Master's Thesis

On the relation between the diversity of players and the stability of Nash equilibria in non-cooperative games

Guidance

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Abstract

Various equilibria are filling our world. In the game theory, such equilibria are determined by game players' strategies. For example, market prices of assets and traffic in networks are equilibria determined by investors' investment and drivers' path selection, respectively.

In the real world, it is meaningful to analyze *stability* of an equilibrium. For example, a central bank adopts a policy that stabilizes market prices of assets when the prices violently fluctuate. If we can find primary factors that stabilize the violent fluctuating, we can apply the result to the policy. Intuitively, *diversity* of players' criteria is one of those factors. For example, one of reasons of the so-called bubble economy in Japan from the late 1980s to the early 1990s, is that a lot of investors had the same criteria that the Japanese economy is peaceful.

The purpose of this paper is to analyze a relation between *diversity* of players' criteria and *stability* of an equilibrium mathematically. To this end, we first formulate versatile Nash equilibrium problems (NEP) that include asset equilibrium problems (AEP) and traffic equilibrium problems (TEP). Then, we define diversity of players as scattering of criteria which appear in players' utility functions of NEP. At the same time, we define stability of an equilibrium as variance of the Nash equilibrium with respect to random parameters, such as asset's rate of return, in players' utility functions. These definitions allow us to present sufficient conditions under which diversity of players' criteria yields stability of the Nash equilibrium. Then, applying the results to AEPs and TEPs, we analyze the relation between the diversity and the stability in each problems. Consequently, we conclude that, in theory, diversity of players' criteria often yields stability of the Nash equilibrium.

Contents

1 Introduction

Various equilibria are filling our world. For example, market prices of assets, such as stocks, are equilibria determined by investors' investment [11], traffic in networks forms an equilibrium that no drivers can decrease their own cost by changing their own path [10], and populations of living things also form an equilibrium on the food chain.

In the game theory, an equilibrium is described to a state that game players take their own optimal strategy each other. This state is called a Nash equilibrium.

In the real world, it is meaningful to analyze *stability* of an equilibrium. In this context, the stability means how the equilibrium does not vary when parameters that form it barely vary. For example, when market prices of assets violently fluctuate, since such fluctuating gives a lot of benefits to a few people, a central bank adopts a policy that stabilizes the prices. If we can find primary factors that stabilize the violent fluctuating, we can apply the result to the policy.

Intuitively, *diversity* of players' criteria yields *stability* of an equilibrium. For example, one of reasons of the sudden rise in asset prices in Japan from the late 1980s to the early 1990s, so called the bubble economy, is that a lot of investors had the same criteria that the Japanese economy is peaceful. On the other hand, a biologist studied that existence of a lot of kinds of living things and a lot of pairs of interspecific relation yield break down of ecosystem based on a mathematical model [6]. However, some researchers have studied that direct questions at it in recent years [7, 12].

In this paper, we formulate a Nash equilibrium problem (NEP) for the above phenomena, such as an asset price and traffic. Then we analyze the relation between the *diversity* of parameters of players' criteria and the *stability* of the Nash equilibrium.

Some researchers have proposed various equilibrium problems, and discussed its existence and uniqueness [1, 5]. However, since they are based on abstract stochastic differential equations, most of such models are too general to analyze how parameters forming an equilibrium affect the equilibrium. Therefore, we need to propose a more concrete model whose equilibrium be can easily calculated.

In this paper, we first suppose that an optimal strategy is a minimizer of a cost function. In other words, it is not a maximizer of the utility function. The cost function is a mathematical model that quantitatively expresses a cost of players' actions. Each player decides his/her action according to it. Then, we propose a non-cooperative game that has a convex quadratic cost function, in which there exist two kinds of parameters: one varying at random and the other one being different on the player. The proposed game model includes both problems for finding an equilibrium of investors' investment distributions (asset equilibria) and of traffic in networks (traffic equilibria), and hence, it is versatile. We use Markowitz's mean-variance model [4] for expressing each player's cost function. At the same time, we use Gabriel and Bernstein's model, in which driver's cost consists of both time and a toll [3], for expressing each player's cost function.

Next, we define stability and diversity. In this paper, we define diversity of players as scattering of criteria which appear in players' cost functions. At the same time, we define stability of an equilibrium as variance of the Nash equilibrium with respect to random parameters, such as assets' rates of return, in players' cost functions. These definitions allow us to present sufficient conditions under which diversity yields stability.

Then, applying the results to asset equilibrium problems (AEP) and traffic equilibrium problems (TEP), we analyze the relation between the diversity of players' criteria and the stability of the Nash equilibrium in each problems. Finally, we conduct numerical experiments for AEPs to show validity and possibility of relaxation of the presented condition.

This paper is organized as follows. In Section 2, we introduce a non-cooperative game, and give sufficient conditions under which the unique equilibrium exists. In Section 3, we formulate a versatile NEP. Moreover, we transform it into a variational inequality problem, and prove that the unique solution exists. In Section 4, we define stability of players and diversity of equilibria, and present sufficient conditions under which diversity yields stability. In Section 5, for each of AEPs and TEPs, we mathematically analyze the relation between the diversity of players' criteria and the stability of the Nash equilibrium. In Section 6, we report some results of numerical experiments, and we conclude this paper in Section 7.

2 Preliminaries

In this section, we introduce a non-cooperative game where M players minimize their cost functions to decide their optimal strategies. Then, we give sufficient conditions under which a unique equilibrium exists at the game.

2.1 Non-cooperative games and Nash equilibria

Suppose that Player j has an n_j-dimension strategy vector, which is denoted by $x^j \in \mathbb{R}^{n_j}$. Moreover, we use the following definitions:

$$\begin{aligned} x^j &:= (x^j_1, \cdots, x^j_{n_j})^\top, \\ x &:= (x^1, \cdots, x^M)^\top, \\ x^{-j} &:= (x^1, \cdots, x^{j-1}, x^{j+1}, \cdots, x^M)^\top. \end{aligned}$$

Note that the vector x is all player's strategies, and x^{-j} is player's strategies except for Player j's one.

First, we formulate the 1-player (Player j's) model. Suppose that all player's strategies except for Player j's one are given, and let $C_j(x^j; x^{-j})$ be Player j's cost function. Then, we can formulate a non-cooperative game where Player j minimizes his/her cost function as follows:

$$\begin{aligned} \min \quad & C_j(x^j; x^{-j}) \\ \text{s.t.} \quad & x^j \in \Omega_j, \end{aligned} \tag{1}$$

where Ω_j denotes the feasible set of Player j's strategy. Next, we extend problem (1) to the M-player non-cooperative game to find an equilibrium.

As formulated 1-player (Player j) model (1), we can formulate M optimal problems as follows:

$$\begin{array}{ll} \min & C_1(x^1; x^{-1}) \\ \text{s.t.} & x^1 \in \Omega_1 \\ \vdots & \\ \min & C_M(x^M; x^{-M}) \\ \text{s.t.} & x^M \in \Omega_M. \end{array}$$

Now we define an equilibrium of the game, called a Nash equilibrium. The Nash equilibrium is a strategy pair that makes no reason for each of players to change their strategy to decrease their cost.

Then, a Nash equilibrium is the solutions pair $x^* \in \Omega_1 \times \cdots \times \Omega_M$ such that

$$\left.\begin{array}{c} C_1(x^{1*}; x^{-1*}) \leq C_1(x^1; x^{-1*}) \\ \vdots \\ C_M(x^{M*}; x^{-M*}) \leq C_M(x^M; x^{-M*}) \end{array}\right. \quad {}^\forall x \in \Omega_1 \times \cdots \times \Omega_M. \tag{2}$$

Problem (2) is called a Nash equilibrium problem (NEP) as an M-player non-cooperative game.

2.2 Sufficient conditions under which a unique Nash equilibrium exists

In this subsection, we first introduce conditions under which NEP (2) can be formulated into a variational inequality problem (VIP). Next, we give sufficient conditions under which a unique solution exists in VIPs.

VIPs are problems defined as follows:

$$\begin{array}{ll} \text{find} & x \in \Omega \\ \text{s.t.} & \langle \boldsymbol{F}(x), y - x \rangle \geq 0 \quad \forall y \in \Omega, \end{array} \tag{3}$$

where $\Omega \subseteq \mathbb{R}^n$ is a non-empty closed convex set and $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued mapping.

We can reformulate a convex programming problem into a VIP by the following lemma:

Lemma 2.1 [2] *Suppose that a set Ω is non-empty closed convex set. Suppose also that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function at a point $\bar{x} \in \Omega$. Then, $\bar{x} \in \Omega$ is the global optimal solution to the convex problem:*

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \Omega \end{array}$$

if and only if

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in \Omega.$$

Suppose that for each j the function $C_j(x^j; x^{-j})$ is convex with respect to x_j and the set Ω_j is convex. Then, problem (1) is a convex programming problem. Lemma 2.1 implies that NEP (2) is equivalent to the following VIP:

$$\begin{array}{ll} \text{find} & x^* \in \Omega_1 \times \cdots \times \Omega_M \\ \text{s.t.} & \langle \boldsymbol{F}(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \Omega_1 \times \cdots \times \Omega_M, \end{array} \tag{4}$$

where

$$\boldsymbol{F} := \begin{pmatrix} \nabla_{x^1} C_1(x^1; x^{-1}) \\ \vdots \\ \nabla_{x^M} C_M(x^M; x^{-M}) \end{pmatrix}. \tag{5}$$

Now we give the sufficient conditions under which n unique solution exists at VIP (3). To this end, we define strong monotonicity of a mapping $\boldsymbol{F}$.

