Master's Thesis

A Branch-and-Bound Method for Absolute Value Programs and Its Application to Facility Location Problems

Guidance

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Abstract

Recently, the absolute value equation (AVE) has attracted a growing attention. The absolute value program (AVP) is an extension of AVE, which contains absolute values of variables in its objective function and constraints. The AVP has an interesting duality result and reduces to a mathematical program with equilibrium constraints. In this paper, we propose an algorithm for the AVP, which is based on the branch-and-bound method. In the branching procedure, we generate two subproblems by restricting the sign of a component of the variable x. In the bounding procedure, we exploit the duality result in AVP. Furthermore, we carry out numerical experiments for nonconvex facility location problems to show the effectiveness of the proposed algorithm.

Contents

1 Introduction

Recently, the *absolute value equation* (AVE) [1, 7, 11, 13–15, 17–21, 25–27] has attracted a growing attention. The *absolute value program* (AVP) is an extension of AVE, which contains absolute values of variables in its objective function and constraints. Formally, the AVP is stated as follows:

$$\begin{array}{lll}(\mathrm{P}) & \min & c^\top x + d^\top |x| \\ & \text{s.t.} & Ax + B|x| = b, \\ & & Hx + K|x| \geq p,\end{array}$$

where $c, d \in \mathbb{R}^n,\ b \in \mathbb{R}^m,\ p \in \mathbb{R}^\ell,\ A, B \in \mathbb{R}^{m\times n},\ H, K \in \mathbb{R}^{\ell\times n}$, and $|x|$ denotes the vector $|x| = (|x_1|, |x_2|, \dots, |x_n|)^\top \in \mathbb{R}^n$. Although this problem is a nonconvex optimization problem, Mangasarian [12] showed an interesting weak duality result and a sufficient optimality condition for the problem. In addition, the AVE that appears in the constraints of the AVP was shown to be equivalent to a *linear complementarity problem* [12,17]. This result indicates that the AVP is equivalent to a *linear program with linear complementarity constraints* [5,6], which is a special case of the *mathematical program with equilibrium constraints* (MPEC) [9]. MPEC has many applications in real life such as economic equilibrium, engineering design, and traffic equilibrium. However, MPEC is in general difficult to deal with, since its feasible region is necessarily nonconvex and even disconnected. The study on AVP is in its infancy and, to the author's knowledge, there have been no work except for the above-mentioned duality results of Mangasarian [12].

In this paper, we first propose an algorithm for the AVP, which is based on the branch-and-bound method. In the branching procedure, we generate two subproblems by restricting the sign of a component of the variable x in (P). In the bounding procedure, we exploit the duality results in AVP to obtain a lower bound for each subproblem. Furthermore, we apply the proposed algorithm to solve *facility location problems* (FLPs). By using the ℓ_1 norm as a distance function, an FLP can naturally be formulated as an AVP. In particular, we can use the AVP formulation to deal with a nonconvex region in which facilities can be located. We stress that such a problem is considerably difficult to solve compared with the conventional FLPs that assume the convexity of the region.

The paper is organized as follows. In the next section, we give some preliminary results about the AVP. Using these results, we propose a branch-and-bound method for solving AVP in Section 3. In Section 4, we consider two types of FLPs and give some numerical results with the proposed algorithm. Finally, we conclude the paper in Section 5.

2 Duality

The dual problem of AVP (P) is defined as follows [12]:

$$
\begin{aligned}
\text{(D)} \qquad \max \quad & b^\top u + p^\top v \\
\text{s.t.} \quad & |A^\top u + H^\top v - c| + B^\top u + K^\top v \leq d, \qquad (1)\\
& v \geq 0.
\end{aligned}
$$

Note that the constraint (1) can be represented as

$$
\begin{aligned}
& |A^\top u + H^\top v - c| + B^\top u + K^\top v \leq d \\
\iff \quad & |A^\top u + H^\top v - c| \leq d - B^\top u - K^\top v \\
\iff \quad & -d + B^\top u + K^\top v \leq A^\top u + H^\top v - c \leq d - B^\top u - K^\top v \\
\iff \quad & \begin{cases} (-A+B)^\top u + (-H+K)^\top v \leq d - c \\ (A+B)^\top u + (H+K)^\top v \leq d + c. \end{cases}
\end{aligned}
$$

Therefore, the dual problem (D) can be rewritten as follows:

$$
\begin{aligned}
\max \quad & b^\top u + p^\top v \\
\text{s.t.} \quad & (-A+B)^\top u + (-H+K)^\top v \leq d - c, \\
& (A+B)^\top u + (H+K)^\top v \leq d + c, \\
& v \geq 0.
\end{aligned}
$$

Notice that the primal problem (P) is not generally convex, but the dual problem (D) is always a convex optimization problem, or more precisely, a linear program. Moreover, a weak duality theorem and a sufficient optimality condition for AVP are shown in [12], which will be useful in our algorithm.

Theorem 1. [12] *If x and (u, v) are feasible solutions of* (P) *and* (D)*, respectively, then the following inequality holds:*

$$c^\top x + d^\top |x| \geq b^\top u + p^\top v.$$

This theorem says that we can get a lower bound of the optimal value of (P) by solving the dual problem (D). The next theorem gives us a sufficient optimality condition for (P).

