Master's Thesis

Smoothing Method for Nonlinear Second-Order Cone Programs with Complementarity Constraints and Its Application to the Smart House Scheduling Problem

Guidance

Professor Masao FUKUSHIMA

Keisuke TAKASU

Department of Applied Mathematics and Physics

Graduate School of Informatics

Kyoto University

February 2012

Abstract

In this paper, we consider an optimization problem with complementarity constraints and second-order cone constraints. The mathematical program with complementarity constraints (MPCC) has extensively been studied because MPCC has wide application such as engineering design, traffic equilibrium and game theory. Recently, second-order cone programming has also been studied intensively in relation to robust optimization under uncertainty. To the author's knowledge, however, theoretical and algorithmic results about problems that contain both complementarity and second-order cone constraints have yet to be reported. In this paper, we propose a method for solving nonlinear second-order cone programs with complementarity constraints, which uses a smoothing technique to deal with complementarity constraints, and show its convergence. Moreover, as an application, we formulate a mathematical model of smart house scheduling as a nonlinear second-order cone program with complementarity constraints, and give numerical results with the proposed method.

Contents

1 Introduction

In this paper, we consider an optimization problem with complementarity constraints and second-order cone constraints. Before describing the detail of the main problem, we briefly review two closely related and important problems. The mathematical program with complementarity constraints (MPCC) has extensively been studied because MPCC has wide application such as engineering design, traffic equilibrium and game theory. On the theoretical side, optimality conditions for MPCC have been studied [7],[11],[15]. As for the algorithmic study, various approaches such as smoothing and relaxation methods have been proposed [6],[7],[10],[16]. Recently, second-order cone programming (SOCP) [1],[4] has also been studied intensively in relation to robust optimization under uncertainty in finance and control engineering. For solving linear second-order cone programs (LSOCP), efficiency of interior point method has been well recognized, and fast and practical solvers have been developed. A number of algorithms for nonlinear second-order cone programs (NSOCP) have also been proposed [8],[9]. To the author's knowledge, however, research results on theory and algorithms for problems with both complementarity and second-order cone constraints have yet to be reported in the literature.

In this paper, we consider the following nonlinear second-order cone program with complementarity constraints:

$$
\begin{array}{lll}
P: & \min & f(x,y,z) \\
 & \text{s.t.} & (x,y,z) \in R^n \times R^m \times R^m, \\
 & & 0 \le y \perp z \ge 0, \\
 & & g_j(x,y,z) = 0, \ \ j = 1,\dots,p, \\
 & & h_l(x,y,z) \in K^{q_l}, \ \ l = 1,\dots,s,
\end{array}
$$

where $f : R^{n+2m} \to R$, $g_j : R^{n+2m} \to R$ $(j = 1,\dots,p)$, and $h_l : R^{n+2m} \to R^{q_l}$ $(l = 1,\dots,s)$ are twice continuously differentiable functions. $y \perp z$ means $y^T z = 0$, namely, vectors y and z are perpendicular to each other. The constraints $0 \le y \perp z \ge 0$ are called complementarity constraints. For each l, K^{q_l} denotes the q_l-dimensional second-order cone defined by $K^{q_l} = \left\{(\zeta_0, \bar{\zeta}) \in R \times R^{q_l-1} : \zeta_0 \ge \|\bar{\zeta}\|\right\}$. Thus, $h_l(w) = (h_{l0}(w), \bar{h}_l(w))^T \in K^{q_l}$ means $h_{l0}(w) \ge \|\bar{h}_l(w)\|$, where $\|\cdot\|$ denotes the Euclidean norm. In the rest of this paper, we denote $w = (x,y,z) \in R^N$, where $N = n + 2m$.

It is well known that the complementarity constraints can be rewritten as nonlinear equality constraints. However, it is difficult to deal with such nonlinear equality constraints numerically, because the functions involved in the constraints are non-smooth. To circumvent this difficulty, smoothing methods which approximate those functions by smooth functions have been proposed [6],[7]. In this paper, we propose an algorithm for problem P, which uses the smoothing technique, and show that the algorithm can find a point that satisfies optimality conditions for P under appropriate assumptions.

Moreover, as an application of problem P, we consider a smart house scheduling problem. A smart house is a house that has various energy resources such as fuel cell and solar panel. Smart houses have been attracting much attention in recent years thanks to their enhanced ability to save energy by good operations of their equipments [12]. We consider an optimization problem where the total operation cost of a smart house during a given period is minimized. We call it the smart house scheduling problem. Various modeling and solution methods for smart house scheduling problems have been proposed [14]. Especially, in [13], the problem is formulated as a two-stage stochastic program under the assumption that electricity and heat demands as well as outputs of solar power are random variables. Two-stage stochastic programming is an optimization method that introduces a recourse to compensate for the violation of the constraints, when the constraints involving random variables are not satisfied, and adds its expected value to the objective function. In the conventional model, the unit cost of a recourse is assumed to be given. However, because the smart house has various energy resources, there are different possibilities of choosing the resources of recourse. For this reason, we consider the resources to be uncertain data, and formulate a second-order cone programming model based on robust optimization [2]. Moreover, since complementarity constraints should be considered because of a regulation for purchase and selling of electricity, we formulate the model as a nonlinear second-order cone program with complementarity constraints and solve it by the algorithm proposed in this paper.

The paper is organized as follows. In Section 2, we discuss optimality conditions for problem P. In Section 3, we construct NSOCP as an approximation to problem P by using a smoothing technique, and show that we can find a solution that satisfies optimality conditions for problem P by solving the approximate problem. In Section 4, we describe an algorithm for solving NSOCP. In Section 5, we formulate a mathematical model of the smart house scheduling problem. We report numerical results in Section 6. Finally, in Section 7 we conclude the paper.

2 Optimality conditions for P

In this section, we discuss optimality conditions for nonlinear second-order cone programs with complementarity constraints. Optimality conditions are necessary or sufficient conditions for a feasible point of a nonlinear program to be an optimal solution. In this section, we focus on necessary conditions. The Karush-Kuhn-Tucker (KKT) conditions are well-known necessary conditions under certain constraint qualifications [5]. In general, we want to find a point that satisfies the KKT conditions when we solve nonlinear programs. However, it is difficult to consider the standard optimality conditions for the Mathematical Program with Complementarity Constraints (MPCC), because no feasible point satisfies standard constraint qualifications such as Mangasarian-Fromovitz constraint qualifications [5]. Therefore, we cannot treat standard KKT conditions as optimality conditions in MPCC.

We discuss concepts of a stationary point that satisfies necessary optimality conditions. Various concepts of a stationary point are known for the standard MPCC without second-order cone constraints. Because especially a B-stationary point is defined by the feasible set of a problem, we can define it for MPCC. In this paper, we consider a B-stationary point of the nonlinear second-order cone program with complementarity constrains. A B-stationary point of the standard constrained optimization problem is defined as follows.

Definition 2.1 (B-stationary point) *Let S be a nonempty closed subset of R^N. We call $\bar{w} \in S$ a B-stationary point of the problem*

$$\begin{array}{ll} \min & f(w) \\ \text{s.t.} & w \in S, \end{array}$$

if it satisfies

$$-\nabla f(\bar{w}) \in N_S(\bar{w}),$$

where $N_S(\bar{w})$ is the polar cone of the tangent cone $T_S(\bar{w})$ of S at $\bar{w}$, in other words $N_S(\bar{w})$ is the normal cone of S [5].

In ordinary nonlinear programming, a B-stationary point can be represented algebraically. For instance, if the feasible set S is defined by smooth functions $g_j : R^N \to R$ $(j = 1, \ldots, p)$ as

$$S = \left\{ w \in R^N \,|\, g_j(w) = 0 \;\; (j = 1, \ldots, p) \right\},$$

then, the tangent cone of S at $\bar{w} \in S$ is represented as

$$\left\{v \in R^N \,|\, \nabla g_j(\bar{w})^T v = 0 \; (j = 1, \ldots, p)\right\}.$$

This implies that a B-stationary point is a point $\bar{w} \in S$ such that there are some $\lambda_j \, (j = 1, \ldots, p)$ satisfying $\nabla f(\bar{w}) + \sum_{j=1}^{p} \lambda_j \nabla g_j(\bar{w}) = 0$. As stated above, we cannot discuss standard constraint qualifications in MPCC. Nevertheless, it is known that, through optimality conditions for a relaxed problem associated with the given MPCC, we can characterize the B-stationary point of the original problem [7],[10],[11],[5]. In the following, by extending those results to the problem with second-order cone constraints, we establish optimality conditions of problem P.

In the rest of this section, let $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ be a feasible point of problem P. We define the index sets

$$\bar{\alpha} \equiv \{i \,|\, \bar{y}_i = 0 < \bar{z}_i\}, \;\; \bar{\beta} \equiv \{i \,|\, \bar{y}_i = 0 = \bar{z}_i\}, \;\; \bar{\gamma} \equiv \{i \,|\, y_i > 0 = \bar{z}_i\}.$$

The sets $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ constitute a partition of the index set $\{1, \ldots, m\}$, that is, we have $\bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma} = \{1, \ldots, m\}$.

Associated with a feasible point $\bar{w}$, we define the following relaxed problem of problem P:

$$\begin{array}{lll} R(\bar{w}): & \min & f(x, y, z) \\ & \text{s.t.} & (x, y, z) \in R^n \times R^m \times R^m, \\ & & y_i = 0, \; z_i \geq 0, \; i \in \bar{\alpha}, \\ & & y_i \geq 0, \; z_i \geq 0, \; i \in \bar{\beta}, \\ & & y_i \geq 0, \; z_i = 0, \; i \in \bar{\gamma}, \\ & & g_j(w) = 0, \; j = 1, \ldots, p, \\ & & h_l(w) \in K^{q_l}, \; l = 1, \ldots, s. \end{array}$$

The relaxed problem $R(\bar{w})$ is a nonlinear second-order cone program without complementarity constraints, and its KKT conditions are represented as follows [4]:

$$\nabla_x f(\bar{w}) + \sum_{j=1}^{p} \lambda_j \nabla_x g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_x h_l(\bar{w}) \nu_l = 0, \tag{1}$$

$$\nabla_y f(\bar{w}) - \xi + \sum_{j=1}^{p} \lambda_j \nabla_y g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_y h_l(\bar{w}) \nu_l = 0, \tag{2}$$

$$\nabla_z f(\bar{w}) - \eta + \sum_{j=1}^{p} \lambda_j \nabla_z g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_z h_l(\bar{w}) \nu_l = 0, \tag{3}$$

$$\xi_i = 0, \;\; \forall i \in \bar{\gamma}, \tag{4}$$

$$\xi_i \geq 0, \; \eta_i \geq 0, \;\; i \in \bar{\beta}, \tag{5}$$

$$\eta_i = 0, \;\; i \in \bar{\alpha}, \tag{6}$$

$$g_j(\bar{w}) = 0, \;\; j = 1, \ldots, p, \tag{7}$$

$$\nu_l \in K^{q_l}, \;\; h_l(\bar{w}) \in K^{q_l}, \;\; \nu_l^T h_l(\bar{w}) = 0, \;\; l = 1, \ldots, s, \tag{8}$$

where $\xi \in R^m, \eta \in R^m, \lambda \in R^p, \nu \in R^q$ are Lagrange multiplier vectors. The KKT conditions are necessary optimality conditions under certain constraint qualifications, namely, if $\bar{w}$ is a local optimal solution, then, under certain constraint qualifications, there are Lagrange multiplier vectors $\xi \in R^m, \eta \in R^m, \lambda \in R^p, \nu \in R^q$ that satisfy the KKT conditions (1)–(8).

