\mathscr{L}|+1)}$,

$$C = \begin{bmatrix} 0_{|R|\times|R|} & 0_{|R|\times(|W|+|R|(|\mathscr{L}|+1))} \\ N & 0_{|R|\times(|W|+|R|(|\mathscr{L}|+1))} \\ 0_{|R|(|\mathscr{L}|+1)\times|R|} & 0_{|R|\times(|W|+|R|(|\mathscr{L}|+1))} \end{bmatrix}, \qquad h = \begin{bmatrix} 0_{|R|} \\ d \\ 0_{|R|(|\mathscr{L}|+1)} \end{bmatrix},$$

$$F(x,\lambda,s,v) := \begin{bmatrix} g(x) + s - N^\top \lambda \\ s_1 \\ \delta_1 D_1 \mathrm{diag}(M_1)\Delta t(Mx) \\ s_2 \\ \delta_2 D_2 \mathrm{diag}(M_2)\Delta t(Mx) \\ \vdots \\ s_{|R|} \\ \delta_{|R|} D_{|R|} \mathrm{diag}(M_{|R|})\Delta t(Mx) \end{bmatrix}, \quad G(x,\lambda,s,v) := \begin{bmatrix} x \\ 1 \\ v^1 \\ 1 \\ v^2 \\ \vdots \\ 1 \\ v^{|R|} \end{bmatrix},$$

where 0_m and $0_{m\times n}$ are the m-dimensional zero vector and the $(m \times n)$-dimensional zero matrix, respectively.

3.2.2 Robust TAP with uncertain BPR function

Next we introduce a more concrete link cost function called the U. S. Bureau of Public Roads (BPR) function [8]. The BPR function $t_l(y)$ is defined as follows:

$$t_l(y) = a_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^\nu\right), \tag{3.17}$$

where ν, a_l, b_l, c_l are positive scalars. More precisely, a_l represents the free-flow travel time, b_l represents the congestion factor, c_l represents the traffic capacity of link l, and ν is usually chosen as a number between 4 and 5. The BPR function is one of the most popular link cost functions employed in a mathematical model for the traffic network. We suppose that for all routes $r \in R$, the cost functions $f_r(x)$ are additive. Then by using the BPR function, we can express the route cost function as follows:

$$f_r(x) = \sum_{l \in \mathscr{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^\nu\right), \tag{3.18}$$

where M_l denotes the l-th row vector of the link-route incidence matrix M.

Now we consider the situation where the data in the BPR function (3.17) involve uncertainties. Then we formulate such a robust TAP as an SOCCP. In the remainder of this section, we suppose that Assumption 3.2 holds for the uncertainty set.

Uncertainty in the traffic capacity We consider the situation that the traffic capacity c_l is uncertain. Specifically we suppose that c_l is expressed as $c_l = \bar{c}_l + \hat{u}_l$ with nominal $\bar{c}_l$ and uncertainty parameter $\hat{u}_l \in \mathbb{R}$.

Then, the link cost and route cost functions can be expressed as

$$t_l^{\hat{u}_l}(y) = a_l \left(1 + b_l \left(\frac{M_l x}{\bar{c}_l + \hat{u}_l}\right)^{\nu}\right), \tag{3.19}$$

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathscr{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l + \hat{u}_l}\right)^{\nu}\right), \tag{3.20}$$

respectively. Here we assume that $\hat{u}_l > -\bar{c}_l$ so that the denominator will not be zero.

In order to obtain the SOCCP reformulation, we had to assume that the uncertain link cost function is expressed as (3.5). However, function $t_l^{\hat{u}_l}$ in (3.19) cannot be written in the form (3.5) in a straightforward manner. We therefore introduce an "approximate link cost function" based on the first-order Taylor expansion as follows:

$$t_l^{\hat{u}_l}(y) := a_l \left(1 + b_l \left(\frac{M_l x}{\bar{c}_l}\right)^{\nu}\right) - \frac{\nu a_l b_l (M_l x)^{\nu}}{\bar{c}_l^{\nu+1}} \hat{u}_l. \tag{3.21}$$

Also the approximate route cost function can be expressed as

$$f_r^{\hat{u}^r}(x) := \sum_{l \in \mathscr{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^{\nu}\right) - \sum_{l \in \mathscr{L}_r} \frac{\nu a_l b_l (M_l x)^{\nu}}{c_l^{\nu+1}} \hat{u}_l. \tag{3.22}$$

Since the uncertainty parameter $\hat{u}_l$ is very small in general, this approximation is reasonable. Now, let

$$\Delta t_l(y) = -\frac{\nu a_l b_l (M_l x)^{\nu}}{c_l^{\nu+1}}.$$

Then, (3.21) and (3.22) correspond to (3.5) and (3.7), respectively. Thus, we can reformulate the robust TAP as an SOCCP by using the results of Subsection 3.1.2.