Theorem 2. [12] *Let $\bar{x}$ be feasible for the primal AVP* (P) *and $(\bar{u}, \bar{v})$ be feasible for the dual AVP* (D) *with equal primal and dual objective values, that is,*

$$c^\top \bar{x} + d^\top |\bar{x}| = b^\top \bar{u} + p^\top \bar{v}.$$

Then $\bar{x}$ and $(\bar{u}, \bar{v})$ are optimal solutions of (P) *and* (D)*, respectively.*

3 Branch-and-Bound Method

In this section, we propose a branch-and-bound method for AVP. Branch-and-bound method is one of global optimization approaches for nonconvex optimization problems and combinatorial optimization problems, and the method consists of branching and bounding procedures [4]. In the branching procedure, we divide the feasible region of the original problem into some subregions to generate subproblems. On the other hand, in the bounding procedure, we check if a current subproblem can be discarded or not, by implementing some fathoming tests. We now give the detail of the branching and bounding procedures used in the proposed branch-and-bound method for AVP.

A subproblem is constructed from (P) by restricting some variables to be either nonpositive or nonnegative:

$$
\begin{array}{lll}
\mathrm{P}(\mathcal{I},\mathcal{J}) & \min & c^\top x + d^\top |x| \\
 & \text{s.t.} & Ax + B|x| = b, \\
 & & Hx + K|x| \geq p, \\
 & & x_i \geq 0 \quad (i \in \mathcal{I}), \\
 & & x_i \leq 0 \quad (i \in \mathcal{J}),
\end{array}
$$

where $\mathcal{I}$ and $\mathcal{J}$ are subsets of $\{1, 2, \ldots, n\}$. Note that (P) $= \mathrm{P}(\emptyset, \emptyset)$. An example of the enumeration tree with $n = 2$ is shown in Fig. 1.

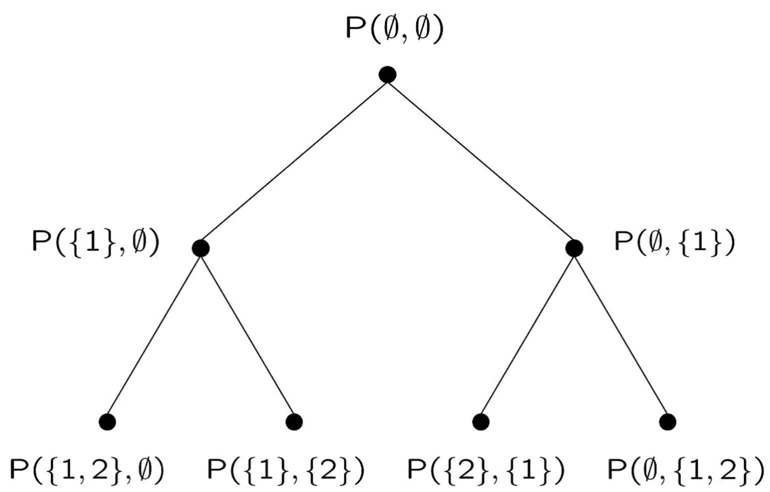

Fig. 1: Enumeration tree ($n = 2$)

At each node of the tree, branching means that we choose a variable x_i and restrict it to be nonnegative or nonpositive in the corresponding subproblem. Therefore, the deepest nodes in the tree correspond to 2^n linear programs, which contain *no* absolute values of the variables. The branch-and-bound method maintains the set of subproblems that can be selected to apply a branching procedure. Such subproblems are called *active*, and the set of the current active subproblems is denoted by $\mathcal{A}$. For example, if we generate two subproblems $\mathrm{P}(\{1\},\emptyset)$, $\mathrm{P}(\emptyset,\{1\})$ at the node $\mathrm{P}(\emptyset,\emptyset)$ in the enumeration tree of Fig. 1, we have $\mathcal{A}=\{\mathrm{P}(\{1\},\emptyset),\ \mathrm{P}(\emptyset,\{1\})\}$.

In the bounding procedure, we consider the dual problem of $\mathrm{P}(\mathcal{I},\mathcal{J})$ in order to get a lower bound of $\mathrm{P}(\mathcal{I},\mathcal{J})$. We now rewrite $\mathrm{P}(\mathcal{I},\mathcal{J})$ in the following way. Let

$$h_i = \begin{array}{c} \begin{array}{ccccccc} & & & \overset{i}{\vee} & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \end{array} \quad i \in \mathcal{I},$$

$$h_i = \begin{array}{c} \begin{array}{ccccccc} & & & \overset{i}{\vee} & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{pmatrix} \end{array} \quad i \in \mathcal{J}.$$

Then, the nonnegativity and nonpositivity constraints on variables x_i in $\mathrm{P}(\mathcal{I},\mathcal{J})$ can be represented as

$$h_i x \geq 0 \quad (i \in \mathcal{I} \cup \mathcal{J}).$$

Now define $\tilde{H} \in \mathbb{R}^{(\ell+|\mathcal{I}|+|\mathcal{J}|)\times n}$, $\tilde{K} \in \mathbb{R}^{(\ell+|\mathcal{I}|+|\mathcal{J}|)\times n}$, $\tilde{p} \in \mathbb{R}^{(\ell+|\mathcal{I}|+|\mathcal{J}|)}$ as

$$\tilde{H} := \begin{bmatrix} H \\ \hline \vdots \\ h_i \\ \vdots \end{bmatrix}, \quad \tilde{K} := \begin{bmatrix} K \\ \hline \vdots \\ 0 \\ \vdots \end{bmatrix}, \quad \tilde{p} := \begin{bmatrix} p \\ \hline \vdots \\ 0 \\ \vdots \end{bmatrix},$$

where $|\mathcal{I}|$ and $|\mathcal{J}|$ denote the numbers of elements of $\mathcal{I}$ and $\mathcal{J}$, respectively. Then, we can rewrite $\mathrm{P}(\mathcal{I},\mathcal{J})$ as follows:

$$\begin{array}{lll} \mathrm{P}(\mathcal{I},\mathcal{J}) & \min & c^\top x + d^\top |x| \\ & \text{s.t.} & Ax + B|x| = b, \\ & & \tilde{H}x + \tilde{K}|x| \geq \tilde{p}. \end{array}$$