In the discussion of optimality conditions for the standard MPCC without second-order cone constraints, the MPEC-linearly independent constraint qualification (MPEC-LICQ) plays an important role.

For example, under the MPEC-LICQ, equivalence between the KKT conditions of the relaxed problem and the B-stationarity of the original MPCC is guaranteed. The following MPEC-nondegenerate constraint qualification (MPEC-NCQ) is an extension of the MPEC-LICQ to the problem with second-order cone constraints.

Definition 2.2 (MPEC-NCQ) *Let $\bar{w}$ be a feasible point of problem P. We say that MPEC-nondegenerate constraint qualification (MPEC-NCQ) holds at $\bar{w}$ if the vectors*

$$\begin{array}{ll} e_{n+i}, & \forall i \in \bar{\alpha} \cup \bar{\beta}, \\ e_{n+m+i}, & \forall i \in \bar{\beta} \cup \bar{\gamma}, \\ \nabla g_j(\bar{w}), & j = 1, \ldots, p, \\ \nabla h_l(\bar{w}) \nabla \psi_l(h_l(\bar{w})), & l = 1, \ldots, s, \end{array}$$

are linearly independent. Here, $e_k \in R^N$ is the k-th unit vector.

Given a vector $t_l \in K^{q_l}$ for each $l = 1, \ldots, s$, the function ψ_l is defined as follows:

$$\begin{aligned} &\psi_l : R^{q_l} \to R^{q_l}, \ \psi_l(t_l) := t_l, \quad \text{if} \ \ t_l = 0, \\ &\psi_l : R^{q_l} \to \emptyset, \ \psi_l(t_l) := \emptyset, \quad \text{if} \ \ t_{l0} > \|\bar{t}_l\|, \\ &\psi_l : R^{q_l} \to R^1, \ \psi_l(t_l) := \|\bar{t}_l\| - t_{l0}, \quad \text{if} \ \ t_{l0} = \|\bar{t}_l\| \neq 0. \end{aligned}$$

Thus, in the definition of the MPEC-NCQ, the function $\nabla h_l(w) \nabla \psi_l(h_l(w))$ is represented as follows:

$$\begin{aligned} &\nabla h_l(w) \nabla \psi_l(h_l(w)) = \nabla h_l(w), \quad \text{if} \ \ h_l(w) = 0, \\ &\nabla h_l(w) \nabla \psi_l(h_l(w)) = -\nabla h_{l0}(w) + \frac{\nabla \bar{h}_l(w) \bar{h}_l(w)}{\|\bar{h}_l(w)\|}, \quad \text{if} \ \ h_{l0}(w) = \|\bar{h}_l(w)\| \neq 0, \\ &\nabla h_l(w) \nabla \psi_l(h_l(w)) = \emptyset, \quad \text{if} \ \ h_{l0}(w) > \|\bar{h}_l(w)\|. \end{aligned}$$

Note that the MPEC-NCQ is a nondegeneracy condition [4] for the relaxed problem $R(\bar{w})$. Thus, under the MPEC-NCQ, the set of Lagrange multipliers satisfying the KKT conditions (1)–(8) is a singleton.

The following theorem says the KKT conditions of nonlinear second-order cone problem $R(\bar{w})$ enable us to characterize a B-stationary point of problem P.

Theorem 2.3 *Let $\bar{w}$ be a feasible point of problem P. If the KKT conditions (1)–(8) of the relaxed problem $R(\bar{w})$ are satisfied at $\bar{w}$, then $\bar{w}$ is a B-stationary point of problem P. Furthermore, if MPEC-NCQ holds at $\bar{w}$, then $\bar{w}$ satisfies the KKT conditions (1)–(8) of the relaxed problem $R(\bar{w})$ if and only if $\bar{w}$ is a B-stationary point of problem P.*

Proof. Associated with an arbitrary partition $(\bar{\beta}_1, \bar{\beta}_2)$ of $\bar{\beta}$, we define the restricted problem of problem P as follows:

$$\begin{array}{lll} Q(\bar{\beta}_1, \bar{\beta}_2): & \min & f(x,y,z) \\ & \text{s.t.} & (x,y,z) \in R^n \times R^m \times R^m, \\ & & y_i = 0, \ z_i \geq 0, \ \ (i \in \bar{\alpha}), \\ & & y_i = 0, \ z_i \geq 0, \ \ (i \in \bar{\beta}_1), \\ & & y_i \geq 0, \ z_i = 0, \ \ (i \in \bar{\beta}_2), \\ & & y_i \geq 0, \ z_i = 0, \ \ (i \in \bar{\gamma}), \\ & & g_j(x,y,z) = 0, \ \ j = 1, \ldots, p, \\ & & h_l(x,y,z) \in K^{q_l}, \ \ l = 1, \ldots, s. \end{array}$$

Problem $Q(\bar{\beta}_1, \bar{\beta}_2)$ is a standard nonlinear second-order cone problem without complementarity constraints. The set of all partitions of the index set $\bar{\beta}$ is denoted as

$$\mathcal{P}(\bar{\beta}) = \left\{ (\bar{\beta}_1, \bar{\beta}_2) \mid \bar{\beta}_1 \cup \bar{\beta}_2 = \bar{\beta},\ \bar{\beta}_1 \cap \bar{\beta}_2 = \emptyset \right\}.$$

Let S be the feasible set of problem P, and $S(\bar{\beta}_1, \bar{\beta}_2)$ be the feasible set of problem $Q(\bar{\beta}_1, \bar{\beta}_2)$. For any $r > 0$ small enough, we have

$$S \cap B(\bar{w}, r) = \bigcup_{(\bar{\beta}_1, \bar{\beta}_2) \in \mathcal{P}(\bar{\beta})} \left(S(\bar{\beta}_1, \bar{\beta}_2) \cap B(\bar{w}, r) \right),$$

where $B(v, r) = \left\{ w \in R^N \mid \|w - v\| \le r \right\}$ is the ball with center $v \in R^N$ and radius $r > 0$. Since the tangent cone is formed on the basis of local information around the point under consideration, we have

$$T_S(\bar{w}) = \bigcup_{(\bar{\beta}_1, \bar{\beta}_2) \in \mathcal{P}(\bar{\beta})} T_{S(\bar{\beta}_1, \bar{\beta}_2)}(\bar{w}).$$

On the other hand, let S_R be the feasible set of the relaxed problem $R(\bar{w})$. Then, for any $r > 0$ small enough, we have

$$S \cap B(\bar{w}, r) \subseteq S_R \cap B(\bar{w}, r).$$

Since

$$\bigcup_{(\bar{\beta}_1, \bar{\beta}_2) \in \mathcal{P}(\bar{\beta})} T_{S(\bar{\beta}_1, \bar{\beta}_2)}(\bar{w}) = T_S(\bar{w}) \subseteq T_{S_R}(\bar{w}),$$

we have the following relation among their normal cones:

$$N_{S_R}(\bar{w}) \subseteq N_S(\bar{w}) = \bigcap_{(\bar{\beta}_1, \bar{\beta}_2) \in P(\bar{\beta})} N_{S(\beta_1, \bar{\beta}_2)}(\bar{w}). \tag{9}$$

Thus, if $-\nabla f(\bar{w}) \in N_{S_R}(\bar{w})$, then we have $-\nabla f(\bar{w}) \in N_S(\bar{w})$. In other words, if $\bar{w}$ is a B-stationary point of the relaxed problem $R(\bar{w})$, then $\bar{w}$ is a B-stationary point of problem P. Since problem $R(\bar{w})$ is a standard nonlinear second-order cone program, if $\bar{w}$ is a KKT point of problem $R(\bar{w})$, then $\bar{w}$ is a B-stationary point of problem $R(\bar{w})$, and hence a B-stationary point of problem P. This shows the first half of the theorem.

From (9), $\bar{w}$ is a B-stationary point of the restricted problem $Q(\bar{\beta}_1, \bar{\beta}_2)$ for an arbitrary partition of $\bar{\beta}$ if and only if $\bar{w}$ is a B-stationary point of problem P. Besides, under the MPEC-NCQ, $\bar{w}$ is a KKT point and also a B-stationary point of the relaxed problem $R(\bar{w})$. Therefore, to show the latter half of the theorem, it suffices to show that $\bar{w}$ satisfies the KKT conditions of the restricted problem $Q(\bar{\beta}_1, \bar{\beta}_2)$ for an arbitrary partition of $\bar{\beta}$ if and only if $\bar{w}$ satisfies the KKT conditions of the relaxed problem $R(\bar{w})$. Here, $\bar{w}$ is said to satisfy the KKT conditions of problem $Q(\bar{\beta}_1, \bar{\beta}_2)$, if there are Lagrange multipliers

$(\xi, \eta, \lambda, \nu)$ satisfying

$$\nabla_x f(\bar{w}) + \sum_{j=1}^{p} \lambda_j \nabla_x g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_x h_l(\bar{w}) \nu_l = 0, \tag{10}$$

$$\nabla_y f(\bar{w}) - \xi + \sum_{j=1}^{p} \lambda_j \nabla_y g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_y h_l(\bar{w}) \nu_l = 0, \tag{11}$$

$$\nabla_z f(\bar{w}) - \eta + \sum_{j=1}^{p} \lambda_j \nabla_z g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_z h_l(\bar{w}) \nu_l = 0, \tag{12}$$

$$\xi_i = 0, \ \forall i \in \bar{\gamma}, \tag{13}$$

$$\xi_i \geq 0, \ \forall i \in \bar{\beta}_2, \tag{14}$$

$$\eta_i \geq 0, \ \forall i \in \bar{\beta}_1, \tag{15}$$

$$\eta_i = 0, \ \forall i \in \bar{\alpha}, \tag{16}$$

$$g_j(\bar{w}) = 0, \ j = 1, \ldots, p, \tag{17}$$

$$\nu_l \in K^{q_l}, \ h_l(\bar{w}) \in K^{q_l}, \ \nu_l^T h_l(\bar{w}) = 0, \ l = 1, \ldots, s. \tag{18}$$