Uncertainty in the congestion factor We consider the situation that the congestion factor b_l is uncertain. Specifically we suppose that b_l is expressed as $b_l = \bar{b}_l + \hat{u}_l$ with nominal $\bar{b}_l$ and uncertainty parameter $\hat{u}_l \in \mathbb{R}$. Then, the link and route cost functions can be expressed as

$$t_l^{\hat{u}_l}(y) = a_l \left(1 + \bar{b}_l \left(\frac{y_l}{c_l}\right)^{\nu}\right) + \hat{u}_l a_l \left(\frac{y_l}{c_l}\right)^{\nu}, \tag{3.23}$$

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathscr{L}_r} a_l \left(1 + \bar{b}_l \left(\frac{M_l x}{c_l}\right)^{\nu}\right) + \sum_{l \in \mathscr{L}_r} a_l \hat{u}_l \left(\frac{M_l x}{c_l}\right)^{\nu}, \tag{3.24}$$

respectively. Let

$$\Delta t_l(y) := a_l \left(\frac{y_l}{c_l}\right)^{\nu}.$$

Then, (3.23) and (3.24) correspond to (3.5) and (3.7), respectively. Thus, we can also reformulate the robust TAP as an SOCCP by using the results of Subsection 3.1.2.

Uncertainty in the free-flow travel time We consider the situation that the free-flow travel time a_l is uncertain. Specifically we suppose that a_l is expressed as $a_l = \bar{a}_l + \hat{u}_l$ with nominal value $\bar{a}_l$ and uncertainty parameter $\hat{u}_l \in \mathbb{R}$. Then, the link and route cost functions can be expressed as

$$t_l^{\hat{u}_l}(y) = \bar{a}_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^{\nu}\right) + \hat{u}_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^{\nu}\right), \tag{3.25}$$

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathscr{L}_r} \bar{a}_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^{\nu}\right) + \sum_{l \in \mathscr{L}_r} \hat{u}_l \left(1 + \left(\frac{M_l x}{c_l}\right)^{\nu}\right), \tag{3.26}$$

respectively. Let

$$\Delta t_l(y) := 1 + \left(\frac{M_l x}{c_l}\right)^{\nu}.$$

Then (3.25) and (3.26) correspond to (3.5) and (3.7), respectively. Thus, we can reformulate the robust TAP as an SOCCP by using the results in Subsection 3.2.1.

4 Numerical experiments

In this section, we introduce two specific traffic models with uncertainty set of various sizes in the cost functions defined by (3.17) and (3.18). We try to compute robust Wardrop equilibria by using the SOCCP reformulation approach studied in Section 3.2 and observe their properties. Throughout this section, we let the uncertainty set U_r ($r \in R_w$) be given by

$$U_r := \left\{ \hat{u}^r \in \mathbb{R}^{|E|} \;\middle|\; \|\hat{u}^r\| \le \rho_w \right\}, \tag{4.1}$$

where ρ_w is a positive constant for each $w \in W$. Notice that OD pair is identified for each $r \in R$. Also, we consider the case with $\nu = 4$ in the cost functions defined by (3.17) and (3.18).

For solving the SOCCPs, we apply the Newton-type method that uses a smoothing technique [13]. All programs are coded in MATLAB 2010a and run on a machine with Intel® Core i5 430M 2.27GHz CPU and 4.00GB memories.