Moreover, the dual problem of $\mathrm{P}(\mathcal{I},\mathcal{J})$ is written as

$$\begin{array}{lll} \mathrm{D}(\mathcal{I},\mathcal{J}) & \max & b^\top u + \tilde{p}^\top v \\ & \text{s.t.} & |A^\top u + \tilde{H}^\top v - c| + B^\top u + \tilde{K}^\top v \leq d, \quad v \geq 0, \end{array}$$

which can further be rewritten as a linear program. From the result of solving the dual problem, subproblem $\mathrm{P}(\mathcal{I},\mathcal{J})$ can be fathomed if one of the following conditions holds:

1. $\mathrm{D}(\mathcal{I}, \mathcal{J})$ is unbounded.

2. The optimal value of $\mathrm{D}(\mathcal{I}, \mathcal{J})$ is greater than the objective value of the incumbent solution, i.e. , the best feasible solution of (P) found so far.

3. There is no duality gap between $\mathrm{P}(\mathcal{I}, \mathcal{J})$ and $\mathrm{D}(\mathcal{I}, \mathcal{J})$.

We now give more details about the bounding operations based on the above three conditions.

If the dual problem $\mathrm{D}(\mathcal{I}, \mathcal{J})$ is unbounded, then the primal problem $\mathrm{P}(\mathcal{I}, \mathcal{J})$ is infeasible from the weak duality theorem. In this case, any subproblem generated from the current subproblem by restricting the sign of some of its variables cannot be feasible. Hence we can discard the current subproblem.

If the optimal value of $\mathrm{D}(\mathcal{I}, \mathcal{J})$, which is a lower bound of the optimal value of $\mathrm{P}(\mathcal{I}, \mathcal{J})$, is greater than the objective value of the incumbent solution, then Theorem 1 ensures that we have no chance to obtain an optimal solution of (P) by generating subproblems from $\mathrm{P}(\mathcal{I}, \mathcal{J})$ further. So we can discard the current subproblem.

If we find out that there is no duality gap between $\mathrm{P}(\mathcal{I}, \mathcal{J})$ and $\mathrm{D}(\mathcal{I}, \mathcal{J})$, then this means subproblem $\mathrm{P}(\mathcal{I}, \mathcal{J})$ is just solved. For this reason, we need not generate new subproblems from $\mathrm{P}(\mathcal{I}, \mathcal{J})$ further, and we can discard the current subproblem. Moreover, if the optimal solution of $\mathrm{P}(\mathcal{I}, \mathcal{J})$ is better than the incumbent solution, then we replace the incumbent solution by the optimal solution of $\mathrm{P}(\mathcal{I}, \mathcal{J})$, since it is a feasible solution of the original problem (P). We can check if there is no duality gap between $\mathrm{P}(\mathcal{I}, \mathcal{J})$ and $\mathrm{D}(\mathcal{I}, \mathcal{J})$ by solving the following system of absolute value equations and inequalities:

$$\begin{aligned} c^\top x + d^\top |x| &= f_d^*, \\ Ax + B|x| &= b, \\ \tilde{H}x + \tilde{K}|x| &\geq \tilde{p}, \end{aligned} \tag{S1}$$

where f_d^* is the optimal value of the dual problem $\mathrm{D}(\mathcal{I}, \mathcal{J})$. If we get a solution of (S1), then the solution is an optimal solution of $\mathrm{P}(\mathcal{I}, \mathcal{J})$ and, in this case, $\mathrm{P}(\mathcal{I}, \mathcal{J})$ and $\mathrm{D}(\mathcal{I}, \mathcal{J})$ have no duality gap.

We now formally state the algorithm.

Branch-and-Bound Method for Absolute Value Program

STEP 0 Let $\mathcal{I} := \emptyset$, $\mathcal{J} := \emptyset$. Find a feasible solution of problem (P) $= \mathrm{P}(\emptyset, \emptyset)$. Let it be the incumbent solution and let f^* be the objective value at the incumbent solution. Set $\mathcal{A} := \{\mathrm{P}(\emptyset, \emptyset)\}$.

STEP 1 Choose a subproblem $\mathrm{P}(\mathcal{I}, \mathcal{J})$ from the set $\mathcal{A}$.

STEP 1-1 If the dual problem $\mathrm{D}(\mathcal{I}, \mathcal{J})$ of $\mathrm{P}(\mathcal{I}, \mathcal{J})$ is infeasible, then go to STEP 2. If $\mathrm{D}(\mathcal{I}, \mathcal{J})$ is unbounded, then fathom $\mathrm{P}(\mathcal{I}, \mathcal{J})$. Set $\mathcal{A} := \mathcal{A} - \{\mathrm{P}(\mathcal{I}, \mathcal{J})\}$ and go to STEP 3.