Definition 2.1 *Suppose that a set $\Omega \subseteq \mathbb{R}^n$ is non-empty closed convex set. Suppose also that a mapping $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued mapping. Then, the mapping $\boldsymbol{F}$ is called strongly monotone in Ω when there exists a constant $\sigma > 0$ such that*

$$x,\ y \in \Omega \quad \Longrightarrow \quad \langle x - y,\ \boldsymbol{F}(x) - \boldsymbol{F}(y) \rangle \geq \sigma ||x - y||^2.$$

Lemma 2.2 [2] *Suppose that $\boldsymbol{F}$ is a continuous mapping. If the mapping $\boldsymbol{F}$ is strongly monotone, then VIP (3) has the unique solution.*

Consequently, if mapping (5) is strongly monotone, then a unique solution exists at VIP (4).

3 Non-cooperative games including convex quadratic cost functions with a random variable and player's individual parameter

In this paper, we present non-cooperative games that have convex quadratic cost functions, which has two kinds of parameters: varying at random one and being different on the player one. In this section, we give details of this game, such as a cost function, and we prove that an equilibrium solution is unique at this game.

3.1 Strategy sets and cost functions

In this paper, we suppose that each player has an N-dimension strategy vector, whose components are non-negative. Suppose also that each player has a strategy resource $p_j > 0$.

Then, we formulate Player j's cost function $C_j(x^j; x^{-j})$ as follows:

$$C_j(x^j; x^{-j}) = \Gamma_1(x^j; x^{-j}) + \frac{1}{\alpha_j} \Gamma_2(x^j; x^{-j}), \tag{6}$$

where

$$\Gamma_1(x^j; x^{-j}) := \left(\sum_{j'=1}^{M} x^{j'}\right)^\top R x^j,$$

$$\Gamma_2(x^j; x^{-j}) := (Qx^j + b)^\top x^j,$$

where

$R \in \mathbb{R}^{N\times N}$	:	a diagonal matrix whose diagonal components are $r_1, \cdots, r_N$;
$\boldsymbol{r} \in \mathbb{R}^N$	:	a random variable vector whose components are $r_i \geq 0$ and satisfy $\boldsymbol{r} \neq 0$;
$Q \in \mathbb{R}^{N\times N}$	:	a positive semidefinite matrix;
$b \in \mathbb{R}^N$	:	an arbitrary constant vector;
α_j	:	a parameter of Player j's criterion.

The conditions that $r_i \geq 0$, $\boldsymbol{r} \neq 0$, and the matrix Q is positive semidefinite allow a unique equilibrium solution of this game to exist. The condition that the matrix Q is positive semidefinite also allows the function C_j to be a convex function. We see an effect of the function $\Gamma_2(x^j; x^{-j})$ value becomes small as the parameter α_j becomes large. In general, each player has each criterion, and hence, it is not real that we suppose that each of them has the same cost function. We introduce the parameter α_j to express the difference of players' criterion.

Now we can express the domain Ω_j of x_j as follows:

$$\Omega_j = \left\{ x^j \in \mathbb{R}^N \mid x^j \geq 0, \ \sum_{i=1}^{N} x_i^j = p^j \right\}. \tag{7}$$

Therefore, since the function C_j is convex and the set Ω_j is convex, problem (1) is a convex programming problem. Note that a solution x^{j^*} of this problem is a function of random variables $r_1, r_2, \cdots, r_N$.

We can apply NEP (2) with cost function (6) and strategy set (7) to asset equilibrium problems (AEP) and traffic equilibrium problems (TEP). AEPs are problems to find an equilibrium of investors' investment distribution. TEPs are problems to find an equilibrium of traffic in networks. In cost function (6), if we determine $b = 0$, the problem becomes an AEP. On the other hand, if we determine $Q = O$, it becomes a TEP. We give these details in Section 5.

3.2 Existence and uniqueness of a Nash equilibrium

In this subsection, we prove that a unique equilibrium solution exists in NEP (2) with cost function (6), and we mention the possibility of calculating the equilibrium solution.

By Lemma 2.1, problem (1) with cost function (6) is equivalent to the following VIP:

$$\begin{aligned} &\text{find} \quad x^{j^*} \in \Omega_j \\ &\text{s.t.} \quad \langle \nabla_{x^j} C_j(x^{j^*}; x^{-j^*}), x^j - x^{j^*} \rangle \geq 0 \quad \forall x^j \in \Omega_j. \end{aligned} \tag{8}$$

Let

$$\begin{aligned}\boldsymbol{F}_j(x^j; x^{-j}) &:= \nabla_{x^j} C_j(x^j; x^{-j}) \\ &= Rx^j + R\sum_{j'=1}^{M} x^{j'} + \frac{2}{\alpha_j} Q x^j + \frac{b}{\alpha_j}.\end{aligned}$$

Then, VIP (8) is written as:

$$\begin{aligned}&\text{find} \quad x^{j*} \in \Omega_j \\ &\text{s.t.} \quad \langle \boldsymbol{F}_j(x^{j*}; x^{-j*}), x^j - x^{j*} \rangle \geq 0 \quad \forall x^j \in \Omega_j.\end{aligned}$$

NEP (2) with cost function (6) is equivalent to the following VIP:

$$\begin{aligned}&\text{find} \quad x^* \in \Omega_1 \times \cdots \times \Omega_M \\ &\text{s.t.} \quad \langle \boldsymbol{F}(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \Omega_1 \times \cdots \times \Omega_M,\end{aligned} \tag{9}$$

where

$$\boldsymbol{F} := \begin{pmatrix} \boldsymbol{F}_1(x^1; x^{-1}) \\ \vdots \\ \boldsymbol{F}_M(x^M; x^{-M}) \end{pmatrix}. \tag{10}$$

Note that this VIP includes MN variables.

Now we prove that a unique equilibrium solution exists in NEP (2).

Lemma 3.1 *Mapping $\boldsymbol{F}$ (10) is strongly monotone.*

Proof. By the definition of the mapping $\boldsymbol{F}$,

$$\boldsymbol{F} = \begin{pmatrix} 2R & R & \cdots & R \\ R & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R \\ R & \cdots & R & 2R \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^M \end{pmatrix} + \begin{pmatrix} \frac{2}{\alpha_1}Q & & O \\ & \ddots & \\ O & & \frac{2}{\alpha_M}Q \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^M \end{pmatrix}.$$

This implies the mapping $\boldsymbol{F}$ is affine. When the mapping $\boldsymbol{F}$ is affine, $\boldsymbol{F}$ is strongly monotone if and only if its Jacobian matrix is positive definite. Therefore, we should prove that the matrix $\nabla_x \boldsymbol{F}$ is positive definite.

For any $z = (z_1, \cdots, z_M)^\top \neq 0 \ \ (z_1, \cdots, z_M \in \mathbb{R}^N)$,

$$\begin{aligned}
z^\top (\nabla_x \boldsymbol{F}) z &= z^\top \begin{pmatrix} 2R & R & \cdots & R \\ R & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R \\ R & \cdots & R & 2R \end{pmatrix} z + z^\top \begin{pmatrix} \frac{2}{\alpha_1} Q & & O \\ & \ddots & \\ O & & \frac{2}{\alpha_m} Q \end{pmatrix} z \\
&> z^\top \begin{pmatrix} R & \cdots & R \\ \vdots & \ddots & \vdots \\ R & \cdots & R \end{pmatrix} z \\
&= (z_1 + \cdots + z_m)^\top R (z_1 + \cdots + z_m) \\
&\geq 0,
\end{aligned}$$

where the second inequality follows from that Q is positive semidefinite, $R \neq O$, and $r_i \geq 0$ $(i = 1, \cdots, N)$. Therefore, the matrix $\nabla_x \boldsymbol{F}$ is positive definite, and we can prove that the mapping $\boldsymbol{F}$ is strongly monotone. ∎

The following theorem ensures presented NEP (9) has a unique equilibrium solution.

Theorem 3.1 *NEP (9) has a unique equilibrium solution.*

Proof. By Lemma 2.2, if mapping $\boldsymbol{F}$ (10) is strongly monotone, VIP (9) has a unique solution. By Lemma 3.1, we can prove that mapping $\boldsymbol{F}$ (10) is strongly monotone. Therefore, NEP (9) has a unique equilibrium solution. ∎

By Lemma 3.1, mapping $\boldsymbol{F}$ (10) is strongly monotone, and the set $\Omega_1 \times \cdots \times \Omega_M$ includes affine (convex) functions as inequality constraints and affine functions as equality constraints. Therefore, we can compute the solution by a reformulation with the Fischer-Burmeister function [2].

3.3 Reducing variables in NEP by classifying players

When we apply the model proposed in the preceding subsection to real phenomena, such as asset prices and traffic, it is difficult for us to calculate the model and analyze the equilibria because the number of players M and the dimension of strategy N (ex. the number of assets, the number of paths, etc.) are very large. Then, we classify M players to T classes. As some players have the same cost parameter α_j and strategy resource p^j, we classify them to the same class. Such a classification allows us to see each class as one virtual player that has a special cost function.