4.1 Relationship between size of uncertainty sets and robust Wardrop equilibria

In this subsection, We consider the traffic model illustrated in Figure 1. Each node denotes an origin, a destination, and an intersection, and each link denotes a road connecting the nodes. The set of OD pairs is given by $W := \{w_1, w_2\}$, where $w_1 = (1 \to 5)$ and $w_2 = (2 \to 6)$. The demands for w_1 and w_2 are given by $d_{w_1} = d_{w_2} = 10$. We suppose that the demands do not depend on λ. We give the routes $r \in R = R_{w_1} \cup R_{w_2}$, and the coefficients a_l, b_l, and c_l of the link functions (3.17) as shown in Table 1 and 2, respectively. Now we consider the case where only a_l is uncertain with uncertainty set U_r expressed by (4.1). Therefore, we use (3.25) and (3.26) as the link cost function and the route cost function with uncertainty, respectively. In this experiment, we vary ρ_{w_1} from 0.001 to 5, and fix ρ_{w_2} at 0.001, and compute a robust Wardrop equilibrium for each ρ_{w_1}. Then, we observe the route flow $\{x_r\}_{r\in R}$ and the minimum cost λ_w at the obtained equilibria of the robust TAPs.

Table 3 shows the obtained values of $\{x_r\}_{r\in R}$ and $\{\lambda_w\}_{w\in W}$ at the equilibrium for each ρ_{w_1}. From the table, we can observe that, as ρ_{w_1} increases, x_{r_1} and λ_{w_1} get larger, but x_{r_2} gets smaller. On the other hand, as to w_2, as ρ_{w_1} increases, x_{r_4} and λ_{w_2} get smaller, but x_{r_3} gets larger. We can interpret these results as follows: Let us consider the drivers who belong to the OD pair $w_1 \in W$. In Figure 1, r_1 has only one link 1, while r_2 has three links 2, 3 and 4, that is, route r_2 is more complicated than r_1. In such a situation, drivers may think that more complicated routes involve more uncertainty and require higher costs than simple routes, and therefore avoid using route r_2. Thus the result of this experiment well reflect such driver's estimation for uncertainty.

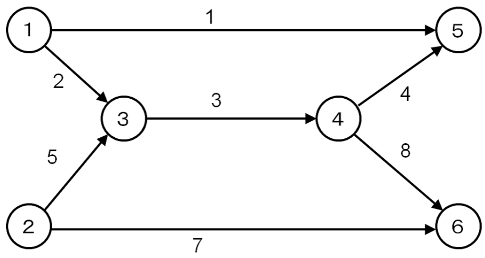

Figure 1: The network in section 4.1

Table 1: Relation of OD pairs, routes and links in Figure 1

OD pair	route	order of links
w_1	r_1	1
	r_2	$2 \to 3 \to 4$
w_2	r_3	$5 \to 3 \to 8$
	r_4	7

4.2 Difference of the minimal cost on equilibrium and actual cost in the network

In this subsection, we consider the traffic model illustrated in Figure 2. This network is taken from [22]. The set of OD pairs is given by $W := \{w_1, w_2, w_3, w_4\}$, where $w_1 = (1 \to 7)$, $w_2 = (2 \to 7)$, $w_2 = (3 \to 7)$, and $w_4 = (4 \to 7)$. The routes for each OD pair and the coefficients a_l, b_l and c_l of the link cost function (3.17) are shown in Tables 4 and 5, respectively. We suppose that, in OD pair w_4, there are six types of drivers. In order to describe such different types of drivers, we introduce the virtual OD pairs w_{4a}, w_{4b}, …, w_{4f} which have the same origin and destination nodes, but different uncertainty parameters. Let the set of the virtual OD pairs be $\tilde{W}_4 := \{w_{4a}, w_{4b}, \ldots, w_{4f}\}$. Then we distinguish each route in OD pair w_4 for six types of

Table 2: Coefficients of link cost functions

link	a_l	b_l	c_l
1	5	0.15	2
2	1	0.15	1
3	1	0.15	1
4	1	0.15	1
5	1	0.15	1
6	1	0.15	1
7	5	0.15	2

Table 3: Uncertainty size, obtained route flow and minimum cost ($\rho_{w_2} = 0.001$)

ρ_{w_1}	x_{r_1}	x_{r_2}	x_{r_3}	x_{r_4}	λ_{w_1}	λ_{w_2}
0	7.329	2.671	2.670	7.330	140.268	140.369
0.001	7.330	2.670	2.670	7.330	140.345	140.345
0.005	7.333	2.667	2.671	7.329	140.654	140.250
0.01	7.336	2.664	2.673	7.327	141.039	140.131
0.05	7.362	2.638	2.685	7.315	144.114	139.206
0.1	7.393	2.607	2.701	7.299	147.938	138.104
0.5	7.603	2.397	2.800	7.200	177.753	131.003
1	7.793	2.207	2.887	7.113	213.425	125.000
5	8.378	1.622	3.137	6.863	471.863	109.038

drivers as follows:

$$\begin{aligned}
R_{w_{4a}} &:= \{r_{8a}, r_{9a}, r_{10a}, r_{11a}, r_{12a}\}, \\
R_{w_{4b}} &:= \{r_{8b}, r_{9b}, r_{10b}, r_{11b}, r_{12b}\}, \\
&\vdots \\
R_{w_{4f}} &:= \{r_{8f}, r_{9f}, r_{10f}, r_{11f}, r_{12f}\}.
\end{aligned}$$