STEP 1-2 If we get an optimal value $\bar{f}$ of the dual problem $\mathrm{D}(\mathcal{I}, \mathcal{J})$ and it satisfies $\bar{f} > f^*$, then fathom $\mathrm{P}(\mathcal{I}, \mathcal{J})$. Set $\mathcal{A} := \mathcal{A} - \{\mathrm{P}(\mathcal{I}, \mathcal{J})\}$ and go to STEP 3.

STEP 1-3 Solve the system (S1) of absolute value equations and inequalities. If we fail to get a solution of (S1), then go to STEP 2. If we get a solution of (S1) and, in addition, the objective function value at the solution, denoted $f_{(\mathcal{I},\mathcal{J})}$, satisfies $f_{(\mathcal{I},\mathcal{J})} \geq f^*$, then $\mathrm{P}(\mathcal{I}, \mathcal{J})$ is fathomed immediately. If $f_{(\mathcal{I},\mathcal{J})} < f^*$ is satisfied, then set $f^* := f_{(\mathcal{I},\mathcal{J})}$, update the incumbent solution, and fathom $\mathrm{P}(\mathcal{I}, \mathcal{J})$. Set $\mathcal{A} := \mathcal{A} - \{\mathrm{P}(\mathcal{I}, \mathcal{J})\}$ and go to STEP 3.

STEP 2 Choose x_i as the branching variable, where $i \notin \mathcal{I} \cup \mathcal{J}$, and generate two subproblems $\mathrm{P}(\mathcal{I} \cup \{i\}, \mathcal{J})$ and $\mathrm{P}(\mathcal{I}, \mathcal{J} \cup \{i\})$ from $\mathrm{P}(\mathcal{I}, \mathcal{J})$. Set $\mathcal{A} := \mathcal{A} \cup \{\mathrm{P}(\mathcal{I} \cup \{i\}, \mathcal{J}), \mathrm{P}(\mathcal{I}, \mathcal{J} \cup \{i\})\} - \{\mathrm{P}(\mathcal{I}, \mathcal{J})\}$, and return to STEP 1.

STEP 3 If $\mathcal{A} = \emptyset$, then terminate. The incumbent solution is an optimal solution of the original problem (P). Otherwise, return to STEP 1.

In order to get a feasible solution of (P) in STEP 0 and to solve (S1) in STEP 1-3, we can use the *Successive Linearization Algorithm* (SLA) for the system of absolute value equations and inequalities. This algorithm was first proposed by Mangasarian [12] to solve AVE, and we extend the algorithm so as to deal with a system that contains *absolute value inequality* (AVI).

Here we describe SLA for the AVE-AVI system (S2) shown below, which represents the constraints of (P). The algorithm can similarly be applied to solve (S1).

$$\begin{cases} Ax + B|x| = b, \\ Hx + K|x| \geq p. \end{cases} \tag{S2}$$

First we give a result that relates the AVE-AVI system (S2) to the following concave minimization problem constructed from (S2):

$$\begin{aligned} \min_{(x,t,s_1,s_2)\in\mathbb{R}^{n+n+m+\ell}} \quad & \epsilon(-e^\top|x| + e^\top t) + e^\top s_1 + e^\top s_2 \\ \text{s.t.} \quad & -s_1 \le Ax + Bt - b \le s_1, \\ & -Hx - Kt + p \le s_2, \\ & 0 \le s_2, \\ & -t \le x \le t, \end{aligned} \tag{2}$$

where $\epsilon > 0$ and e is the vector of ones.

Proposition 3. *If* (S2) *is solvable, then there exists some* $\bar{\epsilon} > 0$ *such that, for any* $\epsilon \in (0, \bar{\epsilon}]$*, any solution* $(\bar{x}, \bar{t}, \bar{s_1}, \bar{s_2})$ *of* (2) *satisfies*

$$\begin{aligned} |\bar{x}| &= \bar{t}, \\ A\bar{x} + B|\bar{x}| &= b, \\ H\bar{x} + K|\bar{x}| &\ge p. \end{aligned}$$

Proof . The proof is analogous to that of Proposition 3 in [12].

From this result, a solution of the AVE-AVI system (S2) can be obtained by solving the concave minimization problem (2) with a sufficiently small $\epsilon > 0$. We now give the algorithm for (S2), which is an extension of the SLA for AVE [12]. Let $z = (x, t, s_1, s_2)^\top$. Denote the feasible region of problem (2) by Z and its objective function by $f(z)$.

Algorithm 1. *Start with any* $z^0 \in Z$*. At the k-th iteration, given* z^k*, find* z^{k+1} *such that*

$$z^{k+1} \in \text{arg vertex} \min_{z\in Z} \xi^\top(z - z^k),$$

where ξ *is a subgradient of* $f(z)$ *at* z^k*, and* arg vertex $\min_{z\in Z} \xi^\top(z - z^k)$ *is the set of vertex solutions of the linear program* $\min_{z\in Z} \xi^\top(z - z^k)$*. Stop if* $\xi^\top(z^{k+1} - z^k) = 0$.

In our numerical experiments, we compute a subgradient ξ as follows:

$$\xi = \begin{pmatrix} -\epsilon g \\ \epsilon e \\ e \\ e \end{pmatrix} \in \mathbb{R}^{n+n+m+\ell}, \quad g_i = \begin{cases} 1 & (x_i^k > 0) \\ 0 & (x_i^k = 0) \\ -1 & (x_i^k < 0) \end{cases}, \quad i = 1, \cdots, n.$$

As is well-known, a concave minimization problem has at least one optimal solution at a vertex in its feasible region, provided a solution exists. Taking this fact into account, the SLA tries to find an optimal solution of (2) by solving a sequence of linear programs formed by linearizing the objective function of problem (2). The sequence generated by the SLA finitely converges to a point that satisfies a necessary optimality condition for the concave minimization problem [10, 12]. Notice that the solution obtained by this algorithm is not guaranteed to be a global optimal solution of (2). However, we can easily check if the computed solution actually satisfies (S2) by direct substitution.