It is clear from the comparison of the KKT conditions (1)–(8) with (10)–(18), that if $\bar{w}$ is a KKT point of problem $R(\bar{w})$, then for any $(\bar{\beta}_1, \bar{\beta}_2)$, $\bar{w}$ satisfies the KKT conditions of problem $Q(\bar{\beta}_1, \bar{\beta}_2)$. Conversely, let $\bar{w}$ satisfy the KKT conditions of problem $Q(\bar{\beta}_1, \bar{\beta}_2)$ for any $(\bar{\beta}_1, \bar{\beta}_2)$. Now, since the MPEC-NCQ holds, the set of Lagrange multipliers satisfying (10)–(12) and (18) is a singlton. Since the Lagrange multipliers $(\bar{w}, \bar{\xi}, \bar{\eta}, \bar{\lambda}, \bar{\nu})$ satisfy the KKT conditions (10)–(18) for any $(\bar{\beta}_1, \bar{\beta}_2)$, we have $\bar{\xi}_i, \bar{\eta}_i \geq 0$ for all $i \in \bar{\beta}$. Therefore, $\bar{w}$ together with the Lagrange multipliers $(\bar{\xi}, \bar{\eta}, \bar{\lambda}, \bar{\nu})$ satisfies the KKT conditions (1)–(8) of problem $R(\bar{w})$. □

3 Smoothing method

In the previous section, we established the optimality condition of problem P through its relaxed problem. In this section, we describe a smoothing technique for solving problem P. It is well known that the complementarity constraints can be rewritten as nonlinear equality constraints. However, it is difficult to deal with such nonlinear equality constraints numerically, because the functions involved in the constraints are non-smooth. To circumvent this difficulty, by using the smoothing technique, we approximate problem P by a smooth nonlinear second-order cone program, and argue the relation between the approximate problem and the original problem.

The Fischer-Burmeister function $\phi : \ R^2 \to R$ is defined by

$$\phi(a, b) \equiv a + b - \sqrt{a^2 + b^2}.$$

The function ϕ has the property

$$\phi(a, b) = 0 \iff a \geq 0, \ b \geq 0, \ ab = 0.$$

With the Fisher-Brumeister function ϕ, we define the function $\Phi : R^m \times R^m \to R^m$ by

$$\Phi(y, z) \equiv (\phi(y_1, z_1), \ldots, \phi(y_m, z_m))^T.$$

Then, by rewriting the complementarity constraints as the nonlinear equality constraints $\Phi(y,z)=0$, we can transform problem P to the following nonlinear second-order cone program:

$$
\begin{array}{lll}
N: & \min & f(x,y,z) \\
& \text{s.t.} & (x,y,z) \in R^n \times R^m \times R^m, \\
& & \Phi(y,z) = \mathbf{0}, \\
& & g_j(x,y,z) = 0, \ \ j = 1,\dots,p, \\
& & h_l(x,y,z) \in K^{q_l}, \ \ l = 1,\dots,s.
\end{array}
$$

However, since the function $\phi(a,b)$ is non-smooth at $(a,b) = (0,0)$, we cannot apply gradient-based algorithms directly to the above problem. To circumvent this difficulty, we use the smoothing Fischer-Brumeister function [6] defined by

$$\phi^\epsilon(a,b) = a + b - \sqrt{a^2+b^2+\epsilon},$$

where $\epsilon \geq 0$. For any $\epsilon > 0$, t the function ϕ^ϵ is smooth everywhere, and its first derivatives are given by

$$\frac{\partial \phi^\epsilon(a,b)}{\partial a} = 1 - \frac{a}{\sqrt{a^2+b^2+\epsilon}}, \tag{19}$$

$$\frac{\partial \phi^\epsilon(a,b)}{\partial b} = 1 - \frac{b}{\sqrt{a^2+b^2+\epsilon}}. \tag{20}$$

Furthermore, the function ϕ^ϵ has the following property:

$$\phi^\epsilon(a,b) = 0 \iff \begin{cases} a \geq 0, \ b \geq 0, \ ab = 0 & \text{if } \ \epsilon = 0, \\ a > 0, \ b > 0, \ ab = \epsilon/2 & \text{if } \ \epsilon > 0. \end{cases}$$

With the function ϕ^ϵ, we define the function $\Phi^\epsilon : R^m \times R^m \to R^m$ by

$$\Phi^\epsilon(y,z) \equiv \left(\phi^\epsilon(y_1,z_1),\dots,\phi^\epsilon(y_m,z_m)\right)^T,$$

and consider the following approximate problem $N(\epsilon)$ to problem P:

$$
\begin{array}{lll}
N(\epsilon): & \min & f(x,y,z) \\
& \text{s.t.} & (x,y,z) \in R^n \times R^m \times R^m, \\
& & \Phi^\epsilon(y,z) = \mathbf{0}, \\
& & g_j(x,y,z) = 0, \ \ j = 1,\dots,p, \\
& & h_l(x,y,z) \in K^{q_l}, \ \ l = 1,\dots,s.
\end{array}
$$

As mentioned above, problem $N(0)$ is equivalent to P. Thus, we solve a sequence of approximate problems $N(\epsilon)$ with the smoothing parameter $\epsilon > 0$ approaching zero.

In the following, we write $\Phi^\epsilon(w) \equiv \Phi^\epsilon(y,z)$ for simplicity. From (19) and (20), the gradient of each component function $\Phi^\epsilon_i(w) \equiv \phi^\epsilon(y_i,z_i)$ is represented as

$$\nabla \Phi^\epsilon_i(w) = \left(1 - \frac{y_i}{\sqrt{(y_i)^2+(z_i)^2+\epsilon}}\right) e_{n+i} + \left(1 - \frac{z_i}{\sqrt{(y_i)^2+(z_i)^2+\epsilon}}\right) e_{n+m+i}. \tag{21}$$

When $\epsilon > 0$, problem $N(\epsilon)$ is a smooth nonlinear second-order cone program, and its KKT conditions are written as

$$\nabla_x f(w) + \sum_{j=1}^{p} \lambda_j \nabla_x g_j(w) - \sum_{l=1}^{s} \nabla_x h_l(w)\nu_l = 0, \tag{22}$$

$$\nabla_y f(w) - \sum_{i=1}^{m} \mu_i \frac{\partial \phi_i^\epsilon}{\partial a} e_i^{(m)} + \sum_{j=1}^{p} \lambda_j \nabla_y g_j(w) - \sum_{l=1}^{s} \nabla_y h_l(w)\nu_l = 0, \tag{23}$$

$$\nabla_z f(w) - \sum_{i=1}^{m} \mu_i \frac{\partial \phi_i^\epsilon}{\partial b} e_i^{(m)} + \sum_{j=1}^{p} \lambda_j \nabla_z g_j(w) - \sum_{l=1}^{s} \nabla_z h_l(w)\nu_l = 0, \tag{24}$$

$$\Phi^\epsilon(w) = 0, \tag{25}$$

$$g_j(w) = 0, \quad j = 1, \dots, p, \tag{26}$$

$$\nu_l \in K^{q_l}, \;\; h_l(w) \in K^{q_l}, \;\; \nu_l^T h_l(w) = 0, \quad l = 1, \dots, s, \tag{27}$$

where (μ, λ, ν) is the Lagrange multiplier vector, $e_k^{(m)} \in R^m$ is k-th unit vector, and

$$\frac{\partial \phi_i^\epsilon}{\partial a} \equiv \frac{\partial \phi^\epsilon}{\partial a}(y_i, z_i), \qquad \frac{\partial \phi_i^\epsilon}{\partial b} \equiv \frac{\partial \phi^\epsilon}{\partial b}(y_i, z_i).$$

In the following, we write $\pi \equiv (\mu, \lambda, \nu)$.

In MPCC, the following condition is often assumed.

Definition 3.1 (Strict complementarity) *A feasible point $\bar{w}$ of problem P is said to be strictly complementary if $\bar{\beta} = \emptyset$.*

Since the function $\phi^\epsilon(a, b)$ is smooth at any point $(a, b) \neq (0, 0)$ even if $\epsilon = 0$, under the strict complementarity assumption, we can discuss local optimality conditions of $N(0)$, or equivalently P, by ignoring complementarity constraints $y_i z_i = 0$. Then, the optimality condition is equivalent to the KKT conditions for the standard nonlinear second-order cone program, and so we can argue theoretical convergence of algorithms in a similar manner to the latter problem. However, strict complementarity is often not satisfied in practice. Therefore we need to consider a weaker condition.

Definition 3.2 (Asymptotic weak nondegeneracy [7]) *Suppose that $\epsilon_k \to 0$ and $w^k \to \bar{w}$ as $k \to \infty$. For each $i \in \bar{\beta}$, any accumulation point $\bar{r}_i$ of $\left\{\nabla \Phi_i^{\epsilon_k}(w^k)\right\}$ is represented as*

$$\bar{r}_i = \bar{u}_i e_{n+i} + \bar{v}_i e_{n+m+i},$$

with $\bar{u}_i, \bar{v}_i \in R$ such that $(1 - \bar{u}_i)^2 + (1 - \bar{v}_i)^2 \leq 1$. If for all $i \in \bar{\beta}$, neither $\bar{u}_i$ nor $\bar{v}_i$ is equal to zero at any accumulation point $\bar{r}_i$, we say that $\left\{w^k\right\}$ is asymptotically weakly nondegenerate.

This condition means that $(1 - \frac{y_i^k}{\sqrt{(y_i^k)^2 + (z_i^k)^2 + \epsilon_k}})$ and $(1 - \frac{z_i^k}{\sqrt{(y_i^k)^2 + (z_i^k)^2 + \epsilon_k}})$ in (21) converge to $\bar{u}_i$ and $\bar{v}_i$, respectively, and neither of them is not equal to zero. In other words, y_i^k and z_i^k approach zero in the same order of magnitude. Besides, if $\bar{w}$ is strictly complementary, namely $\bar{\beta} = \emptyset$, then the asymptotic weak nondegeneracy is guaranteed automatically.

We define the Lagrangian for problem $N(\epsilon)$ by

$$L^\epsilon(w, \pi) := f(w) - \Phi^\epsilon(w)^T \mu - g(w)^T \lambda - h(w)^T \nu.$$

When $\epsilon > 0$, since problem $N(\epsilon)$ is a smooth nonlinear second-order cone program, its second-order optimality condition is defined as follows [4].