We denote the sets of virtual OD pairs and routes as

$$\begin{aligned}
\tilde{R} &:= R_{w_1} \cup R_{w_2} \cup R_{w_3} \cup R_{w_{4a}} \cup R_{w_{4b}} \cup R_{w_{4c}} \cup R_{w_{4d}} \cup R_{w_{4e}} \cup R_{w_{4f}} \\
\tilde{W} &:= \{w_1, w_2, w_3, w_{4a}, w_{4b}, w_{4c}, w_{4d}, w_{4e}, w_{4f}\}.
\end{aligned}$$

We use the link cost functions (3.17) with uncertain data b_l, that is, $t_l^{\hat{u}_l}(y)$ given by (3.23). Then the uncertain route cost function $f_r^{\hat{u}^r}(x)$ is given by (3.24), and the uncertainty set U_r is given by (4.1). For each $w \in \tilde{W}$, the uncertainty radius ρ_w and the traffic demands d_w are given in Table 6. Note that OD pair w_4 has the six types of drivers with the same demands and the different uncertainty radiuses ρ_w. In this experiment, we calculate the minimum worst costs at the robust Wardrop equilibrium. Table 6 shows the route flow $\{\bar{x}_r\}_{r\in\tilde{R}}$ at the robust Wardrop equilibrium and the minimum value $\{\bar{\lambda}_w\}_{w\in\tilde{W}}$ of the worst cost for each (virtual) OD pair.

Based on the calculated robust Wardrop equilibria, we next observe the distribution of each driver's "actual" route cost by means of a simulation based approach. Let $\{\bar{x}_r\}_{r\in\tilde{R}}$ and $\{\bar{\lambda}_w\}_{w\in\tilde{W}}$ be the solution of the robust TAP. Moreover, let F_w be the actual route cost of the driver with OD pair $w \in \tilde{W}$, i.e., $F_w = F_w(r, u^r) := f_r^{u^r}(\bar{x}) = \sum_{l\in\mathscr{L}_r} t_l^{u_l}(M\bar{x})$, where u^r denotes the actual value of the uncertain parameter $\hat{u}^r$. Then, we calculate the value of F_w for each $w \in \tilde{W}$ by the following steps:

1. Let each driver choose his/her route $r \in \tilde{R}_w$ with probability x_r/d_w.
2. Then, let $F_w := \sum_{l\in\mathscr{L}_r} t_l^{u_l}(Mx)$, where each u_l $(l \in \mathscr{L}_r)$ is randomly chosen from the normal distribution with mean 0 and variance 0.03

We carry out the above steps 10,000 times for each virtual OD pair $w \in \tilde{W}_4$, and get the histograms shown in Figures 3–8. In each figure, the horizontal and the vertical axes show the value of F_w and the number of times each value is obtained, respectively. The red vertical line shows the value of $\bar{\lambda}_w$. Table 7 shows the mean value and the standard deviation of F_w for 10,000 trials, and the frequency of the event that the actual cost F_w was larger than the presumed worst route cost $\bar{\lambda}_w$.

From Table 7, we can see that the value of ρ_w makes a very small impact on the mean and the standard deviation of F_w. However we can also observe that the actual route cost F_w becomes smaller than λ_w more often as ρ_w becomes larger. Especially, when $w = w_{4a}$, i.e., $\rho_{w_{4a}} = 0$, the mean value of $F_{w_{4a}}$ is almost equal to the minimum value $\lambda_{w_{4a}}$ of the worst route cost. On the other hand, when $w = w_{4f}$, i.e., $\rho_{w_{4f}} = 0.05$, the mean value of $F_{w_{4f}}$ is much smaller than $\lambda_{w_{4f}}$. In the real networks, drivers who estimate the uncertain events very carefully often experience the situation that the actual cost was smaller than that he/she had presumed. From such a viewpoint, the result of this experiment is intuitive and persuasive.