Niimi [9] proposes such a classification on AEP. As we apply it to our model, we can formulate NEP (2) with cost function (6) as the following VIP:

$$\begin{aligned}
&\text{find} \quad \bar{x}^* \in \bar{\Omega}_1 \times \cdots \times \bar{\Omega}_T \\
&\text{s.t.} \quad \langle \bar{\boldsymbol{F}}(\bar{x}^*), \bar{x} - \bar{x}^* \rangle \geq 0 \quad \forall \bar{x} \in \bar{\Omega}_1 \times \cdots \times \bar{\Omega}_T,
\end{aligned} \tag{11}$$

where

$$\bar{\boldsymbol{F}} := \begin{pmatrix} \bar{\boldsymbol{F}}_1(\bar{x}^1; \bar{x}^{-1}) \\ \vdots \\ \bar{\boldsymbol{F}}_T(\bar{x}^T; \bar{x}^{-T}) \end{pmatrix},$$

$$\begin{aligned} \bar{\boldsymbol{F}}_t(\bar{x}^t; \bar{x}^{-t}) &:= \nabla_{\bar{x}^t} C_t(\bar{x}^t; \bar{x}^{-t}) \\ &= \frac{1}{k_t} R\bar{x}^t + R \sum_{t'=1}^{T} \bar{x}^{t'} + \frac{2}{\alpha_t k_t} Q\bar{x}^t + \frac{b}{\alpha_t}, \end{aligned} \tag{12}$$

$$\bar{\Omega}_t = \left\{ \bar{x}^t \in \mathbb{R}^N \mid \bar{x}^t \geq 0, \sum_{i=1}^{N} \bar{x}^t_i = \bar{p}^t \right\},$$

k_t is the number of players in a class t, x^t is the strategy vector of each of players in a class t[1], α_t is the cost parameter in a class t, p^t is the strategy resource, $\bar{x}^t := k_t x^t$, and $\bar{p}^t := k_t p^t$. This VIP includes TN variables. We know the number of variables from MN to TN. We can prove that the mapping $\bar{\boldsymbol{F}}$ is strongly monotone in the same way as Lemma 3.1. Therefore, classified NEP (11) has a unique solution.

4 Diversity of players and stability of equilibria

In this section, we prepare for mathematical analyses on the relation between diversity of players and stability of a Nash equilibrium. We first show a mathematical expression of an equilibrium solution in the case that the strategy dimension $N = 2$ and the number of class players $T = 2$. Next, we define stability of players and diversity of Nash equilibria, and present sufficient conditions under which the diversity yields the stability.

4.1 A Nash equilibrium in games with 2 variables and 2 players

Since the set $\Omega_1 \times \cdots \times \Omega_M$ is defined with affine functions, the Karush-Kuhn-Tucker conditions of VIP is equivalent to the following mixed complementarity problem (MCP) [2]:

$$\begin{array}{lcl} \text{find} & (x, \lambda) & \\ \text{s.t.} & \bar{\boldsymbol{G}}(x) & = 0 \\ & \bar{\boldsymbol{F}}(x) + \nabla \bar{\boldsymbol{G}}(x)\lambda & \geq 0 \\ & x & \geq 0 \\ & (\bar{\boldsymbol{F}}(x) + \nabla \bar{\boldsymbol{G}}(x)\lambda)^\top x & = 0, \end{array} \tag{13}$$

[1]Since NEP (9) has a unique solution as shown in Theorem 3.1, the strategy vectors of players in the same class are equal each other.

where

$$\bar{\boldsymbol{G}}(x) := \begin{pmatrix} \sum_{i=1}^{N} x_i^1 - \bar{p}^1 \\ \vdots \\ \sum_{i=1}^{N} x_i^T - \bar{p}^T \end{pmatrix}$$

and the vector $\lambda = (\underbrace{\lambda^1, \lambda^1, \cdots, \lambda^1}_{N}, \lambda^2, \lambda^2, \cdots, \lambda^T)^\top$ is the Lagrange multiples vector for the constraint $\bar{\boldsymbol{G}}(x) = 0$.

Now suppose the following assumptions.

Assumption 4.1

(i) *The dimension of strategy $N = 2$ and the number of class players $T = 2$.*

(ii) *The variable $x > 0$.*

(iii) *The matrix $\begin{bmatrix} \frac{2}{\alpha_1 k_1} Q + (\frac{1}{k_1} + 1)R & R \\ R & \frac{2}{\alpha_2 k_2} Q + (\frac{1}{k_2} + 1)R \end{bmatrix}$ is singular.*

(iv) *The matrix Q's components $q_{11} \geq 0,\ q_{12} = q_{21} = q_{22} = 0$.*

(v) *The variable $r_2 = r_0$ (constant).*

Then, by Assumption 4.1 (i) and (ii), we can write MCP (13) as:

$$\bar{\boldsymbol{G}}(x) = \begin{bmatrix} x_1^1 + x_2^1 - \bar{p}^1 \\ x_1^2 + x_2^2 - \bar{p}^2 \end{bmatrix} = 0,$$

$$\begin{aligned} \bar{\boldsymbol{F}}(x) + \nabla \bar{\boldsymbol{G}}(x)\lambda \ &= \ \begin{bmatrix} \frac{2}{\alpha_1 k_1} Q & O \\ O & \frac{2}{\alpha_2 k_2} Q \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} (\frac{1}{k_1} + 1)R & R \\ R & (\frac{1}{k_2} + 1)R \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{b_1}{\alpha_1} \\ \frac{b_2}{\alpha_1} \\ \frac{b_1}{\alpha_2} \\ \frac{b_2}{\alpha_2} \end{bmatrix} + \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{bmatrix} = 0. \end{aligned}$$

Therefore, by Assumption 4.1 (iii), we get

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} = - \begin{bmatrix} \frac{2}{\alpha_1 k_1} Q + (\frac{1}{k_1} + 1)R & R \\ R & \frac{2}{\alpha_2 k_2} Q + (\frac{1}{k_2} + 1)R \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 + \frac{b_1}{\alpha_1} \\ \lambda_1 + \frac{b_2}{\alpha_1} \\ \lambda_2 + \frac{b_1}{\alpha_2} \\ \lambda_2 + \frac{b_2}{\alpha_2} \end{bmatrix}.$$

It then follows from Assumption 4.1 (iv) and (v) that

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} = - \begin{bmatrix} \frac{2}{\alpha_1 k_1} q_{11} + K_1 r_1 & 0 & r_1 & 0 \\ 0 & K_1 r_0 & 0 & r_0 \\ r_1 & 0 & \frac{2}{\alpha_2 k_2} q_{11} + K_2 r_1 & 0 \\ 0 & r_0 & 0 & K_2 r_0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 + \frac{b_1}{\alpha_1} \\ \lambda_1 + \frac{b_2}{\alpha_1} \\ \lambda_2 + \frac{b_1}{\alpha_2} \\ \lambda_2 + \frac{b_2}{\alpha_2} \end{bmatrix}, \quad (14)$$

where

$$\begin{aligned} K_1 &:= \frac{1}{k_1} + 1, \\ K_2 &:= \frac{1}{k_2} + 1. \end{aligned}$$

4.2 Mathematical definitions of diversity and stability

We now define the diversity of players and the stability of an equilibrium on NEP (11). Let X be a group of players, and T_X be the number of players in X. Then, let m_X be $\frac{1}{T_X} \sum_{t_X=1}^{T_X} \alpha_{t_X}$, the average of the players' cost parameters α_{t_X} $(t_X = 1, 2, \cdots, T_X)$, and ${s_X}^2$ be the variance $\frac{1}{T_X} \sum_{t_X=1}^{T_X} (\alpha_{t_X} - m_X)^2$.

We first define the diversity of players.

Definition 4.1 *Suppose that there are players groups U and W. Then, we say that the group U has more diversity than the group W if*

$$s_U > s_W.$$

Next, we define the stability of an equilibrium. Then, in a group X, since the solution of NEP (11) is a function of m_X, s_X and random variables $r_1, r_2, \cdots, r_N$, it can be written as $x_X^*(m_X, s_X, \boldsymbol{r})$. Moreover, let $p(\boldsymbol{r})$ be the concurrent probability density function of random variables $r_1, r_2, \cdots, r_N$, $E_{\boldsymbol{r}}[S_X(m_X, s_X, \boldsymbol{r})_i]$ be $\int_{\boldsymbol{r}} S_X(m_X, s_X, \boldsymbol{r})_i \cdot p(\boldsymbol{r}) d\boldsymbol{r}$, the expectation of $S_X(m_X, s_X, \boldsymbol{r})_i := \sum_j x_X^*(m_X, s_X, \boldsymbol{r})_i^j$, and $V_{\boldsymbol{r}}[S_X(m_X, s_X, \boldsymbol{r})_i]$ be the variance $\int_{\boldsymbol{r}} (S_X(m_X, s_X, \boldsymbol{r})_i - E_{\boldsymbol{r}}[S_X(m_X, s_X, \boldsymbol{r})_i]^2 \cdot p(\boldsymbol{r}) d\boldsymbol{r}$.

Definition 4.2 *Suppose that there are players groups U and W. Then, we say that the group U has more equilibria stability than the group W if, for any $i = 1, 2, \cdots, N$,*

$$V_{\boldsymbol{r}}[S_U(m_U, s_U, \boldsymbol{r})_i] < V_{\boldsymbol{r}}[S_W(m_W, s_W, \boldsymbol{r})_i].$$

Definition 4.1 means that diversity of players is scattering of players' criteria. Definition 4.2 defines stability of an equilibrium solution by how the solution does not vary when the random variables vary.

From the definition, we will study the behavior of the variance of equilibria. More precisely, let m be the average of cost parameters $(\alpha_1 \leq \alpha_2)$ and let s^2 be the variance. Then, if we can mathematically prove that $V_{r_1}[S(s, r_1; m)]$, the variance of $S(s, r_1; m) := x_1^1 + x_1^2$ with respect to r_1, monotonically decreases with respect to s, then we can calculate that diversity yields stability under the assumption that m is constant.