Definition 3.3 (second-order sufficient condition) *Suppose $\epsilon > 0$. We say that (w, π) satisfies the second-order sufficient condition if (w, π) satisfies the KKT conditions (22)–(27) of problem $N(\epsilon)$ and the following condition:*

$$d^T \left(\nabla_w^2 L^\epsilon(w, \pi) + \sum_{l=1}^{s} H_l(w, \pi) \right) d \geq 0, \quad \forall d \in C^\epsilon,$$

where

$$C^\epsilon := \left\{ d \in R^N \;\middle|\; \begin{array}{ll} \nabla \Phi^\epsilon(w)^T d = 0 & \\ \nabla g_j(w)^T d = 0, & j = 1, \ldots, p \\ \nabla h_l(w)^T d = 0, & l : \nu_{l0} > \bar{\nu}_l \\ \nabla h_l(w)^T d \in span\{R_l \nu_l\}, & l : \nu_{l0} = \|\bar{\nu}_l\| \neq 0,\; h_l(w) = 0 \\ d^T \nabla h_l(w) \nu_l = 0, & l : \nu_{l0} = \|\bar{\nu}_l\| \neq 0,\; h_{l0}(w) = \|\bar{h}_l(w)\| \neq 0 \end{array} \right\},$$

$$H_l(w, \pi) := \begin{cases} -\frac{\nu_{l0}}{h_{l0}(w)} \nabla h_l(w) R_l \nabla h_l(w)^T & \text{if } h_{l0}(w) = \|\bar{h}_l(w)\| \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $R_l := \begin{pmatrix} 1 & 0^T \\ 0 & -I_{q_l - 1} \end{pmatrix} \in R^{q_l \times q_l}$.

Finally, we show that, by solving a sequence of approximate problems $N(\epsilon_k)$ and letting $\epsilon_k \to 0$, we obtain a B-stationary point of problem P.

Theorem 3.4 *Suppose $\epsilon_k \to 0$ as $k \to \infty$. For each k, let (w^k, π^k) satisfy the KKT conditions (22)–(27) and the second-order sufficient condition for problem $N(\epsilon_k)$. If $w^k \to \bar{w}$, the MPEC-NCQ holds at $\bar{w}$, and $\{w^k\}$ is asymptotically weakly nondegenerate, then $\bar{w}$ satisfies the KKT conditions of the relaxed problem $R(\bar{w})$, namely, $\bar{w}$ is a B-stationary point of problem P.*

Proof. Let $\bar{r}_i$ be an arbitrary accumulation point of $\{\nabla \Phi_i^{\epsilon_k}(w^k)\}$ for each $i \in \bar{\beta}$. Since $\{w^k\}$ is asymptotically weakly nondegenerate, for all $i \in \bar{\beta}$, there exist $\bar{u}_i > 0$ and $\bar{v}_i > 0$ that satisfy $(1 - \bar{u}_i)^2 + (1 - \bar{v}_i)^2 \leq 1$ and

$$\bar{r}_i = \bar{u}_i e_{n+i} + \bar{v}_i e_{n+m+i}.$$

Besides, since we have

$$\lim_{k \to \infty} \nabla \Phi_i^{\epsilon_k}(w^k) = e_{n+i} \qquad \forall i \in \bar{\alpha},$$
$$\lim_{k \to \infty} \nabla \Phi_i^{\epsilon_k}(w^k) = e_{n+m+i} \quad \forall i \in \bar{\gamma},$$

letting (w^k, π^k) be a KKT point of problem $N(\epsilon_k)$ and taking the limit yield

$$\nabla_x f(\bar{w}) + \sum_{j=1}^{p} \bar{\lambda}_j \nabla_x g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_x h_l(\bar{w}) \nu_l = 0,$$

$$\nabla_y f(\bar{w}) - \sum_{i \in \bar{\alpha}} \bar{\mu}_i e_i^{(m)} - \sum_{i \in \bar{\beta}} \bar{\mu}_i \bar{u}_i e_i^{(m)} + \sum_{j=1}^{p} \bar{\lambda}_j \nabla_y g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_y h_l(\bar{w}) \bar{\nu}_l = 0,$$

$$\nabla_z f(\bar{w}) - \sum_{i \in \bar{\gamma}} \bar{\mu}_i e_i^{(m)} - \sum_{i \in \bar{\beta}} \bar{\mu}_i \bar{v}_i e_i^{(m)} + \sum_{j=1}^{p} \bar{\lambda}_j \nabla_z g_j(\bar{w}) - \sum_{l=1}^{s} \nabla_z h_l(\bar{w}) \bar{\nu}_l = 0.$$

Now, we want to show that

$$\bar{\xi}_i \equiv \bar{\mu}_i \bar{u}_i \geq 0, \;\; \forall i \in \bar{\beta}, \tag{28}$$

$$\bar{\eta}_i \equiv \bar{\mu}_i \bar{v}_i \geq 0, \;\; \forall i \in \bar{\beta}, \tag{29}$$

which implies that the accumulation point $\bar{w}$ satisfies the KKT conditions (1)–(8) of problem $R(\bar{w})$. For the purpose of contradiction, let us assume that $\bar{\xi}_{i'} < 0$ for some $i' \in \bar{\beta}$. Since $\bar{u}_{i'}, \bar{v}_{i'} > 0$, it means $\bar{\mu}_{i'} < 0$, which in turn implies $\bar{\eta}_{i'} < 0$. We choose a vector $d^k \in R^N$ with $\|d^k\| = 1$ such that

$$\begin{array}{ll}
(d^k)_{n+i} = 0, & i \in \bar{\alpha} \cup \bar{\beta} \backslash \{i'\}, \\
(d^k)_{n+m+i} = 0, & i \in \bar{\gamma} \cup \bar{\beta} \backslash \{i'\}, \\
(d^k)_{n+i'} = \frac{\partial}{\partial b}\phi^{\epsilon_k}(y^k_{i'}, z^k_{i'}), & \\
(d^k)_{n+m+i'} = -\frac{\partial}{\partial a}\phi^{\epsilon_k}(y^k_{i'}, z^k_{i'}), & \\
\nabla g_j(w^k)^T d^k = 0, & j = 1, \ldots, p, \\
\nabla h_l(w^k)^T d^k = 0, & l : h_l(w^k) = 0, \\
\left(-\nabla h_{l0}(w^k) + \frac{\nabla \bar{h}_l(w^k)\bar{h}_l(w^k)}{\|\bar{h}_l(w^k)\|}\right)^T d^k = 0, & l : h_{l0}(w^k) = \|\bar{h}_l(w^k)\| \neq 0.
\end{array}$$

Since the MPEC-NCQ holds at $\bar{w}$, such choice of d^k is guaranteed for sufficiently large k. Moreover we have $d^k \in C^\epsilon$. The rest of the proof is analogous to that of Theorem 3.1 in [7], and we have (28) and (29). Therefore, $\bar{w}$ is a KKT point of problem $R(\bar{w})$, and hence, $\bar{w}$ is a B-stationary point of problem P by Theorem 2.3. $\square$

If $\bar{w}$ is strictly complementary, namely $\bar{\beta} = \emptyset$, we need not consider the index $i \in \bar{\beta}$ in the proof of Theorem 3.4. Therefore, the result of the theorem holds without the second-order sufficient condition and the MPEC-NCQ.

Based on the results of Theorem 3.4, we consider the following algorithm for problem P that solves problem $N(\epsilon_k)$ approximately at each iteration while letting $\epsilon_k \to 0$.

Algorithm 3.5
Step 0. Choose $w^0 = (x^0, y^0, z^0)$, $\delta \in (0,1)$, $\epsilon_0 > 0$ and set $k := 0$.
Step 1. Find an approximate solution w^k of problem $N(\epsilon_k)$.
Step 2. If ϵ_k is sufficiently small, stop. Otherwise, set $\epsilon_{k+1} := \delta\epsilon_k$, $k := k+1$, and go to Step 1.

4 Algorithm for nonlinear second-order cone programs

From Theorem 3.4, we need to solve a nonlinear second-order cone program at every iteration of the smoothing method,. In this section, we describe an SQP-type algorithm [9] for solving problem $N(\epsilon)$.

The SQP-type algorithm solves the following subproblem at every iteration:

$$\begin{array}{lll}
QP(v^\tau): & \min & \nabla f(v^\tau)^T d + \frac{1}{2} d^T M_\tau d \\
& \text{s.t.} & \Phi^\epsilon(v^\tau) + \nabla \Phi^\epsilon(v^\tau)^T d = 0, \\
& & g(v^\tau) + \nabla g(v^\tau)^T d = 0, \\
& & h(v^\tau) + \nabla h(v^\tau)^T d \in K,
\end{array}$$

where v^τ is a current iterate and M_τ is a symmetric positive definite matrix. Problem $QP(v^\tau)$ is a strictly convex quadratic program with second-order cone constraints. Since we can transform problem $QP(v^\tau)$ into a linear second-order cone program, we can use fast and practical solvers such as SDPT3 [17]. The next proposition is established in relation to $QP(v^\tau)$ [9].

Proposition 4.1 *Under a certain constraint qualification, v^τ satisfies the KKT conditions of problem $N(\epsilon)$ if and only if d^τ is 0.*

We formally state the algorithm as follows.

Algorithm 4.2 (SQP-type)

Step 0. Choose v^0, $\sigma \in (0,1)$, $\rho_{-1}, \kappa > 0$. Set $\tau := 0$.

Step 1. Choose a symmetric positive definite matrix M_τ. Find the optimal solution d^τ and corresponding Lagrange multiplier u^τ of $QP(v^\tau)$. If $d^\tau = 0$, then stop. Otherwise, go to Step 2.

Step 2. Let $\rho_\tau := \max\{\|u^\tau\|_\infty + \kappa, \rho_{\tau-1}\}$, and define a penalty function by

$$P_\tau(v) \equiv f(v) + \rho_\tau \left(\sum_{i=1}^{m} |\Phi_i^\epsilon(v)| + \sum_{j=1}^{p} |g_j(v)| + \sum_{l=1}^{s} \max\left\{0, -(h_{l0}(v) - \|\bar{h}_l(v)\|)\right\} \right).$$

Using Armijo rule, choose a step size $\alpha_\tau > 0$ that satisfies

$$P_\tau(v^\tau) - P_\tau(v^\tau + \alpha_\tau d^\tau) \geq \sigma \alpha_\tau d^{\tau T} M_\tau d^\tau. \tag{30}$$

Step 3. Let $v^{\tau+1} := v^\tau + \alpha_\tau d^\tau$, $\tau := \tau + 1$. Go to Step 1.