We now give the way to generate subproblems in STEP 2. Note that we go to STEP 2 after either of the following two cases occurs.

Case 1. In STEP 1-1, $\mathrm{D}(\mathcal{I}, \mathcal{J})$ is infeasible.

Case 2. In STEP 1-3, (S2) cannot be solved.

If Case 1 occurs, then we generate two subproblems by choosing any variable x_i such that $i \notin \mathcal{I} \cup \mathcal{J}$ as the branching variable. In Case 2, we do not have a solution of (S2), but a local optimal solution of problem (2) is available. In this case, we choose as the branching variable a variable x_i ($i \notin \mathcal{I} \cup \mathcal{J}$) such that $|x_i| \geq |x_{i'}|$ for all $i' \notin \mathcal{I} \cup \mathcal{J}$ at the local optimal solution of (2).

In STEP 1, some kinds of rules can be used for choosing an active subproblem $\mathrm{P}(\mathcal{I}, \mathcal{J}) \in \mathcal{A}$. In the numerical experiments conducted in the next section, we use the depth-first search, which generally chooses an active subproblem corresponding to the farthest node from the root node in the enumeration tree. In particular, when we return to STEP 1 after generating two subproblems, we choose one of these subproblems. In this case, the problem we choose depends on the above-mentioned two cases. If we generate two subproblems in STEP 2 after Case 1 occurs, then we choose any of the two subproblems. In Case 2, as we mentioned above, we have a local optimal solution of (2). In this case, if the branching variable x_i in the local optimal solution takes a positive value, then we choose subproblem $\mathrm{P}(\mathcal{I} \cup \{i\}, \mathcal{J})$. Otherwise, we choose $\mathrm{P}(\mathcal{I}, \mathcal{J} \cup \{i\})$.

4 Numerical Experiments

In this section, we consider facility location problems (FLPs) as an application of AVP. Moreover, we show some numerical results with the proposed branch-and-bound algorithm applied to some examples of FLPs. All computations were carried out on an Intel(R) Core(TM)^2 Duo 3Ghz×2 machine with a MATLAB code. The CPLEX was used to solve linear programs in SLA.

In general, FLP is the problem of finding optimal locations of facilities, and it can be formulated as various kinds of mathematical programs depending on the type of constraints and optimization criteria [2]. In general, there are two kinds of facilities from the resident's point of view. The first category is a *desirable* facility such as schools, libraries and fire stations. Such a facility should be located as closely as possible to the residents. The other category is an *undesirable* facility, which includes incineration plants, electric power stations, chemical factories and so on. These facilities should be located far from the residential area. From the viewpoint of geography, there are three kinds of areas in which facilities can be located, i.e., continuous spaces, discrete spaces, and networks. Furthermore, the distance between two facilities can be measured by using various norms such as the Euclidean, ℓ_1, and ℓ_∞ norms.

In our numerical experiments, we consider two types of FLP on a continuous space with ℓ_1 metric, which can be represented as AVP. Note that the ℓ_1 norm distance between two points x and y can be represented as $e^\top|x-y|$.

4.1 Minimax Location Problem

A minimax multi-facility location problem can be formulated as a mathematical program as follows [2]:

$$\begin{array}{ll} \min & \max \left\{ \max_{i\in I, j\in J} \alpha_{ij} e^\top|x^i - P^j|,\ \max_{i,k\in I, i\neq k} \beta_{ik} e^\top|x^i - x^k| \right\} \\ \text{s.t.} & x^i \in X \ (i \in I), \end{array} \tag{3}$$

where $x^i \in \mathbb{R}^2$ $(i \in I)$ and $P^j \in \mathbb{R}^2$ $(j \in J)$ denote the locations of the new and the existing facilities, respectively, I and J are finite index sets, α_{ij} and β_{ik} are positive weighting factors, and $X \subset \mathbb{R}^2$ is the locatable region for facilities.

This problem is to minimize the maximum weighted distance between new and existing facilities, and between new facilities themselves. This represents a mathematical model of locating desirable facilities, such as schools and fire stations, in a residential area. This kind of problems has been well-studied for the past decades. In particular, using ℓ_1 norm as a distance function, Konforty and Tamir [8] studied the minimax single facility location problem with a forbidden region around each existing facility.

Problem (3) can be rewritten as the following problem by introducing a new variable $z \in \mathbb{R}$:

$$\begin{array}{lll} (\mathrm{P_a}) & \displaystyle\min_{x,z} & z \\ & \text{s.t.} & z \geq \alpha_{ij} e^\top|x^i - P^j| \quad (i \in I, j \in J), \\ & & z \geq \beta_{ik} e^\top|x^i - x^k| \quad (i,k \in I, i \neq k), \\ & & x^i \in X \quad (i \in I). \end{array}$$

If X is a convex polyhedron, $(\mathrm{P_a})$ is easy to solve because $(\mathrm{P_a})$ reduces to a linear program. Here, we deal with the more general case where X is a nonconvex region.