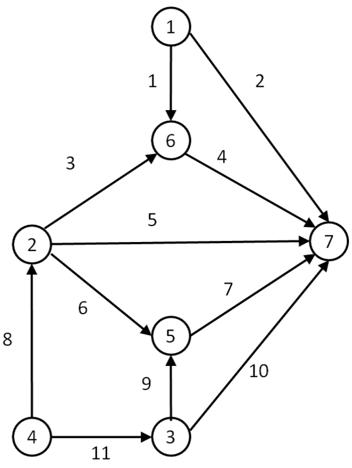

Figure 2: The network in Section 4.2

Table 4: Relation between routes and links for each OD pair

OD pair	route: order of links
w_1	$r_1 : 2,\ r_2 : 1 \to 4$
w_2	$r_3 : 3 \to 4,\ r_4 : 5,\ r_5 : 6 \to 7$
w_3	$r_6 : 10,\ r_7 : 9 \to 7$
w_4	$r_8 : 11 \to 10,\ r_9 : 11 \to 9 \to 7,\ r_{10} : 8 \to 6 \to 7,\ r_{11} : 8 \to 5,\ r_{12} : 8 \to 3 \to 4$

Table 5: Coefficients of link cost functions

link	1	2	3	4	5	6	7	8	9	10	11
a_l	5	11	6	6	15	5	7	6	1	11	10
b_l	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15
c_l	150	160	200	200	150	200	200	100	100	160	100

Table 6: Obtained solutions of robust TAPs

OD pair	class	ρ_w	d_w	$\bar{\lambda}_w$	$\bar{x}_r$
w_1		0.001	500	67.924	$(x_1, x_2) = (387.124, 112.876)$
w_2		0.001	600	91.699	$(x_3, x_4, x_5) = (140.226, 361.844, 97.929)$
w_3		0.001	400	107.726	$(x_6, x_7) = (400, 0)$
w_4	w_{4a}	0	140	464.219	$(x_{8a}, x_{9a}, x_{10a}, x_{11a}, x_{12a}) = (0, 0, 140, 0, 0)$
	w_{4b}	0.01	140	489.134	$(x_{8b}, x_{9b}, x_{10b}, x_{11b}, x_{12b}) = (0, 0, 0, 0, 140)$
	w_{4c}	0.02	140	513.937	$(x_{8c}, x_{9c}, x_{10c}, x_{11c}, x_{12c}) = (0, 0, 0, 0, 140)$
	w_{4d}	0.03	140	538.740	$(x_{8d}, x_{9d}, x_{10d}, x_{11d}, x_{12d}) = (41.991, 68.626, 0, 0, 29.383)$
	w_{4e}	0.04	140	562.636	$(x_{8e}, x_{9e}, x_{10e}, x_{11e}, x_{12e}) = (0, 140, 0, 0, 0)$
	w_{4f}	0.05	140	586.532	$(x_{8f}, x_{9f}, x_{10f}, x_{11f}, x_{12f}) = (0, 140, 0, 0, 0)$

Table 7: Actual route cost F_w

class of w_4	mean	deviation	$F_w > \bar{\lambda}_w$ (%)
w_{4a}	464.302	74.324	50.1
w_{4b}	464.315	74.323	37
w_{4c}	464.235	74.704	25.3
w_{4d}	463.966	74.312	15.7
w_{4e}	464.704	74.487	9.5
w_{4f}	464.380	74.686	5.1