In practice, we stop the algorithm in Step 1, if $\|d^\tau\| < TOL$ for a sufficiently small $TOL > 0$. We make the following assumptions.

Assumption 4.3
(A1) For each τ, subproblem $QP(v^\tau)$ has a solution.
(A2) The generated sequence $\left\{(v^{\tau+1}, u^\tau)\right\}$ is bounded.
(A3) There exist some positive constants γ and Γ that satisfy

$$\gamma\|d\|^2 \leq d^T M_\tau d \leq \Gamma\|d\|^2,\ \forall d \in R^{n+2m},\ \forall \tau \in \{0,1,2,\ldots\}.$$

The next theorem establishes global convergence of Algorithm 4.2 [9].

Theorem 4.4 *Suppose that assumptions (A1)–(A3) hold. Any accumulation point of the sequence generated by Algorithm 4.2 is a KKT point of problem $N(\epsilon)$.*

5 Smart house scheduling problem

In this section, we formulate the smart house scheduling problem as a nonlinear second-order cone program with complementarity constraints. The problem is to minimize the total operation cost of a smart house during a given period under the constraints on electrical devices, energy balances and electricity trading. The scheduling period is divided into H subperiods of equal length and the h-th subperiod ($h = 1, 2, \ldots, H$) is simply called period h. The set of periods is denoted as $\mathcal{H} = \{1, 2, \ldots, H\}$.

As shown in Figure 1, the smart house treated in this section is composed of four devices; a solar panel, a fuel cell, a heat storage tank and a gas boiler. The solar panel converts solar energy into electric energy. The fuel cell is a device which makes the town gas react to generate electricity and heat. We can store heat as hot water in the heat storage tank. The gas boiler also produces hot water by burning the town gas purchased from a gas company. In this model, we can buy electricity from an electricity company. Moreover, we are allowed to sell surplus electricity generated by the solar panel. However, because of a regulation, we cannot sell electricity when we buy it, namely at least one of $z^-(h)$ and $z^+(h)$ must be equal to 0. Because of this fact, we must consider complementarity constraints.

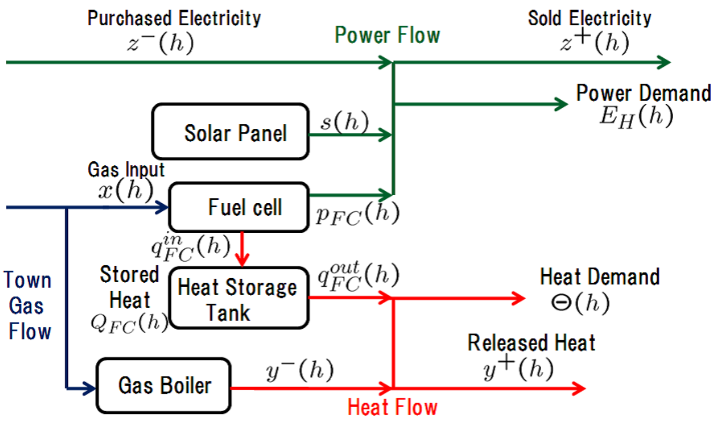

Figure 1: Model of smart house

5.1 Mathematical model of smart house

5.1.1 Constraints on fuel cell

The mathematical model described in this section is based on the previous work [13].

Let $x(h)$ denote the gas consumed by the fuel cell during period h. The variable $x(h)$ takes a value between 0 and a given upper bound $x_{\max}$, that is

$$0 \le x(h) \le x_{\max}, \quad \forall h \in \mathcal{H}. \tag{31}$$

We consider the relation between the gas consumption $x(h)$ and the power output $p_{FC}(h)$ of the fuel cell. Because of an inherent property of fuel cell, it cannot generate a good amount of electricity with a small quantity of gas consumption. Thus, we use a function such that $p_{FC}(h) = 0$ when $x(h) = 0$, and approaches asymptotically the linear function $p_{FC}(h) = a(x(h) - \alpha)$ as $x(h)$ increases. An example of such functions is given by

$$p_{FC}(h) = a\alpha \left(\sqrt{\frac{x(h)^2}{\alpha^2} + 1} - 1 \right), \quad \forall h \in \mathcal{H}. \tag{32}$$

Similarly, the relation between the gas consumption $x(h)$ and the hot water output $q_{FC}^{in}(h)$ of the fuel cell is represented as

$$q_{FC}^{in}(h) = b\beta \left(\sqrt{\frac{x(h)^2}{\beta^2} + 1} - 1 \right), \quad \forall h \in \mathcal{H}. \tag{33}$$

Fig. 2 shows the function given by (32) with a=1.254 and α =13.66411. In Figure 2, the solid line represents the function $p_{FC}(h) = a\alpha \left(\sqrt{\frac{x(h)^2}{\alpha^2} + 1} - 1 \right)$ and the broken line represents the function $p_{FC}(h) = a(x(h) - \alpha)$.

Let $Q_{FC}(h)$ and $Q_{FC}(H+1)$ denote the stored heat in the heat storage tank during period h and at the end of the scheduling period, respectively. The variable $Q_{FC}(h)$ takes a value between 0 and a given upper bound $Q_{\max}^{FC}$, that is,

$$0 \le Q_{FC}(h) \le Q_{\max}^{FC}, \quad h = 1, 2, \ldots, H+1. \tag{34}$$

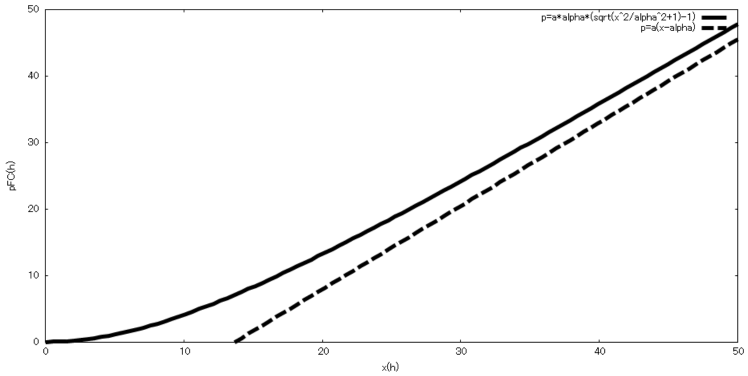

Figure 2: Relation between $x(h)$ and $p_{FC}(h)$

Let $q_{FC}^{in}(h)$ denote the hot water output from the fuel cell and $q_{FC}^{out}(h)$ denote the hot water output from the heat storage tank. Then we have the following recurrence relation:

$$Q_{FC}(h+1) = Q_0^{FC} + \sum_{k=1}^{h} q_{FC}^{in}(k) - \sum_{k=1}^{h} q_{FC}^{out}(k), \quad \forall h \in \mathcal{H}, \tag{35}$$

$$q_{FC}^{in}(h), q_{FC}^{out}(h) \geq 0, \quad \forall h \in \mathcal{H}, \tag{36}$$

where Q_0^{FC} is the stored heat at the beginning of the scheduling period.

5.1.2 Constraints on hot water output and released heat

Since we cannot generate and release heat at the same time, at least one of the hot water output $y^-(h)$ and the released heat $y^+(h)$ during period h must be equal to 0. Therefore, we must have

$$y^-(h) \cdot y^+(h) = 0, \quad \forall h \in \mathcal{H}, \tag{37}$$

$$y^+(h), y^-(h) \geq 0, \quad \forall h \in \mathcal{H}. \tag{38}$$

Taking into account the constraints (44) and the objective function (50) introduced later, it is clear that the complementarity constraint (37) holds at an optimal solution without imposing the constraints (37) explicitly.

5.1.3 Constraints on electricity and heat balances

Let $p_{FC}(h)$ denote the power output of the fuel cell. Let $z^-(h)$ and $z^+(h)$ denote, respectively, the purchased and sold electricity. In addition, let $s(h)$ and $E_H(h)$ denote given constants which represent the solar power and the electricity demand. Since, electricity balance must be satisfied at each period, we have the following constraints:

$$p_{FC}(h) + s(h) + z^-(h) = z^+(h) + E_H(h), \quad \forall h \in \mathcal{H}, \tag{39}$$

$$p_{FC}(h), z^+(h), z^-(h) \geq 0, \quad \forall h \in \mathcal{H}. \tag{40}$$

Let $q_{FC}^{out}(h)$ denote the hot water output from the heat storage tank. Let $y^-(h)$ and $y^+(h)$ denote the hot water output of the gas boiler and the released heat, respectively. In addition, $\Theta(h)$ denote a given constant which represents the heat demand. Since heat balance must be satisfied at each period, we have the following constraints:

$$q_{FC}^{out}(h) + y^-(h) = \Theta(h) + y^+(h), \quad \forall h \in \mathcal{H}. \tag{41}$$

However, electricity demand $E_H(h)$, heat demand $\Theta(h)$ and solar power $s(h)$ are uncertain in reality. For this reason, we treat them as random variables, and assume that they follow a discrete probability distribution with a finite sample space. Specifically, we assume that the vector of these random variables

$$(E_H(1), \ldots, E_H(H), \Theta(1), \ldots, \Theta(H), s(1), \ldots, s(H))$$

takes a finite number of realizations

$$(E_{Hi}(1), \ldots, E_{Hi}(H), \Theta_i(1), \ldots, \Theta_i(H), s_i(1), \ldots, s_i(H))$$

with probability p_i, where i denotes an index of scenario. Let n be the number of scenarios and $\mathcal{I}$ be the set of scenario indices, namely, $\mathcal{I} = \{1, 2, \ldots, n\}$.