We now give the detail of the problem that we solve in numerical experiments. We define the facility locatable region X as the set of points $x = (x_1, x_2)^\top \in \mathbb{R}^2$ that satisfy the following inequality:

$$|0.15x_2 + ||x_1| - 6|| + 0.5|x_2| + |0.5x_1 + ||x_2| - 6|| + 0.1|x_1| \le 10.5.$$

The region X is nonconvex, as shown in Fig. 2. The above inequality can be represented by the following AVE-AVI system by introducing artificial variables $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R}$:

$$\begin{aligned} &|x_1| - \theta_1 = 6, \\ &|x_2| - \theta_2 = 6, \\ &0.15x_2 + |\theta_1| - \theta_3 = 0, \\ &0.5x_1 + |\theta_2| - \theta_4 = 0, \\ &0.1|x_1| + 0.5|x_2| + |\theta_3| + |\theta_4| \le 10.5. \end{aligned} \tag{4}$$

Notice that if the region X is described by (4), then problem ($\mathrm{P_a}$) is formulated as AVP.

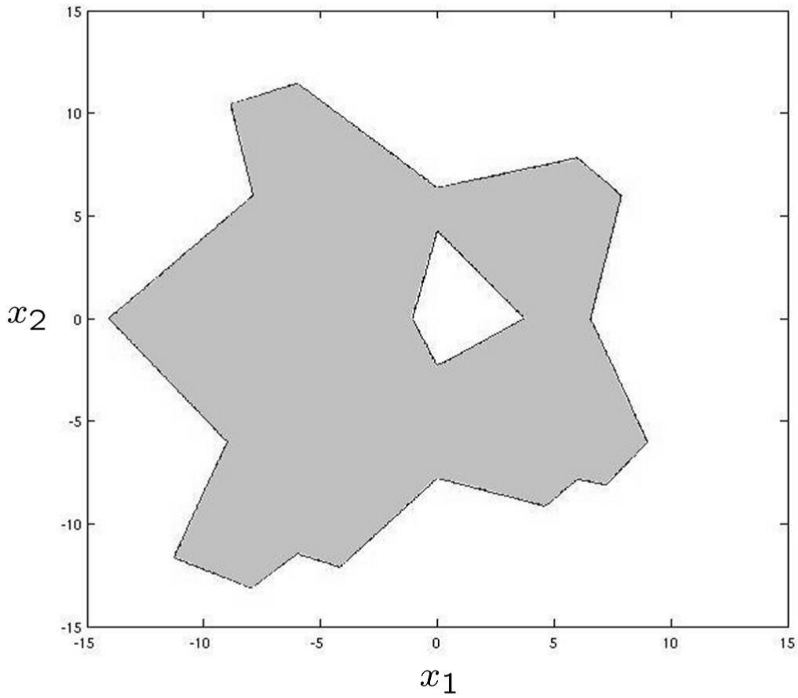

Fig. 2: Region X where the facilities are located.

In the numerical experiments, we let $I = \{1, 2\}$, $J = \{1, 2, \ldots, 7\}$ and set the locations of the existing facilities as $P^1 = (-10, 0)$, $P^2 = (-6, 7)$, $P^3 = (5, 5)$, $P^4 = (4, -5)$, $P^5 = (-10, -10)$, $P^6 = (-4, -10)$, $P^7 = (-2, 0)$. Moreover, we choose the positive weight $\beta_{12} = 0.8$, and use two data sets for the weights α_{ij} as follows:

$$
\begin{aligned}
(\alpha_{11},\ \alpha_{12},\ \alpha_{13},\ \alpha_{14},\ \alpha_{15},\ \alpha_{16},\ \alpha_{17}) &= (1.0,\ 0.8,\ 0.6,\ 0.6,\ 1.0,\ 0.8,\ 0.8),\\
(\alpha_{21},\ \alpha_{22},\ \alpha_{23},\ \alpha_{24},\ \alpha_{25},\ \alpha_{26},\ \alpha_{27}) &= (0.6,\ 0.8,\ 1.0,\ 1.0,\ 0.6,\ 0.8,\ 0.8)
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
(\alpha_{11},\ \alpha_{12},\ \alpha_{13},\ \alpha_{14},\ \alpha_{15},\ \alpha_{16},\ \alpha_{17}) &= (0.8,\ 1.0,\ 1.0,\ 0.6,\ 0.6,\ 0.6,\ 0.8),\\
(\alpha_{21},\ \alpha_{22},\ \alpha_{23},\ \alpha_{24},\ \alpha_{25},\ \alpha_{26},\ \alpha_{27}) &= (0.8,\ 0.6,\ 0.6,\ 1.0,\ 1.0,\ 1.0,\ 0.8).
\end{aligned} \tag{6}
$$

The problems with α_{ij}'s given by (5) and (6) will be called *Minimax-1* and *Minimax-2*, respectively. The branch-and-bound method was able to find solutions of *Minimax-1* and *Minimax-2*, which are given by $x^1 = (-6.35, -2.39)$, $x^2 = (-0.91, -0.33)$ and $x^1 = (-4, 2.75)$, $x^2 = (-5.5, -3.25)$, respectively. The solutions are depicted in Figs. 3 and 4. For each problem, the CPUtime, the number of subproblems fathomed in STEP 1-1, STEP 1-2, STEP 1-3, and the number of nodes explored are summarized in Table 1.