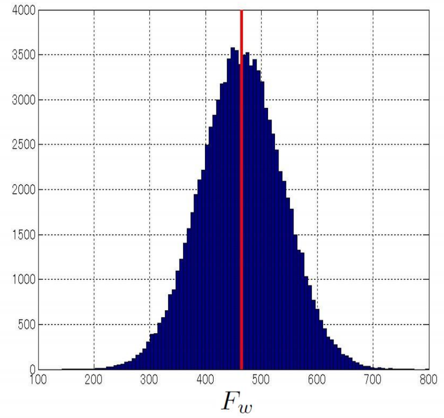

Figure 3: Cost of drivers with $\rho_{w_{4a}} = 0$

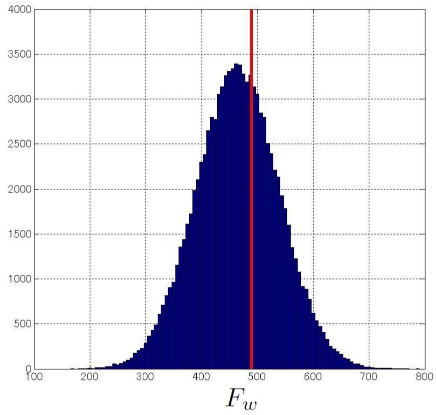

Figure 4: Cost of drivers with $\rho_{w_{4b}} = 0.01$

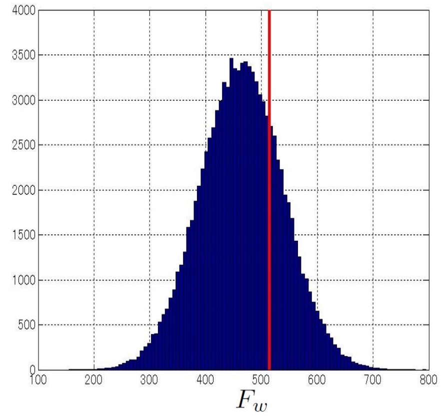

Figure 5: Cost of drivers with $\rho_{w_{4c}} = 0.02$

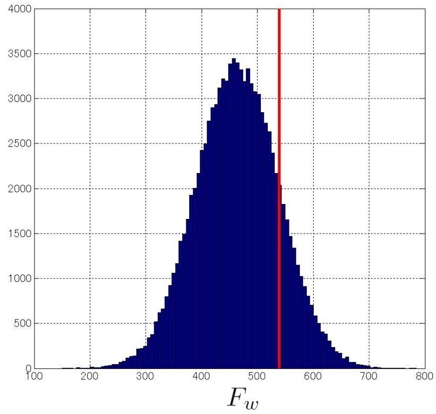

Figure 6: Cost of drivers with $\rho_{w_{4d}} = 0.03$

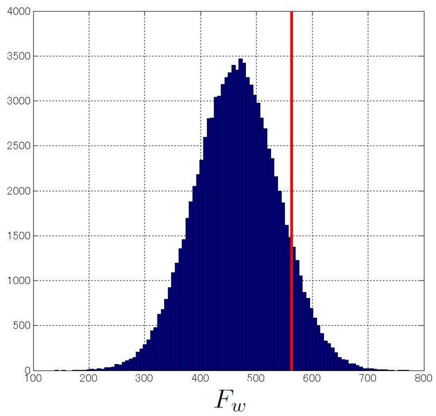

Figure 7: Cost of drivers with $\rho_{w_{4e}} = 0.04$

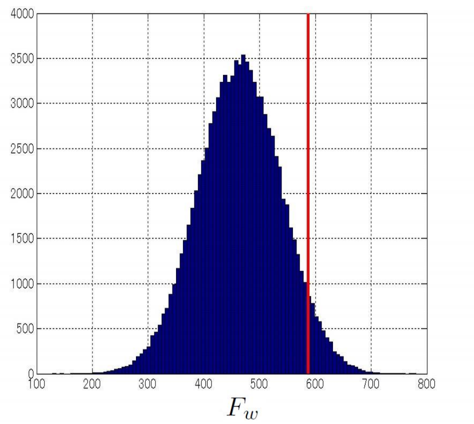

Figure 8: Cost of drivers with $\rho_{w_{4f}} = 0.05$

5 Conclusion and remarks

In this paper, we consider the robust TAP for the traffic network with uncertain data. We have shown a condition for the existence of a robust Wardrop equilibrium. Furthermore we have shown that the robust TAP can be formulated as an SOCCP under the assumption that the route cost functions are additive, the link cost functions are non-separable and the uncertainty sets are ellipsoidal. In the numerical experiments, we have observed that our model well explains each user's route selection under uncertain situations.

We still have some future issues to be addressed. First of all, we have not given conditions for uniqueness of the robust Wardrop equilibrium. It is known that the TAP without uncertainty has a unique link flow y satisfying Wardrop's user equlibrium principle under a certain monotonity assumption. So, it would be also interesting to study under which condition a similar uniqueness property is satisfied for the robust TAP. Second, we have not focused on non-additive route cost functions with uncertainty. Since the non-additive route cost functions play an important role in practical situations, it would be important to formulate robust TAPs with non-additive route cost functions as computable classes of traffic equilibrium problems.