We consider the electricity balance constraint (39) under the above assumption. Since the realizations of $E_{Hi}(h)$ and $s_i(h)$ are different among scenarios, we cannot expect that electricity balance (39) is satisfied for all scenarios i and periods h. Thus, we compensate for an imbalance in the constraint (39) by introducing recourse with an additional recourse cost. If the left-hand side of (39) is larger than the right-hand side, electricity is in short supply. In this case, to complement the supply, we can take actions such as purchasing electricity, increasing power output of fuel cell, or reducing electricity sale. We must choose one or some of these actions. Let $e_i^-(h)$ denote the recourse variables to increase power supply in scenario i. On the other hand, if electricity is in excess supply, we must reduce power supply. Let $e_i^+(h)$ denote the recourse variables to reduce power supply in scenario i. Then, the recourse constraints on electricity balance are written as

$$e_i^-(h) - e_i^+(h) = z^+(h) + E_{Hi}(h) - p_{FC}(h) - s_i(h) - z^-(h), \quad \forall h \in \mathcal{H}, \quad \forall i \in \mathcal{I}, \tag{42}$$

$$e_i^-(h), e_i^+(h) \geq 0, \quad \forall h \in \mathcal{H}, \quad \forall i \in \mathcal{I}. \tag{43}$$

We discuss the heat balance constraint (41) in a similar way. Let θ_i^- denote the recourse variables to increase heat supply in scenario i. Let θ_i^+ denote the recourse variables to reduce heat supply in scenario i. Then, the recourse constraints on heat balance are written as

$$\theta_i^-(h) - \theta_i^+(h) = \Theta_i(h) + y^+(h) - q_{FC}^{out}(h) - y^-(h), \quad \forall h \in \mathcal{H}, \ \forall i \in \mathcal{I}, \tag{44}$$

$$\theta_i^-(h), \theta_i^+(h) \geq 0, \quad \forall h \in \mathcal{H}, \quad \forall i \in \mathcal{I}. \tag{45}$$

5.1.4 Constraints on purchased electricity and sold electricity

Because of a regulation, we cannot sell electricity when we buy it. Therefore, at least one of $z^-(h)$ and $z^+(h)$ must be equal to 0, that is,

$$z^-(h) \cdot z^+(h) = 0, \quad \forall h \in \mathcal{H}. \tag{46}$$

Moreover, because of a regulation, we cannot sell electricity more than the amount produced by solar power. Hence, we have

$$z^+(h) \leq s(h), \quad \forall h \in \mathcal{H}. \tag{47}$$

Since $s(h)$ follows a discrete probability distribution, we introduce the recourse variables in a way similar to the case of electricity and heat balance constraints. Let $\zeta_i(h)$ denote the recourse variables to reduce electricity sale in scenario i. Then, we can write the recourse constraints associated with solar power and power sale as

$$z^+(h) - \zeta_i(h) \le s_i(h), \quad \forall h \in \mathcal{H}, \ \forall i \in \mathcal{I}, \tag{48}$$

$$\zeta_i(h) \ge 0, \quad \forall h \in \mathcal{H}, \ \forall i \in \mathcal{I}. \tag{49}$$

5.1.5 Objective function

The objective function to be minimized is represented as

$$\begin{aligned} C_1 \sum_{h=1}^{H} x(h) + & C_2 \sum_{h=1}^{H} z^-(h) - C_3 \sum_{h=1}^{H} z^+(h) + C_4 \sum_{h=1}^{H} y^-(h) \\ & + \sum_{i=1}^{n} p_i \left\{ \sum_{h=1}^{H} C_{e^-}(h) e_i^-(h) + \sum_{h=1}^{H} C_{e^+}(h) e_i^+(h) \right. \\ & \left. + \sum_{h=1}^{H} C_{\theta^-}(h) \theta_i^-(h) + \sum_{h=1}^{H} C_{\theta^+}(h) \theta_i^+(h) + \sum_{h=1}^{H} C_{\zeta}(h) \zeta_i(h) \right\}. \end{aligned} \tag{50}$$

Here, the constants C_1, C_2, C_3, and C_4 denote the unit costs of gas, purchased electricity, sold electricity, and heat, respectively. $C_{e^-}(h)$ and $C_{e^+}(h)$ denote the unit costs of recourses $e^-(h)$ and $e^+(h)$, respectively. $C_{\theta^-}(h)$ and $C_{\theta^+}(h)$ denote the unit costs of recourses $\theta^-(h)$ and $\theta^+(h)$, respectively. $C_\zeta(h)$ denotes the unit cost of recourse $\zeta(h)$. The terms in (50) represent the total gas cost for the fuel cell, the total cost of purchased electricity, the total cost of sold electricity, the total cost of the gas boiler and the expected value of the total recourse cost. Note that the third term is negated since it corresponds to a profit.

5.2 Robust optimization

In the previous section, the unit costs of recourse are treated as given constants. However, in the case of short supply of electricity, there are various ways of implementing recourses, for example, purchasing electricity, reducing electricity, increasing power output of the fuel cell, and their mixture. For this reason, it is not practical to consider the unit costs of recourse to be given constants. Thus we suppose that the unit costs $C_{e^-}(h)$ of recourse $e^-(h)$, as well as the unit costs of other recourses, are uncertain.

To deal with uncertainty of the recourse costs, we adopt a robust optimization model [2]. Robust optimization is a modeling concept based on the worst case analysis. If a rectangle $C_{e^-}^{\min} \le C_{e^-}(h) \le C_{e^-}^{\max}$, $h \in \mathcal{H}$, is used as the uncertainty set for the unit cost of recourses, then we tend to estimate an extreme case such as $C_{e^-}(h) = C_{e^-}^{\max}$ for all periods, which is considered to be extremely conservative. Thus, an ellipsoid is preferred as the uncertainty set [3]. Rectangular and ellipsoidal uncertainty sets are shown in Figure 3 and Figure 4, respectively.

We assume that the uncertain vector $C_{e^-} = (C_{e^-}(1), \ldots, C_{e^-}(H))^T$ belongs to the ellipsoidal uncertainty set given by

$$\Omega_{e^-} = \{ k_{e^-} + A_{e^-} u : \|u\| \le 1 \}, \tag{51}$$

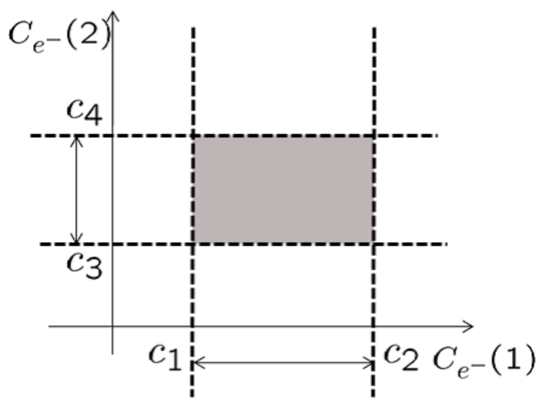

Figure 3: Rectangular uncertainty set

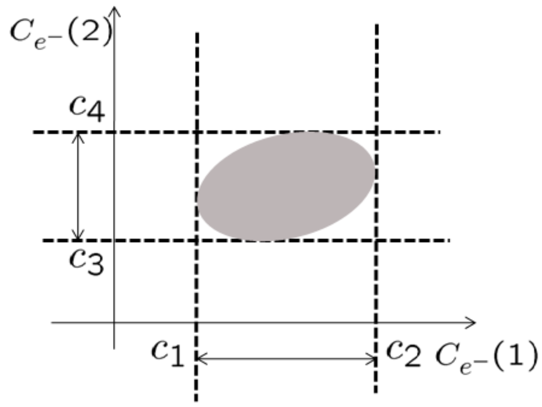

Figure 4: Ellipsoidal uncertainty set

where $u \in R^H$, $A_{e^-} \in R^{H\times H}$ and $k_{e^-} \in R^H$. In the special case where $A_{e^-} = \delta I$ with a positive constant and the unit matrix $I \in R^{H\times H}$, the uncertainty set Ω_{e^-} is the sphere with center k_{e^-} and radius δ. We specify the uncertainty sets for $C_{e^+}, C_{\theta^-}, C_{\theta^+}, C_\zeta$ in a similar way. Because we consider the worst case, the terms related to the recourse costs in the objective function (50)

$$\sum_{i=1}^n p_i \left\{ \sum_{h=1}^H C_{e^-}(h)e_i^-(h) + \sum_{h=1}^H C_{e^+}(h)e_i^+(h) + \sum_{h=1}^H C_{\theta^-}(h)\theta_i^-(h) + \sum_{h=1}^H C_{\theta^+}(h)\theta_i^+(h) + \sum_{h=1}^H C_\zeta(h)\zeta_i(h) \right\},$$

are written as follows:

$$\max_{C_{e^-}\in\Omega_{e^-}} C_{e^-}^T \sum_{i=1}^n p_i e_i^- + \max_{C_{e^+}\in\Omega_{e^+}} C_{e^+}^T \sum_{i=1}^n p_i e_i^+ + \max_{C_{\theta^-}\in\Omega_{\theta^-}} C_{\theta^-}^T \sum_{i=1}^n p_i \theta_i^- + \max_{C_{\theta^+}\in\Omega_{\theta^+}} C_{\theta^+}^T \sum_{i=1}^n p_i \theta_i^+ + \max_{C_\zeta\in\Omega_\zeta} C_\zeta^T \sum_{i=1}^n p_i \zeta_i, \qquad (52)$$

where e_i^- are the vector with elements $e_i^-(h)$, $h = 1, \ldots, H$, and e_i^+, θ_i^-, θ_i^+ ζ_i are defined similarly. By the special structure (51) of the uncertainty sets, (52) is rewritten as follows:

$$k_{e^-}^T t_{e^-} + \|A_{e^-}^T t_{e^-}\| + \bar{k}_{e^+}^T t_{e^+} + \|A_{e^+}^T t_{e^+}\| + k_{\theta^-}^T t_{\theta^-} + \|A_{\theta^-}^T t_{\theta^-}\| + k_{\theta^+}^T t_{\theta^+} + \|A_{\theta^+}^T t_{\theta^+}\| + k_\zeta^T t_\zeta + \|A_\zeta^T t_\zeta\|,$$

where

$$t_{e^-} = \sum_{i=1}^n p_i e_i^-,\ t_{e^+} = \sum_{i=1}^n p_i e_i^+,\ t_{\theta^-} = \sum_{i=1}^n p_i \theta_i^-,\ t_{\theta^+} = \sum_{i=1}^n p_i \theta_i^+,\ t_\zeta = \sum_{i=1}^n p_i \zeta_i. \qquad (53)$$

Summing up the discussion so far, the mathematical model of the smart house scheduling problem can be represented as follows:

$$\begin{array}{lll} SH: & \min & C_1 \sum_{h=1}^H x(h) + C_2 \sum_{h=1}^H z^-(h) - C_3 \sum_{h=1}^H z^+(h) + C_4 \sum_{h=1}^H y^-(h) \\ & & + k_{e^-}^T t_{e^-} + s_{e^-} + \bar{k}_{e^+}^T t_{e^+} + s_{e^+} + k_{\theta^-}^T t_{\theta^-} + s_{\theta^-} + k_{\theta^+}^T t_{\theta^+} + s_{\theta^+} + k_\zeta^T t_\zeta + s_\zeta \\ & \text{s.t.} & \text{second-order cone constraints } \|A_{e^-}^T t_{e^-}\| \le s_{e^-},\ \|A_{e^+}^T t_{e^+}\| \le s_{e^+},\ \|A_{\theta^-}^T t_{\theta^-}\| \le s_{\theta^-}, \\ & & \qquad \|A_{\theta^+}^T t_{\theta^+}\| \le s_{\theta^+},\ \|A_\zeta^T t_\zeta\| \le s_\zeta, \\ & & \text{linear constraints } (31), (34)-(36), (38), (40), (42)-(45), (48), (49), (53), \\ & & \text{complementarity constraints } (46), \\ & & \text{nonlinear equality constraints } (32), (33). \end{array}$$

6 Numerical experiments

In this section, we show some numerical results with Algorithm 3.5 applied to the smart house scheduling problem SH formulated in the previous section. All computations were carried out on a Core 2 Duo 1.2GHz $\times$ 2 machine. We used the SDPT3 [17] to solve linear second-order cone programs within Algorithm 4.2, and all programs were coded in Matlab 7.10.0.