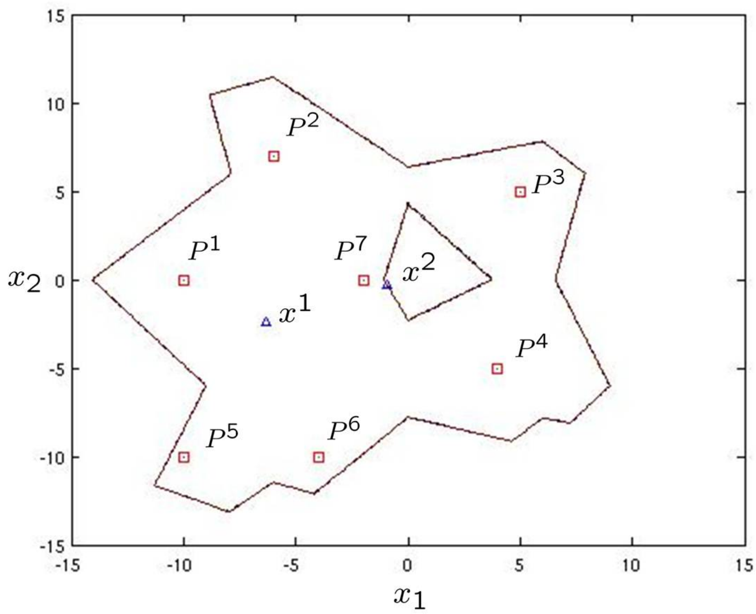

Fig. 3: Solution of Minimax-1

Table 1: Results for minimax location problems

	CPUtime (sec)	STEP 1-1	STEP 1-2	STEP 1-3	No. of nodes explored
Minimax-1	0.7	14	38	4	110
Minimax-2	0.94	14	60	4	154

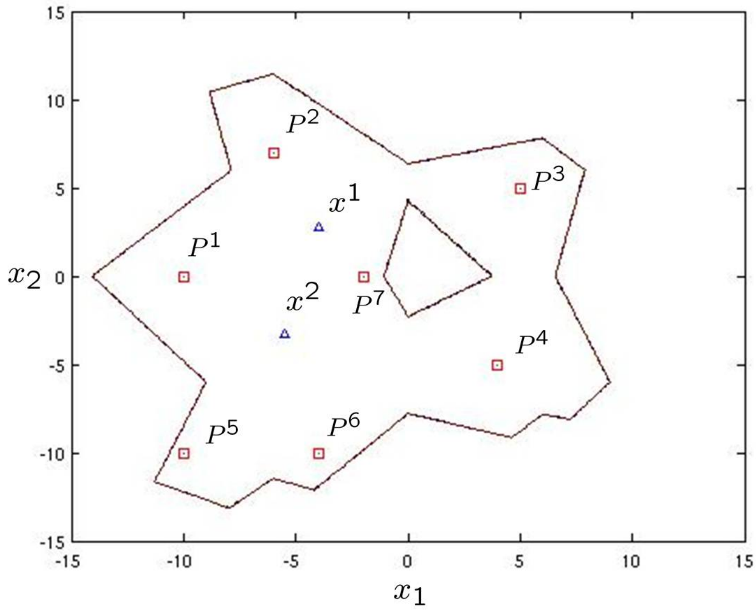

Fig. 4: Solution of Minimax-2

4.2 Maximin Location Problem

A maximin multi-facility location problem is generally formulated as follows [2]:

$$\begin{array}{ll} \max & \min \ \{\min_{i\in I, j\in J} \alpha_{ij} e^\top |x^i - P^j|, \ \min_{i,k\in I, i\neq k} \beta_{ik} e^\top |x^i - x^k| \ \} \\ \text{s.t.} & \qquad\qquad\qquad x^i \in X \ (i \in I), \end{array} \tag{7}$$

where x^i, P^j, α_{ij}, β_{ij} and X represent the same stuffs as in Section 4.1. Unlike the minimax location problem in the previous section, this problem maximizes the minimum weighted distances between new and existing facilities, and between new facilities themselves. For example, this problem will be useful in locating competing facilities such as convenience stores and gas stations.

Sayin [22] and Nadirler and Karasakal [16] reformulated single facility maximin location problems on a convex region with ℓ_1 norm as a mixed integer program. Tamir [23] proposed an algorithm for two-facility maximin location problems on a convex region with ℓ_1 norm. Guerrero García et al. [3] studied an algorithm for a single facility maximin location problem on a convex region with any norm. In these approaches, the region for locating facilities is assumed to be convex. Here we solve multi-facility location problems on a nonconvex region.

Problem (7) can be rewritten as the following problem by introducing a new variable

$z \in \mathbb{R}$ [23, 24]:

$$
\begin{array}{llll}
(\mathrm{P_b}) & \max\limits_{x,z} & z & \\
& \text{s.t.} & z \leq \alpha_{ij} e^\top |x^i - P^j| & (i \in I, j \in J), \\
& & z \leq \beta_{ik} e^\top |x^i - x^k| & (i, k \in I, i \neq k), \\
& & x^i \in X \quad (i \in I). &
\end{array}
$$

Notice that, unlike the inequality constraints in $(\mathrm{P_a})$, those in this problem are nonconvex.