Acknowledgments

I am deeply grateful to Assistant Professor Shunsuke Hayashi, who supervised me throughout my bachelor's and master's course, for receiving generous support from him. Especially, he reads my poor draft carefully. I would also like to express my gratitude to Professor Masao Fukushima for giving me a lot of constructive and precise advices. I would like to thank Associate Professor Nobuo Yamashita for his valuable comments on my paper. I would also like to thank Mr. Takayuki Okuno for reading my draft and giving many valuable comments. I would like to express special thanks to Hitoshi Takahashi, who is the former member of my laboratory. My study is motivated by his bachelor thesis. Finally, I would like to thank all members of Fukushima Laboratory.

References

[1] H. Z. Aashtiani and T. L. Magnanti. Equilibria on a congested transportation network. *SIAM Journal on Algebraic and Discrete Methods*, 2:213–226, 1981.

[2] R. P. Agdeppa, N. Yamashita, and M. Fukushima. The traffic equilibrium problem with nonadditive costs and its monotone mixed complementarity problem formulation. *Transportation Research B*, 41(8):862–874, 2007.

[3] J.-P. Aubin and H. Frankowska. *Set Valued Analysis*. Birkhauser, 1990.

[4] I. Averbakh and Y. B. Zhao. Explicit reformulations for robust optimization problems with general uncertainty sets. *SIAM Journal on Optimization*, 18(4):1436–1466, 2007.

[5] M. Beckmann, C. McGuire, and C. Winsten. *Studies in the Economics of Transportation*. Yale University Press, 1956.

[6] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton Univ Press, 2009.

[7] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[8] Bureau of Public Roads. *Traffic Assignment Manual*. U.S. Department of Commerce, Urban Planning Division, Washington, DC, 1964.

[9] J. S. Chen, X. Chen, and P. Tseng. Analysis of nonsmooth vector-valued functions associated with second-order cones. *Mathematical Programming*, 101(1):95–117, 2004.

[10] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems, I and II*. Springer, 2003.

[11] M. Fukushima, Z. Q. Luo, and P. Tseng. Smoothing functions for second-order-cone complementarity problems. *SIAM Journal on Optimization*, 12:436–460, 2002.

[12] S. A. Gabriel and D. Bernstein. The traffic equilibrium problem with nonadditive path costs. *Transportation Science*, 31:337–348, 1997.

[13] S. Hayashi, N. Yamashita, and M. Fukushima. A combined smoothing and regularization method for monotone second-order cone complementarity problems. *SIAM Journal on Optimization*, 15(2):593–615, 2005.

[14] R. Nishimura, S. Hayashi, and M. Fukushima. SDP reformulation for robust optimization problems based on nonconvex QP duality. Technical report 2009-014, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, 2009.

[15] F. Ordóñez and N. E. Stier-Moses. Robust wardrop equilibrium. *Network Control and Optimization*, 4465:247–256, 2007.

[16] F. Ordóñez and N. E. Stier-Moses. Wardrop equilibria with risk-averse users. *Transportation Science*, 44(1):63–86, 2010.

[17] S. Pan and J. S. Chen. A damped Gauss-Newton method for the second-order cone complementarity problem. *Applied Mathematics and Optimization*, 59(3):293–318, 2009.

[18] M. Patriksson. *Traffic Assignment Problems: Models and Methods*. V.S.P. Intl Science, 10 1994.

[19] Y. Sheffi. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice Hall, 1985.

[20] H. Takahashi. Robust Wardrop equilibria for traffic model with uncertainty. Bachelor Thesis (in Japanese), Applied Mathematics and Physics Course in the School of Informatics and Mathematical Science, Faculty of Engineering, Kyoto University, 2010.

[21] J. G. Wardrop. Some theoretical aspects of road traffic research. *Proceedings of Institute of Civil Engineers Part II*, 1:325–378, 1952.

[22] H. Yang and M. G. H. Bell. Traffic restraint, road pricing and network equilibrium. *Transportation Research Part B: Methodological*, 31(4):303–314, 1997.

[23] C. Zhang, X. Chen, and A. Sumalee. Robust Wardrop's user equilibrium assignment under stochastic demand and supply: Expected residual minimization approach. *Transportation Research B*, doi:10.1016/j.trb.2010.09.008.