4.3 Sufficient conditions for monotonically decreasing of the variance of a function including non-negative random variables

In this subsection, including negative random variables, we present sufficient conditions under which variance $V_r[f(s, r; m)]$ with respect to r monotonically decreases with respect to s, where a function $f(\cdot, \cdot; m) : \mathbb{R}^2 \to \mathbb{R}$ with a give m and r is a non-negative random variable.

Suppose the following holds.

- The function $f(\cdot, \cdot; m)$ is defined on a subset H of the set $H_0 := \{(s, r) \in \mathbb{R}^2 \mid r \geq 0\}$, where H includes $r \geq 0$.
- The variable r is a random variable whose probability distribution can be arbitrarily defined on the interval $[0, \infty)$.

For our purpose, we need the following 4 assumptions.

Assumption 4.2 *For each m, the functions $f(s, r; m)$ and $f(s, r; m)^2$ are continuous and partial differentiable on H.*

Assumption 4.3 **(i)** *The limit $\lim_{r\to\infty} f(s, r; m)$ exists.*

(ii) *The function $(f(s, r; m) - E_r[f(s, r; m)]) \cdot p(r)$ is continuous on the set H, where $E_r[f(s, r; m)]$ is the expectation of $f(s, r; m)$ with respect to r, and $p(r)$ is a probability density function of r.*

(iii) *The function $f(s, r; m)$ is a strictly monotonically decreasing function with respect to r.*

Assumption 4.4 **(i)** *The function $\frac{\partial}{\partial r} f(s, r; m)$ is continuous on H.*

(ii) *$\frac{\partial}{\partial r}(\frac{\partial}{\partial s} f(s, r; m)) = \frac{\partial}{\partial s}(\frac{\partial}{\partial r} f(s, r; m))$ on the set H.*

Assumption 4.5 *For each m, $\frac{\partial}{\partial s}(\frac{\partial}{\partial r} f(s, r; m)) \geq 0$ on H.*

Under Assumption 4.3, we get the following lemma.

Lemma 4.1 *Suppose Assumption 4.3 holds. There exists a unique $r > 0$ where the sign of the function $(f(s, r; m) - E_r[f(s, r; m)]) \cdot p(r)$ changes, and the sign changes from positive to negative then.*

Proof. Let $h(r) := f(s, r; m) - E_r[f(s, r; m)]$. Then, from Assumption 4.3 (i) and (ii), we have

$$\begin{aligned}
h(0) &= f(s,0;m) - \int_0^\infty f(s,r;m)\cdot p(r)dr \\
&> f(s,0;m) - \int_0^\infty f(s,0;m)\cdot p(r)dr \\
&= f(s,0;m) - f(s,0;m)\int_0^\infty p(r)dr \\
&= f(s,0;m) - f(s,0;m) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{r\to\infty} h(r) &= \lim_{r\to\infty} f(s,r;m) - \int_0^\infty f(s,r;m)\cdot p(r)dr \\
&< \lim_{r\to\infty} f(s,r;m) - \lim_{r\to\infty} f(s,r;m)\int_0^\infty p(r)dr \\
&= \lim_{r\to\infty} f(s,r;m) - \lim_{r\to\infty} f(s,r;m) \\
&= 0.
\end{aligned}$$

Moreover, since $f(s,r;m)$ is a strictly monotonically decreasing function with respect to r, it is clear that $h(r)$ is also a strictly monotonically decreasing function with respect to r.

From the above, as the function $f(s,r;m)$ is the strictly monotonically decreasing function with respect to $r \geq 0$, there exists a unique $r > 0$ where the sign of the function $h(r)$ changes, and the sign changes from positive to negative then.

Now $p(r) \geq 0$ where $r \geq 0$ because an arbitrary probability density function is non-negative in the domain. Therefore, how the sign of $h(r)\cdot p(r)$ changes is same with that of $h(r)$. ∎

Next, we give a technical lemma.

Lemma 4.2 *Let $g_1(r), g_2(r) : \mathbb{R} \to \mathbb{R}$ be functions such that*

$g_1(r)$ **:** *on the interval $[0,\infty)$, the function that is continuous and has a unique $r > 0$ where the sign changes, and the sign changes from positive to negative then;*

$g_2(r)$ **:** *on the interval $[0,\infty)$, the function that is continuous and monotonically increasing.*

Then, there exists a constant U such that

$$\int_0^\infty g_1(r)g_2(r)dr \leq U \int_0^\infty g_1(r)dr.$$

Proof. Let $\bar{r}$ be a variable r on the interval $[0, \infty)$ where the sign of the function $g_1(r)$ changes.

If $0 \leq r \leq \bar{r}$, $g_1(r) \leq 0$, and $g_2(\bar{r}) \leq g_2(r)$, then

$$\int_0^{\bar{r}} g_1(r)g_2(r)dr \leq g_2(\bar{r}) \int_0^{\bar{r}} g_1(r)dr. \tag{15}$$

If $\bar{r} \leq r$, $g_1(r) \geq 0$, and $g_2(r) \leq g_2(\bar{r})$, then

$$\int_{\bar{r}}^{\infty} g_1(r)g_2(r)dr \leq g_2(\bar{r}) \int_{\bar{r}}^{\infty} g_1(r)dr. \tag{16}$$

Therefore, as we sum up formulas (15) and (16),

$$\int_0^{\infty} g_1(r)g_2(r)dr \leq g_2(\bar{r}) \int_0^{\infty} g_1(r)dr.$$

Therefore, $U = g_2(\bar{r})$ satisfies

$$\int_0^{\infty} g_1(r)g_2(r)dr \leq U \int_0^{\infty} g_1(r)dr.$$

∎

Note that the function $(f(s,r;m) - E_r[f(s,r;m)]) \cdot p(r)$ satisfies conditions on $g_1(r)$ under Assumption 4.3, and the function $\frac{\partial}{\partial s} f(s,r;m)$ satisfies conditions on $g_2(r)$ under Assumptions 4.4 and 4.5.

From this series of lemmas, we get the following theorem.

Theorem 4.1 *Suppose Assumption 4.2, 4.3, 4.4, and 4.5 hold. Then, $V_r[f(s,r;m)]$, the variance of $f(s,r;m)$ with respect to $r \geq 0$, monotonically decreases with respect to s for any m, on the set H.*

Proof. By Assumption 4.2, an order of differential calculus and integral calculus is exchangeable. Therefore,

$$\begin{aligned}
\frac{\partial}{\partial s}\left(V[f(s,r;m)]\right) &= \frac{\partial}{\partial s}\left(E_r[f(s,r;m)^2] - (E_r[f(s,r;m)])^2\right) \\
&= \frac{\partial}{\partial s}\left(\int_0^{\infty} (f(s,r;m))^2 \cdot p(r)dr - \left(\int_0^{\infty} f(s,r;m) \cdot p(r)dr\right)^2\right) \\
&= \int_0^{\infty} \frac{\partial}{\partial s}(f(s,r;m))^2 \cdot p(r)dr - 2\left(\int_0^{\infty} f(s,r;m) \cdot p(r)dr\right) \cdot \\
&\qquad\qquad \left(\int_0^{\infty} \frac{\partial}{\partial s} f(s,r;m) \cdot p(r)dr\right) \\
&= \int_0^{\infty} 2f(s,r;m)\frac{\partial}{\partial s} f(s,r;m) \cdot p(r)dr - 2E_r[f(s,r;m)] \cdot \\
&\qquad\qquad \left(\int_0^{\infty} \frac{\partial}{\partial s} f(s,r;m) \cdot p(r)dr\right) \\
&= 2\int_0^{\infty} (f(s,r;m) - E_r[f(s,r;m)]) \cdot p(r) \cdot \frac{\partial}{\partial s} f(s,r;m)dr = (*).
\end{aligned}$$

By Assumption 4.3, we can use Lemma 4.1. Moreover, by Assumption 4.4 and 4.5, $\frac{\partial}{\partial r}(\frac{\partial}{\partial s}f(s,r;m)) \leq 0$ on the set H. Therefore, by Lemma 4.2 and $(*)$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(V[f(s,r;m)]\right) &\leq 2\cdot U \cdot \int_0^\infty (f(s,r;m) - E_r[f(s,r;m)])\cdot p(r)dr \\ &= 2\cdot U\cdot (E_r[f(s,r;m)] - E_r[f(s,r;m)]) = 0.\end{aligned}$$

∎

Theorem 4.1 implies Assumption 4.2, 4.3, 4.4 and 4.5 are sufficient conditions under which $V_r[f(s,r;m)]$, the variance of the function $f(m,s,r)$ with respect to r, monotonically decreases with respect to s when we fix the variable m. In other words, if $S(s,r_1;m) = x_1^1 + x_1^2$ satisfies Assumption 4.2, 4.3, 4.4 and 4.5, then the diversity of players yields the stability of equilibria.

5 Diversity and stability in AEPs and TEPs

In this section, we apply the preceding results in Section 4 to AEPs and TEPs. First, we formulate the problem and show that it is a special case of NEP (11). Moreover, we present a mathematical expression of the equilibrium solution of that problem. Next, we consider conditions for satisfying the assumptions in Section 4.3, and get more concrete sufficient conditions under which diversity of players yields stability of Nash equilibria. Consequently, we analyze the relation between the diversity and the stability in AEPs and TEPs.

5.1 AEPs with Markowitz's mean-variance model

We use Markowitz's mean-variance model [4] for expressing each player's cost function.

5.1.1 Cost functions in AEPs

Suppose that M investors want to decide how they distribute their investment to N kinds of assets. Suppose also that their investment is non-negative, and let the following:

x_i^j : Investor j's investment for Asset i;
X_i : the sum of investment for Asset i ($i.e.$ $\sum_j x_i^j$);
p^j : Investor j's investment capital (a positive constant),
$(i = 1, \cdots, N,\ j = 1, \cdots, M)$.