The values of the constants used in the experiments are shown in Table 1. We assume that the uncertainty set Ω_{e^-} associated with the unit costs $C_{e^-} = (C_{e^-}(1), \ldots, C_{e^-}(H))^T$ of recourses $e_i^- = (e_i^-(1), \ldots, e_i^-(H))^T$ is given by the sphere with

$$k_{e^-} = \frac{1}{2}(C_{e^-}^{\max} + C_{e^-}^{\min})\mathbf{1}, \tag{54}$$

$$A_{e^-} = \delta_{e^-} I,$$

$$\delta_{e^-} = \frac{1}{2}(C_{e^-}^{\max} - C_{e^-}^{\min}), \tag{55}$$

in (51), where $\mathbf{1}$ is the vector of which all elements are one. We treat the uncertainty sets of C_{e^+}, C_{θ^-}, C_{θ^+} and C_ζ in a similar way. We make three scenarios of electricity demands, head demands and solar power based on the real data from November 1st to 3rd in 2007 (see Figure 5, Figure 6 and Figure 7), and assume that these three scenarios occur with probabilities $p_1 = 1/3$, $p_2 = 1/3$, and $p_3 = 1/3$.

The optimal solutions of problem SH with the above data are shown in Figure 8–Figure 14. The optimal value obtained was -219.0056 and the total number of subproblems solved within Algorithm 4.2 was 88. The hot water output $y^-(h)$, the released heat $y^+(h)$ and recourses to reduce heat supply $\theta_i^-(h)$ were 0 constantly.

Table 1: Constants used in the experiments

constant	explanation	Value
H	The number of subperiods	6
C_1	Unit cost of gas[yen/Wh]	0.006997
C_2	Unit cost of purchased electricity[yen/Wh]	0.0191
C_3	Unit cost of sold electricity[yen/Wh]	0.049
C_4	Unit cost of heat[yen/Wh]	0.00897
$C_{e^-}^{\max}$	Maximum value of the unit cost of e^-[yen/Wh]	0.02865
$C_{e^-}^{\min}$	Minimum value of the unit cost of e^-[yen/Wh]	0.013032
$C_{e^+}^{\max}$	Maximum value of the unit cost of e^+[yen/Wh]	-0.002895
$C_{e^+}^{\min}$	Minimum value of the unit cost of e^+[yen/Wh]	-0.006365
$C_{\theta^-}^{\max}$	Maximum value of the unit cost of θ^-[yen/Wh]	0.013455
$C_{\theta^-}^{\min}$	Minimum value of the unit cost of θ^-[yen/Wh]	0
$C_{\theta^+}^{\max}$	Maximum value of the unit cost of θ^+[yen/Wh]	0
$C_{\theta^+}^{\min}$	Minimum value of the unit cost of θ^+[yen/Wh]	-0.00299
$C_{\zeta}^{\max}$	Maximum value of the unit cost of ζ[yen/Wh]	0.060468
$C_{\zeta}^{\min}$	Minimum value of the unit cost of ζ[yen/Wh]	0.04485
$x_{\max}$	Upper bound of gas input[Wh]	2300
$Q_{\max}^{FC}$	Upper bound of stored heat [Wh]	10467
Q_0^{FC}	Stored heat at the beginning of the scheduling period[Wh]	0
a	Slope of power output of the fuel cell[Wh/W]	1.254
b	Slope of heat output of the fuel cell[Wh/W]	2.26
α	Constant of power output of the fuel cell[W]	13.66411
β	Constant of heat output of the fuel cell[W]	308.4071

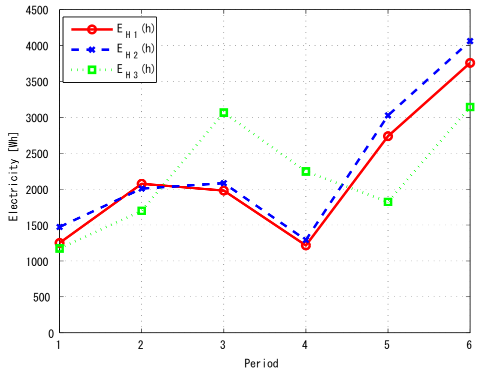

Figure 5: Scenarios of electricity demand $E_H(h)$

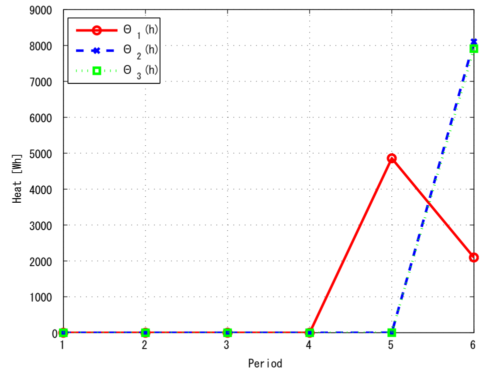

Figure 6: Scenarios of heat demand $\Theta(h)$

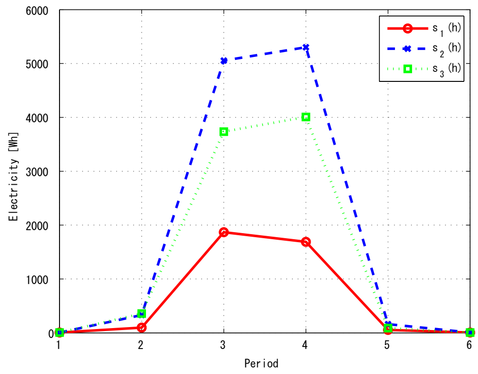

Figure 7: Scenarios of solar power $s(h)$

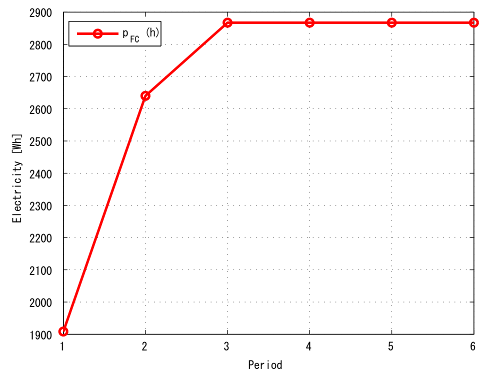

Figure 8: Electricity output p_{FC} from the fuel cell at the solution

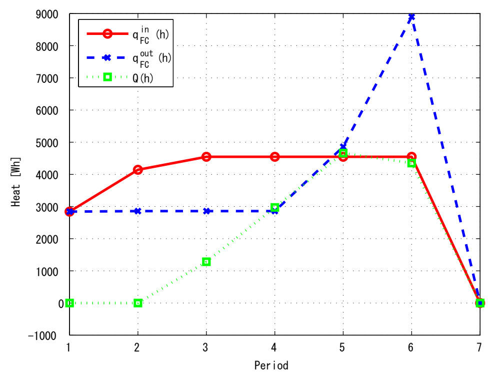

Figure 9: Heat output $q^{in}_{FC}(h)$ from the fuel cell, hot water output $q^{out}_{FC}(h)$ from the heat storage tank, and stored heat $Q_{FC}(h)$ in the heat storage tank at the solution

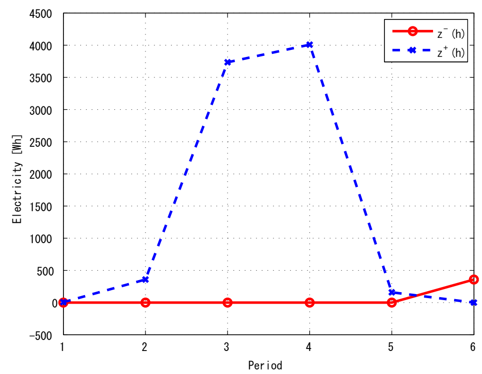

Figure 10: Purchased electricity $z^-(h)$ and sold electricity $z^+(h)$ at the solution

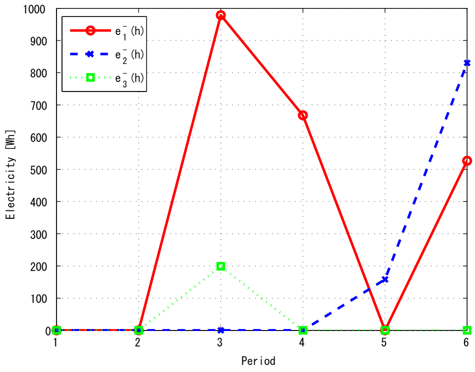

Figure 11: Recourses $e_1^-(h)$, $e_2^-(h)$, $e_3^-(h)$ to increase power supply at the solution

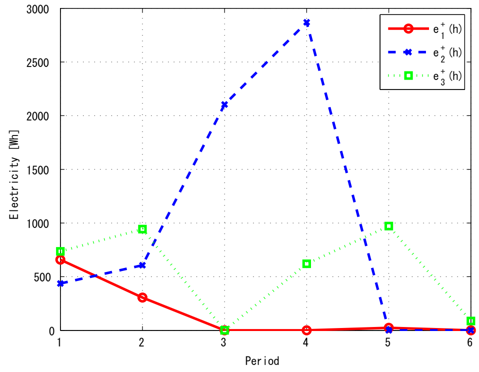

Figure 12: Recourses $e_1^+(h)$, $e_2^+(h)$, $e_3^+(h)$ to reduce power supply at the solution

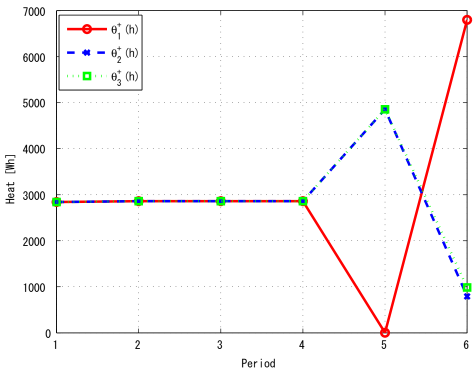

Figure 13: Recourses $\theta_1^+(h)$, $\theta_2^+(h)$, $\theta_3^+(h)$ to reduce heat supply at the solution

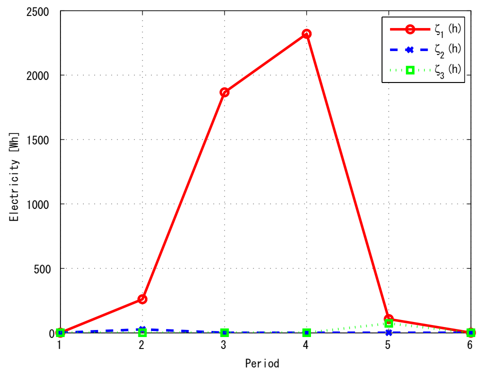

Figure 14: Recourses $\zeta_1(h)$, $\zeta_2(h)$, $\zeta_3(h)$ to reduce electricity sale at the solution

Next, we examine the influence of the size of the uncertainty sets on the solutions, by varying the radius of the uncertainty sets. For example, we define the radius of the uncertainty set associated with the unit cost C_{e^-} as

$$\delta_{e^-} = \frac{1}{2} r (C_{e^-}^{\max} - C_{e^-}^{\min}),$$

and vary the value of parameter r. Similarly, we define the radius of the uncertainty sets associated with the unit costs of the other recourses. We examine the behavior of the solutions of problem SH and show the optimal values for each r in Figure 15. We can observe that the optimal value increases, as r increases. This result is appropriate because the costs in the worst case will be large, when the uncertainty sets are large.