In the numerical experiments, we let the index sets of the new and the existing facilities be $I = \{1, 2\}$ and $J = \{1, 2, \dots, 7\}$, respectively. In addition, we set all the positive weights α_{ij} and β_{12} to be 1. The region X is the nonconvex region described by (4). Moreover, the locations of the existing facilities are given in the following two data sets:

$$
\begin{array}{llll}
P^1 = (-10, 0), & P^2 = (-9, -3), & P^3 = (-6, 2), & P^4 = (-6, -7), \\
P^5 = (-2, 0), & P^6 = (3, 4), & P^7 = (5, 5) &
\end{array}
\tag{8}
$$

and

$$
\begin{array}{llll}
P^1 = (-10, 0), & P^2 = (-8, 10), & P^3 = (-7, -5), & P^4 = (-7, 4), \\
P^5 = (-5, -2), & P^6 = (4, -3), & P^7 = (5, -7). &
\end{array}
\tag{9}
$$

The problems with the data sets (8) and (9) are called *Maximin-1* and *Maximin-2*, respectively. By using the proposed branch-and-bound method, we obtained a solution $x^1 = (7.93, -6)$, $x^2 = (-4.18, -12.11)$ for *Maximin-1* and a solution $x^1 = (6, 6)$, $x^2 = (-11.29, -11.64)$ for *Maximin-2*. Those solutions are shown in Figs. 5 and 6. The CPUtime, the number subproblems fathomed in STEP 1-1, STEP 1-2, STEP 1-3, and the number of nodes explored in each problem are shown in Table 2.

Table 2: Results for maximin location problems

	CPUtime (sec)	STEP 1-1	STEP 1-2	STEP 1-3	No. of nodes explored
Maximin-1	35.3	1647	1402	20	6136
Maximin-2	78.6	3799	2595	41	12868

In all the AVPs of the form (P) formed from the above examples *Minimax*-1, 2, and *Maximin*-1, 2, the number of variables is 43 and the number of constraints is 55. From the results shown in Sections 4.1 and 4.2, we observe that we were able to find a global optimal solution of each problem by exploring only a small number of nodes compared with the number of all possible nodes $(2^{44} - 1)$ in the enumeration tree. Although $(\mathrm{P_a})$ and $(\mathrm{P_b})$ have the same numbers of variables and constraints, there is a big difference in the CPUtime between these two problems as shown in Table 1 and Table 2. The reason for this phenomenon may be explained as follows. The minimax location problem (3) has a convex objective function, although the feasible region is nonconvex. On the other hand, the maximin location problem (7) has a nonconvex objective function in addition to a nonconvex feasible region. Such a problem is extremely difficult to deal with in practice.

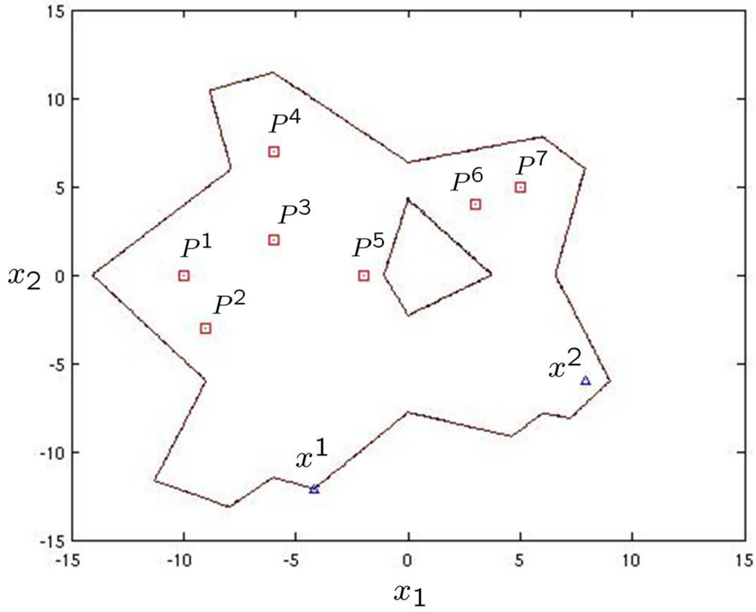

Fig. 5: Solution of Maximin-1

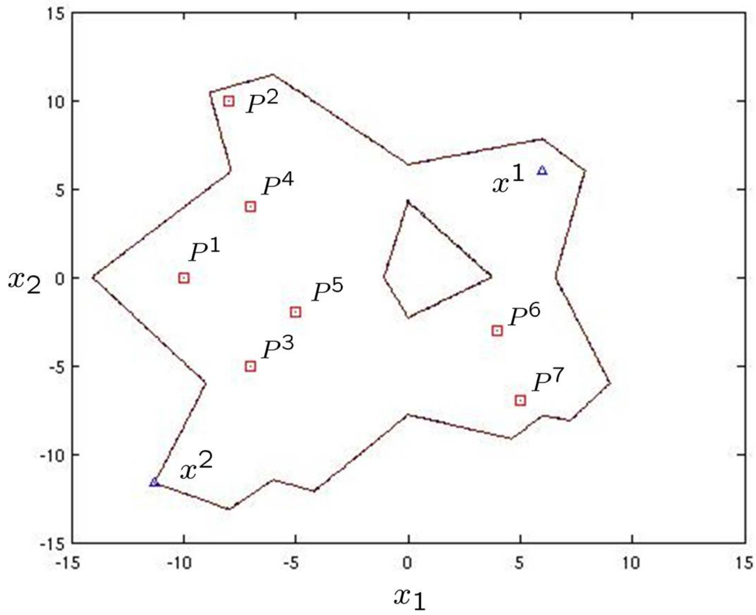

Fig. 6: Solution of Maximin-2

5 Conclusion

In this paper, we have proposed an algorithm for the AVP, which is based on the branch-and-bound method. We have also carried out numerical experiments for nonconvex multi-facility location problems with ℓ_1 norm, which may naturally be reformulated as AVP. The numerical results demonstrate the effectiveness of the proposed algorithm.

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