Then, we define Asset i's return $\rho_i(x)$ as follows[2]:

[2] In general, we should define the return by $\rho_i(x) = \frac{\text{(Asset } i\text{'s future price)} - \text{(Asset } i\text{'s equilibrium price)}}{\text{(Asset } i\text{'s equilibrium price)}}$, however, the model does not become a special case of NEP (11) then.

$$\rho_i(x) \quad = \quad \frac{\text{(Asset } i\text{'s future price)} - \text{(Asset } i\text{'s equilibrium price)}}{\text{(Asset } i\text{'s future price)}}. \tag{17}$$

Moreover, we define Asset i's future price and Asset i's equilibrium price S_i as follows:

$$\begin{aligned}
\text{(Asset } i\text{'s future price)} &= (1+R_i)\tilde{S}_i, \\
\text{(Asset } i\text{'s future price) } S_i &= (\textstyle\sum_{i'=1}^{N} \tilde{S}_{i'})\frac{X_i}{\sum_{i'=1}^{N} X_{i'}} = \frac{\sum_{i'=1}^{N} \tilde{S}_{i'}}{\sum_{j'=1}^{M} p^{j'}} X_i,
\end{aligned}$$

where

$$\begin{aligned}
R_i &: \text{ Asset } i\text{'s rate of return;} \\
\tilde{S}_i &: \text{ Asset } i\text{'s true value.}
\end{aligned}$$

Therefore, we can transform $\rho_i(x)$ as follows:

$$\begin{aligned}
\rho_i(x) &= \frac{(1+R_i)\tilde{S}_i - S_i}{(1+R_i)\tilde{S}_i} \\
&= 1 - \frac{S_i}{(1+R_i)\tilde{S}_i} \\
&= 1 - \frac{\sum_{i'} \tilde{S}_{i'}}{(1+R_i)\tilde{S}_i(\sum_{j'} p^{j'})} X_i \\
&= 1 - r_i X_i, \qquad (18)
\end{aligned}$$

where

$$r_i = \frac{\sum_{i'} \tilde{S}_{i'}}{(1+R_i)\tilde{S}_i(\sum_{j'} p^{j'})}$$

is a positive random variable.

Now suppose that each investor decides his/her investment strategy according to the mean-variance model. Then, we can define the cost function $C_j(x^j; x^{-j})$ that Investor j minimizes as follows:

$$\begin{aligned}
C_j(x^j; x^{-j}) &= \text{(investment risk)} - \text{(investment return)} \\
&= \frac{1}{\alpha_j} \left(\frac{x^j}{p^j}\right)^\top Q \left(\frac{x^j}{p^j}\right) - \rho(x^j; x^{-j})^\top \left(\frac{x^j}{p^j}\right), \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_j &: \text{ Investor } j\text{'s risk preference;} \\
Q &: \text{ the variance-covariance matrix for a rate of assets' growth (semidefinite).}
\end{aligned}$$

The risk preference is a weight for an investment risk, which we innovate to quantitatively express a difference of values on the risk between investors.

As we substitute formula (18) for formula (19),

$$\begin{aligned} C_j(x^j;x^{-j}) &= \frac{1}{\alpha_j {p^j}^2} {x^j}^\top Q x^j + \frac{1}{p^j}\left(\sum_{j'=1}^{M} x^{j'}\right)^\top R x^j - \frac{x_1^j+\cdots+x_n^j}{p^j} \\ &= \frac{1}{\alpha_j {p^j}^2} {x^j}^\top Q x^j + \frac{1}{p^j}\left(\sum_{j'=1}^{M} x^{j'}\right)^\top R x^j - 1, \end{aligned}$$

where R is a diagonal matrix whose diagonal components are $r_1, \cdots, r_n$. We can add and multiply constants to a cost function because the optimal solution does not vary, and hence, we define the cost function of this problem as follows:

$$C_j(x^j;x^{-j}) = \left(\sum_{j'=1}^{M} x^{j'}\right)^\top R x^j + \frac{1}{\alpha_j p^j} {x^j}^\top Q x^j.$$

5.1.2 Mathematical expression of market prices equilibrium

In the same way as Section 3.3, as we get some investors who have the same risk preference α_j and investment capital p^j together, and classify M investors to virtual T class investors, we can reformulate the model as the following $\bar{\boldsymbol{F}}_t(\bar{x}^t;\bar{x}^{-t})$ in this case:

$$\bar{\boldsymbol{F}}_t(\bar{x}^t;\bar{x}^{-t}) = \frac{1}{k_t} R\bar{x}^t + R\sum_{t'=1}^{T} \bar{x}^{t'} + \frac{2}{\alpha_t \bar{p}^t} Q\bar{x}^t. \tag{20}$$

As we compare formula (12) to formula (20), we see that this problem is the case that $b=0$ and the third term $\frac{1}{\alpha_t k_t}$ is replaced to $\frac{1}{\alpha_t \bar{p}_t}$. Therefore, as we substitute $b=0$ and replace $\frac{1}{\alpha_t k_t}$ to $\frac{1}{\alpha_t \bar{p}_t}$ in formula (14), we get the equilibrium solution of this AEP as follows under Assumption 4.1[3]:

$$\begin{aligned} \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} &= -\begin{bmatrix} \frac{2}{\alpha_1 \bar{p}^1} q_{11} + K_1 r_1 & 0 & r_1 & 0 \\ 0 & K_1 r_0 & 0 & r_0 \\ r_1 & 0 & \frac{2}{\alpha_2 \bar{p}^2} q_{11} + K_2 r_1 & 0 \\ 0 & r_0 & 0 & K_2 r_0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{bmatrix} \\ &= -r_0 \begin{bmatrix} \frac{2}{\alpha_1}\cdot\frac{q_{11}}{\bar{p}^1 r_0} + K_1 \frac{r_1}{r_0} & 0 & \frac{r_1}{r_0} & 0 \\ 0 & K_1 & 0 & 1 \\ \frac{r_1}{r_0} & 0 & \frac{2}{\alpha_2}\cdot\frac{q_{11}}{\bar{p}^2 r_0} + K_2\frac{r_1}{r_0} & 0 \\ 0 & 1 & 0 & K_2 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{bmatrix}. \end{aligned}$$

[3]These assumptions correspond to regarding Asset 2 as an asset that has a constant growth rate and no risk. (ex. cash)

Let $m > 0$ be the average of 2 class investors' risk preferences $(0 < \alpha_1 \leq \alpha_2)$, and s^2 be the variance. Then, we get

$$\alpha_1 = m - s,$$

$$\alpha_2 = m + s.$$

Suppose also that Class 1's investment capital is equal to Class 2's investment capital,

$$\bar{p}^1 = \bar{p}^2 = p, \tag{21}$$

$q := \frac{q_{11}}{pr_0}$, and $r := \frac{r_1}{r_0}$. Then, we get the equilibrium solution as follows:

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} = -r_0 \begin{bmatrix} \frac{2q}{m-s} + K_1 r & 0 & r & 0 \\ 0 & K_1 & 0 & 1 \\ r & 0 & \frac{2q}{m+s} + K_2 r & 0 \\ 0 & 1 & 0 & K_2 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{bmatrix}.$$

As we calculate λ_1, λ_2 by $\bar{\boldsymbol{G}}(x) = 0$ and the inverse matrix, we get $S(s, r; m) := x_1^1 + x_1^2$, the variable part of Asset 1's equilibrium price $S_1 = \frac{\sum_{i'=1}^{2} \tilde{S}_{i'}}{\sum_{j'=1}^{2} p^{j'}} (x_1^1 + x_1^2)$, as follows:

$$S(s, r; m) = \frac{D(s; m)r + E(s; m)}{A(s; m)r^2 + B(s; m)r + C(s; m)}, \tag{22}$$

where

$$\begin{aligned} A(s; m) &= (m^2 - s^2)(K_1 K_2 - 1), \\ B(s; m) &= 2q\{(m + s)K_2 + (m - s)K_1\} + 2(m^2 - s^2)(K_1 K_2 - 1), \\ C(s; m) &= 4q^2 + 2q\{(m + s)K_2 + (m - s)K_1\} + (m^2 - s^2)(K_1 K_2 - 1), \\ D(s; m) &= 2p(m^2 - s^2)(K_1 K_2 - 1), \\ E(s; m) &= 2pq\{(m - s)(K_1 + 1) + (m + s)(K_2 + 1)\} + 2p(m^2 - s^2)(K_1 K_2 - 1). \end{aligned}$$

5.1.3 Diversity of investors and stability of market prices equilibrium

From the formula meaning, in AEPs, diversity means the diversity of investors' risk preferences, and stability means the stability of market prices equilibrium. We want to analyze the relation between them, and hence, we first organize relational formulas generalized from parameter conditions.

We define a domain of function $S(s, r; m)$ (22) by H. The parameters m, s, and r exist on the set H, and hence,

$$m > 0,\ 0 \leq s < m,\ r \geq 0. \tag{23}$$

By $K_t = \frac{1}{k_t} + 1$ and $k_t \geq 1$,

$$1 < K_1 \leq 2,\ 1 < K_2 \leq 2. \tag{24}$$

We can suppose $q > 0$ because the case $q = 0$ is included in the proof on TEPs in the next subsection. Moreover, by $p > 0$, for any m and s on the set H,

$$A(s;m) = (m^2 - s^2)(K_1K_2 - 1) > 0, \tag{25}$$
$$B(s;m) = 2q\{(m+s)K_2 + (m-s)K_1\} + 2(m^2 - s^2)(K_1K_2 - 1) > 0, \tag{26}$$
$$C(s;m) = 4q^2 + 2q\{(m+s)K_2 + (m-s)K_1\} + (m^2 - s^2)(K_1K_2 - 1) > 0, \tag{27}$$
$$D(s;m) = 2p(m^2 - s^2)(K_1K_2 - 1) > 0, \tag{28}$$
$$E(s;m) = 2pq\{(m-s)(K_1+1) + (m+s)(K_2+1)\} + 2p(m^2 - s^2)(K_1K_2 - 1) > 0. \tag{29}$$

In the following, for example, let A be $A(s;m)$.