In addition, we check the behavior of the amount of recourses when parameter r varies. For example, the expected value of the amount of recourses e^- is computed by

$$\sum_{i=1}^{n} p_i \sum_{h=1}^{H} e_i^-(h),$$

and those of the other recourses are computed in a similar way. The results are shown in Figure 16 – Figure 19. We can observe that the amount of recourses decreases, as r increases. Thus, we can deduce that one tends to avoid using recourses when the possibility of large unit costs of recourses becomes high.

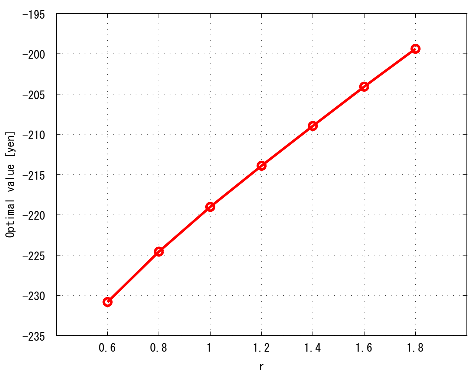

Figure 15: Behavior of the optimal value for various values of parameter r

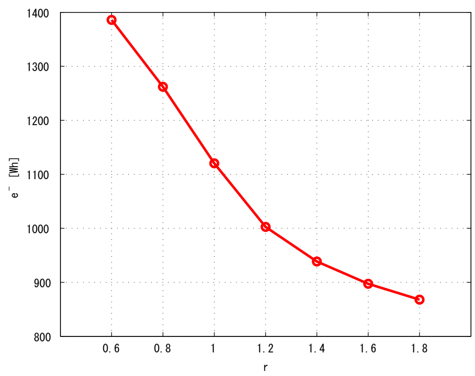

Figure 16: Behavior of the total amount of recourses e^- for various values of parameter r

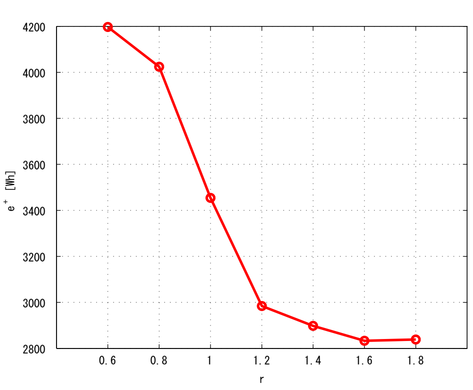

Figure 17: Behavior of the total amount of recourses e^{+} for various values of parameter r

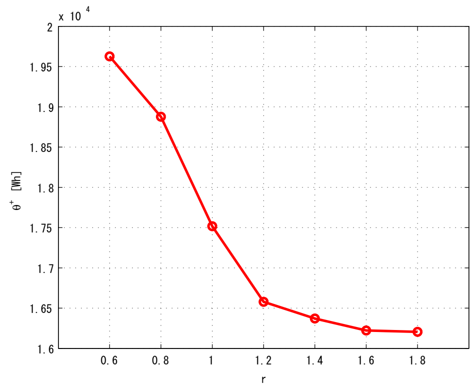

Figure 18: Behavior of the total amount of recourses θ^{+} for various values of parameter r

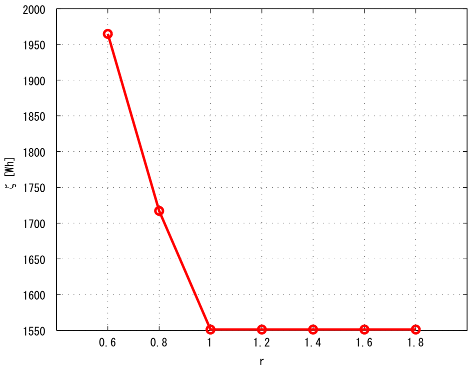

Figure 19: Behavior of the total amount of recourses ζ for various values of parameter r

We may also specify the uncertainty set as a rectangle (see Figure 3). Here, we assume that the uncertain set associated with the unit costs $C_{e^-} = (C_{e^-}(1), \ldots, C_{e^-}(H))^T$ of recourses $e^- = (e^-(1), \ldots, e^-(H))^T$ is given by the cube with center k_{e^-} and side length $2\delta_{e^-}$, i.e.,

$$\Omega_{e^-} = \{k_{e^-} + \delta_{e^-} u \mid \|u\|_\infty \leq 1\},$$

where $\|u\|_\infty := \max\{|u_1|, \ldots, |u_H|\}$. Then, the first term of (52)

$$\max_{C_{e^-} \in \Omega_{e^-}} C_{e^-}^T \sum_{i=1}^n p_i e_i^-$$

is rewritten as

$$k_{e^-}^T \sum_{i=1}^n p_i e_i^- + \delta_{e^-} \sum_{h=1}^H \left| \sum_{i=1}^n p_i e_i^-(h) \right|.$$

We give the uncertainty sets associated with the unit costs of the other recourses in a similar way. Under this situation, the optimal objective value was -195.0368. This value amounts to the optimal objective value of problem SH with a spherical uncertainty set of considerably large radius. This indicates that the costs tend to be estimated higher in the case of rectangular uncertainty sets, than the case of spherical uncertainty sets.

7 Conclusion

In this paper, we have considered a nonlinear second-order cone program with complementarity constraints, and showed that, under certain assumptions, we can find a point that satisfies optimality conditions of the problem by using a smoothing technique. Furthermore, as an application, we have formulated a mathematical model of smart house scheduling with uncertainty in the unit costs of recourses. Through numerical experiments, we have confirmed the validity of the model.

Acknowledgements

First of all, I would like to express my sincere gratitude to Professor Masao Fukushima for his accurate guidance, insightful advices and kind encouragement. The way to view problems I learned from him will help me to be a good researcher in my future. I am also deeply grateful to Ph.D. Yoichi Tanaka from TOHO GAS Co., Ltd. He gave me a chance to tackle this subject and practical advices for my research. I would like to thank Associate Professor Nobuo Yamashita. I have greatly benefited from his advices about my numerical experiment. I would like to express my thanks to Assistant Professor Shunsuke Hayashi for his kind guidance. I would like to express my gratitude to all members in Fukushima Laboratory for their encouragement. Finally, I would like to thank my family for their constant support.

References

[1] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Mathematical Programming*, Vol. 95, pp. 3–51, 2003.

[2] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization.* Princeton Univ Press, 2009.

[3] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, Vol. 23, pp. 769–805, 1998.

[4] J.F. Bonnans and H. Ramírez. Perturbation analysis of second-order cone programming problems. *Mathematical Programming*, Vol. 104, pp. 205–227, 2005.

[5] M. Fukushima. *Fundamentals of Nonlinear Optimization (in Japanese)*. Asakura Shoten, 2001.

[6] M. Fukushima, Z.Q. Luo, and J.S. Pang. A globally convergent sequential quadratic programming algorithm for mathematical programs with linear complementarity constraints. *Computational Optimization and Applications*, Vol. 10, pp. 5–34, 1998.

[7] M. Fukushima and J.S. Pang. Convergence of a smoothing continuation method for mathematical programs with complementarity constraints. In: Thera M., Tichatschke R. (eds). *Ill-posed Variational Problems and Regularization Techniques*, Springer, Berlin. pp. 99–110, 1999.

[8] C. Kanzow, C. Nagel, H. Kato, and M. Fukushima. Successive linearization method for nonlinear semidefinite programs. *Computational Optimization and Application*, Vol. 31, pp. 251–273, 2005.

[9] H. Kato and M. Fukushima. An SQP-type algorithm for nonlinear second-order cone programs. *Optimization Letters*, Vol. 1, pp. 129–144, 2007.

[10] X. Liu, G. Perakis, and J. Sun. A robust SQP method for mathematical programs with linear complementarity constraints. *Computational Optimization and Appricaticns*, Vol. 34, pp. 5–33, 2006.

[11] Z.Q. Luo, J.S. Pang, and D. Ralph. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press, Cambridge, 1996.

[12] Ministry of Economy, Trade and Industry. *Report on Smart House Demonstration Project in 2009 (in Japanese)*. 2010.

[13] S. Ozoe, Y. Tanaka, and M. Fukushima. A two-stage stochastic mixed-integer programming approach to the smart house scheduling problem (in Japanese). *IEEJ Transactions on Power and Energy*, Vol. 131, pp. 885–895, 2011.

[14] M.A.A. Pedrasa, T.D. Spooner, and I.F. MacGill. Coordinated scheduling of residential distributed energy resources to optimize smart home energy services. *IEEE Transactions on Smart Grid*, Vol. 1, pp. 134–142, 2010.

[15] H. Scheel and S. Scholtes. Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity. *Mathematics of Operations Research*, Vol. 25, pp. 1–22, 2000.

[16] S. Scholtes. Convergence properties of a regularization scheme for mathematical programs with complementarity constraints. *SIAM Journal on Optimization*, Vol. 11, pp. 918–936, 2001.

[17] R.H. Tütüncü, K.C. Toh, and M.J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming*, Vol. 95, pp. 189–217, 2003.