Now $S(s,r;m)$ (22), the variable part of Asset 1's equilibrium price, satisfies Assumption 4.2, 4.3, and 4.4.

Lemma 5.1 *Function $S(s,r;m)$ (22) satisfies Assumption 4.2, 4.3 and 4.4.*

Proof. In general, a rational function is a regular function expect for points that the denominator becomes 0. If a function is a regular function, the function is clearly continuous and partial differentiable at any points in the domain. By $r \geq 0$, $Ar^2 + Br + C > 0$. Therefore, each of $S(s,r;m)$ and $S(s,r;m)^2$ is a regular function on the set H. Therefore, $S(s,r;m)$ satisfies Assumption 4.2.

Regarding Assumption 4.3,

$$\lim_{r\to\infty} S(s,r;m) = \lim_{r\to\infty} \frac{Dr + E}{Ar^2 + Br + C} = 0 \qquad (\because (25) - (29)),$$

and hence, $S(s,r;m)$ satisfies (i), and it is clear that $S(s,r;m)$ satisfies (ii). We prove that $S(s,r;m)$ satisfies (iii) as follows.

$$\frac{\partial}{\partial r} S(s,r;m) = \frac{1}{(Ar^2 + Br + C)^2}\{D(Ar^2 + Br + C) - (Dr + E)(2Ar + B)\},$$
$$(\text{numerator}) = -ADr^2 - 2AEr + CD - BE,$$

then

$$\begin{aligned}
CD - BE = {} & 8pq^2(m^2 - s^2)(K_1K_2 - 1) \\
& - 8pq^2\{(m-s)K_1 + (m+s)K_2\}\{(m-s)(K_1+1) + (m+s)(K_2+1)\} \\
& + 4pq\{(m-s)K_1 + (m+s)K_2\}(m^2 - s^2)(K_1K_2 - 1) \\
& - 4pq\{(m-s)(2K_1+1) + (m+s)(2K_2+1)\}(m^2 - s^2)(K_1K_2 - 1) \\
& + 2p(m^2 - s^2)^2(K_1K_2 - 1)^2 \\
& - 4p(m^2 - s^2)^2(K_1K_2 - 1)^2 \\
< {} & 0 \qquad (\because p > 0,\ q > 0,\ (24)).
\end{aligned}$$

Therefore, by $r \geq 0$ and formulas (25)-(29), $\frac{\partial}{\partial r} S(s,r;m) < 0$. Accordingly, $S(s,r;m)$ satisfies Assumption 4.3.

Regarding Assumption 4.4, to prove (i), $\frac{\partial}{\partial r}S(s,r;m)$ should be continuous at any points on the set H, and to prove (ii), each of $\frac{\partial}{\partial r}(\frac{\partial}{\partial s}S(s,r;m))$ and $\frac{\partial}{\partial s}(\frac{\partial}{\partial r}S(s,r;m))$ should be continuous at any points on the set H. The function $\frac{\partial}{\partial r}S(s,r;m)$ is a rational function whose denominator is $(Ar^2+Br+C)^2$. Each of $\frac{\partial}{\partial r}(\frac{\partial}{\partial s}S(s,r;m))$ and $\frac{\partial}{\partial s}(\frac{\partial}{\partial r}S(s,r;m))$ is a rational function whose denominator is $(Ar^2+Br+C)^3$. Therefore, in the same way as the above proof for Assumption 4.2, we get that they are regular functions, and we can prove that $S(s,r;m)$ satisfies Assumption 4.4. ∎

Function $S(s,r;m)$ (22), on the other hand, does not always satisfy Assumption 4.5. However, we can show a sufficient condition for the satisfaction.

Lemma 5.2 *Let k_1 and k_2 be the number of investors in Class 1 and Class 2, respectively, m be the average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$), q_{11} be the $(1,1)$ component of the 2×2 matrix Q, p be the investment capital in each of the 2 classes, and r_0 be the $(2,2)$ component of the 2×2 matrix R. If $k_1 \geq k_2$ and $m \geq \frac{2q_{11}}{pr_0}$, then function $S(s,r;m)$ (22) satisfies Assumption 4.5.*

Proof.

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial r}S(s,r;m)\right) = \frac{N_1+N_2+N_3+N_4}{(Ar^2+Br+C)^3},$$

where for example, let A' be the partial differentiation of $A(s;m)$ by s,

$$\begin{aligned} N_1 &= -(AD)'B - 2(AE)'A + 2ADB' + 4AEA', \\ N_2 &= -(AD)'C - 2(AE)'B + (CD-BE)'A + 2ADC' + 4AEB' - 2(CD-BE)A', \\ N_3 &= -2(AE)' + (CD-BE)'B + 4AEC' - 2(CD-BE)B', \\ N_4 &= (CD-BE)'C - 2(CD-BE)C'. \end{aligned}$$

Then, if the formulas N_1, N_2, N_3, and N_4 are non-negative under $k_1 \geq k_2$ and $m \geq \frac{2q_{11}}{p}$, this proof will be completed. Therefore, we give sufficient conditions under which each of the formulas N_1, N_2, N_3, and N_4 become non-negative.

$$\begin{aligned} N_1 &= -(AD)'B - 2(AE)'A + 2ADB' + 4AEA' \\ &= 2pA\{AB_2' - A'(B_2 - 4mq)\}, \end{aligned}$$

where B_2 is the first term $2q\{(m+s)K_2 + (m-s)K_1\}$ of B. By

$$\begin{aligned} B_2 - 4mq &= 2q\{(m+s)K_2 + (m-s)K_1 - 2m\} \\ &> 2q\{(m+s)\cdot 1 + (m-s)\cdot 1 - 2m\} \quad (\because (24)) \\ &= 0, \end{aligned}$$

$p > 0$, $A > 0$, and clearly $A' \leq 0$, if $B_2' \geq 0$, then $N_1 \geq 0$.

$$
\begin{aligned}
N_2 &= -(AD)^{'}C - 2(AE)^{'}B + (CD-BE)^{'}A + 2ADC^{'} + 4AEB^{'} - 2(CD-BE)A^{'} \\
&= -6pA\{AB_2^{'} + 2mqB_2^{'} - A^{'}(B_2 - 4mq)\} - 24pq^2AA^{'} \\
&\geq -6pA\{AB_2^{'} + 2mqB_2^{'} - A^{'}(B_2 - 4mq)\} \quad (\because p > 0,\ q > 0,\ A > 0,\ A^{'} \leq 0).
\end{aligned}
$$

By $p > 0,\ A > 0,\ q \geq 0,\ m > 0,\ B_2 - 4mq > 0$, and $A^{'} \leq 0$, if $B_2^{'} \geq 0$, then $N_2 \geq 0$.

$$
\begin{aligned}
N_3 &= -2(AE)^{'} + (CD-BE)^{'}B + 4AEC^{'} - 2(CD-BE)B^{'} \\
&= 2p\{-16mq^3A^{'} + 2mqB_2B_2^{'} + 3A^2B_2^{'} - 3AA^{'}(-4mq + 8q^2 + B_2) + 12q(m-q)AB_2^{'}\},
\end{aligned}
$$

$$
\begin{aligned}
-4mq + 8q^2 + B_2 &> 4q(-m + 2q + m) \quad (\because B_2 > 4mq) \\
&= 8q^2 \ > \ 0 \quad (\because q > 0).
\end{aligned}
$$

By $p > 0,\ A > 0,\ q > 0,\ m > 0,\ A^{'} \leq 0$, and $B_2 > 4mq > 0$, if $B_2^{'} \geq 0$ and $m \geq q$, then $N_3 \geq 0$.

$$
\begin{aligned}
N_4 &= (CD-BE)^{'}C - 2(CD-BE)C^{'} \\
&= 2p\{4q^2B_2^{'}(B_2 - 2mq) - 16q^3(m-q)A^{'} + 6q(m-2q)B_2^{'}A + 2mqB_2^{'}B_2 \\
&\qquad -A^{'}A(-4mq + 12q^2 + B_2) + B_2^{'}A^2\},
\end{aligned}
$$

$$
\begin{aligned}
B_2 - 2mq &> 4mq - 2mq \quad (\because B_2 > 4qm) \\
&= 2mq \ > \ 0 \quad (\because m > 0,\ q > 0).
\end{aligned}
$$

By

$$
\begin{aligned}
-4mq + 12q^2 + B_2 &> 4q(-m + 3q + m) \quad (\because B_2 > 4mq) \\
&= 12q^2 \ > \ 0 \quad (\because q > 0),
\end{aligned}
$$

$p > 0,\ A > 0,\ q > 0,\ m > 0,\ A^{'} \leq 0$, and $B_2 > 0$, if $B_2^{'} \geq 0$ and $m \geq 2q$, then $N_4 \geq 0$. From the above, if $B_2^{'} \geq 0$ and $m \geq 2q$, then each of the formulas N_1, N_2, N_3, and N_4 become non-negative. By $B_2^{'} = 2q(K_2 - K_1)$, we get $B_2^{'} \geq 0 \Leftrightarrow K_2 \geq K_1 \Leftrightarrow k_1 \geq k_2$, and we defined $q = \frac{q_{11}}{pr_0}$. Therefore, this proof is completed. ∎

The following theorem gives sufficient conditions under which the diversity of investors' risk preferences yields the stability of the equilibrium price.

Theorem 5.1 *Let k_1 and k_2 be the number of investors in Class 1 and Class 2, respectively, m be the average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$), q_{11} be the $(1,1)$ component of the 2×2 matrix Q, p be the investment capital in each of the 2 classes, and r_0 be the $(2,2)$ component of the 2×2 matrix R. If $k_1 \geq k_2$ and $m \geq \frac{2q_{11}}{pr_0}$, then the diversity of investors' risk preferences yields the stability of the equilibrium price under the assumption that m is constant.*

Proof. By Lemmas 5.1 and 5.2, function $S(s, r; m)$ (22) satisfies the assumptions of Theorem 4.1. Therefore, $V_r[S(s, r; m)]$, the variance of the variable part of Asset 1's equilibrium price $S(s, r; m)$ with respect to $r \geq 0$, monotonically decreases with respect to s, the standard deviation of 2 class investors' risk preferences, in the 2 classes for any m in the domain H.

By Definitions 4.1 and 4.2, we get this result implies that the diversity of investors' risk preferences yields the stability of the equilibrium price under the assumption that m is constant. ∎

From the above, we can consider the relation of diversity and stability on AEPs as follows:

- Under the assumption that class's investment capital is equal among investors who have the same risk preference and investment capital, if the number of investors who like risk is more than that of investors who dislike it, then existence of various investors makes assets' market prices stable.
- Even if the number of investors who like risk is less than that of investors who dislike it, then there exist some cases that existence of various investors makes assets' market prices stable.

When market prices of assets violently fluctuate, since it gives a lot of benefits to a few people, a central bank adopts a policy that stabilizes the fluctuating. From the above result, under the assumption that the class's investment capital is equal among investors who have the same risk preference and investment capital, we can often make the prices stable by acting contrary to general investors. If a central bank adopts such a policy, we may be able to prevent the next violence fluctuating.

5.2 TEPs with Gabriel and Bernstein's model

We use Gabriel and Bernstein's model [3], in which driver's cost consists of both time and a toll, for expressing each player's cost function.

5.2.1 Cost functions in TEPs

Suppose that there are M managers who want their cars to go from a place A to B, and N paths between A and B that are not intersecting each other. Let the following for any $i = 1, \cdots, N$ and $j = 1, \cdots, M$:

x_i^j : the number of Manager j's cars in Path i;
p^j : Manager j's traffic demand, i.e. the number of cars that he/she wants to go (positive), $(i = 1, \cdots, N,\ j = 1, \cdots, M)$.

Note that x_i^j is continuous. Suppose also the following for any $i = 1, \cdots, N$ and $j = 1, \cdots, M$.

- The variable r_i is a positive random variable, and $\sum_{j=1}^M x_i^j$ is Path i's traffic. Now we define the required time in Path i by $r_i \sum_{j=1}^M x_i^j$.
- The p^j cars that Manager j wants to go are in the same class.
- Path i's toll for Manager i's cars is $\frac{b_i'}{\alpha_j} \geq 0$. That is, α_j is the toll weight for the Manager j's car class.

Then, suppose that each manager wants to decide how they distribute their cars to N paths.

Suppose that each manager decides his/her investment strategy according to Gabriel and Bernstein's model. Then, we can define the cost function $C_j(x^j; x^{-j})$ that Manager j minimize as follows:

$$\begin{aligned} C_j(x^j; x^{-j}) &= \text{(sum of required time)} + \text{(sum of tolls that is converted into the value of time)} \\ &= \left(\sum_{j'=1}^{M} x^{j'} \right)^\top R x^j + \frac{1}{\alpha_j} b^\top x^j, \end{aligned}$$

where suppose that the toll vector $\frac{b'}{\alpha_j}$ is converted into the vector $\frac{b}{\alpha_j}$, and R is a diagonal matrix whose diagonal components are $r_1, \cdots, r_n$.

5.2.2 Mathematical expression of traffic equilibrium

In the same way as Section 3.3, as we get some managers that have the same toll weight α_j and traffic demand p^j together, and classify M managers to virtual T class managers, we can reformulate the model as the following $\bar{\boldsymbol{F}}_t(\bar{x}^t; \bar{x}^{-t})$ in this case:

$$\bar{\boldsymbol{F}}_t(\bar{x}^t; \bar{x}^{-t}) = \frac{1}{k_t} R \bar{x}^t + R \sum_{t'=1}^{T} \bar{x}^{t'} + \frac{b}{\alpha_t}. \tag{30}$$

As we compare formula (12) to formula (30), we see that this problem is the case that $Q = O$. Moreover, suppose $b \geq 0$ by the problem settings. Therefore, as we substitute

$q_{11} = 0$ in formula (14), we get the equilibrium solution of this TEP as follows under Assumption 4.1[4]:

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_1^2 \\ x_2^2 \end{bmatrix} = - \begin{bmatrix} K_1 r & 0 & r & 0 \\ 0 & K_1 r_0 & 0 & r_0 \\ r & 0 & K_2 r & 0 \\ 0 & r_0 & 0 & K_2 r_0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 + \frac{b_1}{\alpha_1} \\ \lambda_1 + \frac{b_2}{\alpha_1} \\ \lambda_2 + \frac{b_1}{\alpha_2} \\ \lambda_2 + \frac{b_2}{\alpha_2} \end{bmatrix},$$

where $r := r_1$. In the same way as Section 5.1.2, let $m > 0$ be the average of 2 class managers' toll weights $(0 < \alpha_1 \leq \alpha_2)$, and s^2 the variance. As we calculate λ_1, λ_2 by $\bar{\boldsymbol{G}}(x) = 0$ and the inverse matrix, we get $S(s, r; m) := x_1^1 + x_1^2$, the variable part of Path 1's traffic equilibrium, as follows:

$$S(s, r; m) = \frac{\hat{B}(s; m)}{\hat{A}(s; m)(r + r_0)}, \tag{31}$$

where

$$\begin{aligned} \hat{A}(s; m) &= (m^2 - s^2)(K_1 K_2 - 1), \\ \hat{B}(s; m) &= r_0(m^2 - s^2)(K_1 K_2 - 1)(\bar{p}^1 + \bar{p}^2) - (b_1 - b_2)\{(m + s)(K_2 - 1) + (m - s)(K_1 - 1)\}. \end{aligned}$$

5.2.3 Diversity of tolls for car classes and stability of traffic equilibrium

From the formula meaning, in TEPs, diversity means the diversity of tolls for car classes, and stability means the stability of traffic equilibrium. We want to analyze the relation between them, and hence, we first organize relational formulas generalized from parameter conditions.

We define a domain of function $S(s, r; m)$ (31) by H. The parameters m, s, and r exist on the set H, and hence,

$$m > 0,\ 0 \leq s < m,\ r \geq 0.$$

By $K_t = \frac{1}{k_t} + 1$, and $k_t \geq 1$,

$$1 < K_1 \leq 2,\ 1 < K_2 \leq 2.$$

By Assumption 4.1 (ii) $(x > 0)$,

$$x_1^1 = \frac{r_0 K_1 K_2 (m^2 - s^2)\bar{p}^1 - (b_1 - b_2)K_2(m + s) - r_0(m^2 - s^2)\bar{p}^1 + (b_1 - b_2)(m - s)}{r_0(m^2 - s^2)(r + r_0)(K_1 K_2 - 1)} > 0,$$

$$x_1^2 = \frac{r_0 K_1 K_2 (m^2 - s^2)\bar{p}^2 - (b_1 - b_2)K_1(m - s) - r_0(m^2 - s^2)\bar{p}^2 + (b_1 - b_2)(m + s)}{r_0(m^2 - s^2)(r + r_0)(K_1 K_2 - 1)} > 0,$$

[4]These assumptions correspond to regarding the required time in Path 2 as proportional only to the traffic.

and hence,

$$r_0(m^2 - s^2)\bar{p}^1(K_1K_2 - 1) > (b_1 - b_2)(K_2(m+s) - (m-s)),$$
$$r_0(m^2 - s^2)\bar{p}^2(K_1K_2 - 1) > (b_1 - b_2)(K_1(m-s) - (m+s)).$$

As we add each side,

$$r_0(m^2 - s^2)(K_1K_2 - 1)(\bar{p}^1 + \bar{p}^2) > (b_1 - b_2)\{(m+s)(K_2 - 1) + (m-s)(K_1 - 1)\}.$$

From the above, for any m and s on the set H,

$$\hat{A}(s;m) = (m^2 - s^2)(K_1K_2 - 1) > 0, \quad (32)$$
$$\hat{B}(s;m) = r_0(m^2 - s^2)(K_1K_2 - 1)(\bar{p}^1 + \bar{p}^2) - (b_1 - b_2)\{(m+s)(K_2 - 1) + (m-s)(K_1 - 1)\} > 0. \quad (33)$$

In the following, for example, let $\hat{A}$ be $\hat{A}(s;m)$.

Now $S(s,r;m)$ (31), the variable part of Path 1's traffic equilibrium, satisfies Assumption 4.2, 4.3, and 4.4.

Lemma 5.3 *Function $S(s,r;m)$ (31) satisfies Assumption 4.2, 4.3 and 4.4.*

Proof. In general, a rational function is a regular function expect for points that the denominator becomes 0. If a function is a regular function, the function is clearly continuous and partial differentiable at any points in the domain. By $r \geq 0$, $\hat{A}(r+r_0) > 0$, and hence, each of $S(s,r;m)$ and $S(s,r;m)^2$ is a regular function on the set H. Therefore, $S(s,r;m)$ satisfies Assumption 4.2.

Regarding Assumption 4.3,

$$\lim_{r\to\infty} S(s,r;m) = \lim_{r\to\infty} \frac{\hat{B}}{\hat{A}(r+r_0)} = 0 \qquad (\because (32), (33)),$$

and hence, $S(s,r;m)$ satisfies (i), and it is clear that $S(s,r;m)$ satisfies (ii). We prove that $S(s,r;m)$ satisfies (iii) as follows.

$$\frac{\partial}{\partial r} S(s,r;m) = -\frac{\hat{B}}{\hat{A}(r+r_0)^2} < 0 \quad (\because \hat{A} > 0,\ \hat{B} > 0).$$

Accordingly, we proved that $S(s,r;m)$ satisfies Assumption 4.3.

Regarding Assumption 4.4, to prove (i), $\frac{\partial}{\partial r}S(s,r;m)$ should be continuous at any points on the set H, and to prove (ii), each of $\frac{\partial}{\partial r}(\frac{\partial}{\partial s}S(s,r;m))$ and $\frac{\partial}{\partial s}(\frac{\partial}{\partial r}S(s,r;m))$ should be continuous at any points on the set H. The function $\frac{\partial}{\partial r}S(s,r;m)$ is a rational function whose denominator is $\hat{A}(r+r_0)^2$. Each of $\frac{\partial}{\partial r}(\frac{\partial}{\partial s}S(s,r;m))$ and $\frac{\partial}{\partial s}(\frac{\partial}{\partial r}S(s,r;m))$ is a rational function whose denominator is $\hat{A}^2(r+r_0)^2$. Therefore, in the same way as the above proof for Assumption 4.2, we get that they are regular functions, and we can prove that $S(s,r;m)$ satisfies Assumption 4.4. ∎

Function $S(s,r;m)$ (31), on the other hand, does not always satisfy Assumption 4.5. However, we can show a sufficient condition for the satisfaction.

Lemma 5.4 *Let k_1 and k_2 be the number of investors in Class 1 and Class 2, respectively, b_1 and b_2 be the toll of Path 1 and Path 2, respectively. If $k_1 \geq k_2$ and $b_1 \geq b_2$, then function $S(s, r; m)$ (31) satisfies Assumption 4.5.*

Proof.

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial r}S(s,r;m)\right) = \frac{(b_1 - b_2)(K_1K_2 - 1)\{(K_2 - K_1)(m^2 + s^2) + 2(K_1 + K_2 - 2)ms\}}{\hat{A}^2(r + r_0)^2}.$$

By $m > 0$, $s > 0$, $1 < K_1 \leq 2$, and $1 < K_2 \leq 2$, if $b_1 \geq b_2$ and $K_2 \geq K_1$, then $\frac{\partial}{\partial s}(\frac{\partial}{\partial r}S(s,r;m)) \geq 0$. By $K_2 \geq K_1 \Leftrightarrow k_1 \geq k_2$, this proof is completed. ∎

The following theorem gives sufficient conditions under which the diversity of tolls for car classes yields the stability of traffic equilibrium.

Theorem 5.2 *Let k_1 and k_2 be the number of investors in Class 1 and Class 2, respectively, b_1 and b_2 be the toll of Path 1 and Path 2, respectively. If $k_1 \geq k_2$ and $b_1 \geq b_2$, then the diversity of tolls for car classes yields the stability of traffic equilibrium under the assumption that m is constant.*

Proof. By Lemmas 5.3 and 5.4, function $S(s, r; m)$ (31) satisfies the assumptions of Theorem 4.1. Therefore, $V_r[S(s, r; m)]$, the variance of the variable part of Path 1's traffic equilibrium $S(s, r; m)$ with respect to $r \geq 0$, monotonically decreases with respect to s, the standard deviation of the toll weight, in the 2 classes for any m in the domain H.

By Definitions 4.1 and 4.2, we get this result implies that, under the assumption that m is constant, the diversity of tolls for car classes yields the stability of the traffic equilibrium. ∎

From the above, we can consider the relation of diversity and stability on TEPs as follows:

- When a toll of the path that whose required time randomly varies is more expensive than the other, if the number of managers whose car class require a more toll is more than that of the other managers, then scattering of tolls for car classes makes path traffic stable.
- Even if the number of managers whose car class require a more toll is less than that of the other managers, then there exist some cases that scattering of tolls for car classes makes path traffic stable.

Traffic stabilization allows us to forecast traffic congestion and required time, and hence, it is meaningful to realize it. From the above result, when we compare the path that the required time randomly varies to the path that it does not vary, if the former path's toll is more expensive, then a road company can often make traffic stable by setting scattering of tolls for car classes large. On the other hand, if the former path's toll is cheaper, then it is clear from the deriving process that we can derive the exact opposite result. That is, a road company can often make traffic stable by setting the scattering of tolls for car classes small.

In TEPs, therefore, the relation between the diversity of players and the stability of Nash equilibria is largely dependent on parameters, that is why we cannot point out a general relation. For example, however, when we compare a toll expressway to a free open road, the required time in the expressway varies more randomly because the expressway has more risk of being closed when an traffic accident happen. Therefore, the assumption that a toll of the path whose required time randomly varies is more expensive than the other is satisfied then. We can consider that the assumption is thus often satisfied, and scattering of tolls for car classes often makes path traffic stable in the real world.

6 Numerical experiments

In this section, we conduct numerical experiments for AEPs to show validity of the presented conditions and possibility of relaxation of the presented assumption. More precisely, we numerically solve the AEP, i.e. NEP (11) whose $\bar{\boldsymbol{F}}_t(\bar{x}^t; \bar{x}^{-t})$ is defined by formula (20), for some pairs of each investor's risk preference α_j and each of many enough samples of the rate of return R_i. After that, from the equilibrium solution, we calculate the variance of asset's market price for each variance of risk preference α_j among class investors.

As given in Section 4.1, we can reformulate the NEP into the VIP, and can transform the VIP into MCP (13). In this experiments, in addition, we transform MCP (13) into an unconstrained minimization problem to solve the AEP. The MCP (13) is equivalent to the following problem [2]:

$$\begin{aligned} \min \quad & ||\Phi(x,\lambda)||^2 \\ \text{s.t.} \quad & \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^{TN+T}, \end{aligned} \tag{34}$$

where

$$\Phi(x,\lambda) := \begin{pmatrix} \bar{\boldsymbol{G}}(x) \\ \phi_{FB}(x_1^1, (\bar{\boldsymbol{F}}_1(x))_1 + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_1) \\ \vdots \\ \phi_{FB}(x_N^1, (\bar{\boldsymbol{F}}_1(x))_N + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_N) \\ \phi_{FB}(x_1^2, (\bar{\boldsymbol{F}}_2(x))_1 + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_{N+1}) \\ \vdots \\ \phi_{FB}(x_N^2, (\bar{\boldsymbol{F}}_2(x))_N + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_{2N}) \\ \phi_{FB}(x_1^3, (\bar{\boldsymbol{F}}_3(x))_1 + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_{2N+1}) \\ \vdots \\ \\ \vdots \\ \phi_{FB}(x_N^T, (\bar{\boldsymbol{F}}_T(x))_N + (\nabla\bar{\boldsymbol{G}}(x)\lambda)_{TN}) \end{pmatrix},$$

and suppose that ϕ_{FB} is the Fischer-Burmeister function, i.e. $\phi_{FB}(a,b) := a + b - \sqrt{a^2+b^2}$.

We can solve unconstrained minimization problem (34) by the Quasi-Newton method, etc. [2]. Moreover, as shown in Lemma 3.1, since $\boldsymbol{F}$ is strongly monotone and $\boldsymbol{G}$ is linear, a stationary point of problem (34) is a solution of VIP (9) [2].

We conducted all numerical experiments in MATLAB2012, and used the command *fminunc* included in MATLAB to solve unconstrained minimization problem (34).

6.1 Validity of Theorem 5.1

In this subsection, we confirm that the following sufficient conditions under which the variance of assets' market prices monotonically decreases with respect to the variance of 2 class investors' risk preferences seems correct.

- Let k_1 and k_2 be the number of investors in Class 1 and Class 2, respectively. Then,

$$k_1 \geq k_2. \tag{35}$$

- Let m be the average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$), q_{11} be the $(1,1)$ component of the 2×2 matrix Q, p be the investment capital in each of the 2 classes, and r_0 be the $(2,2)$ component of the 2×2 matrix R. Then,

$$m \geq \frac{2q_{11}}{pr_0}. \tag{36}$$

In relation to that the above conditions, and we conduct 3 experiments. In each experiment, the following parameters are common.

- The average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$) $m = 20$.
- 2 class investors' investment capitals $\bar{p}^1 = \bar{p}^2 = 1$.
- 2 assets' true values $\tilde{S}_1 = \tilde{S}_2 = 1$.

Moreover, we set other parameters as Table 1, and solve problem (34) under formula (20) and Assumption 4.1. Then, suppose that Asset 1's rate of return R_1 accords with $[-0.5,\ 1.5]$ uniform distribution, and we divide the interval $[-0.5,\ 1.5]$ into 100 parts and calculate for each of the 100 samples $-0.48, -0.46, \cdots, 1.5$. Suppose also that Asset 2 is cash, i.e. $R_2 = 0$.

We divide the assumptions in Theorem 5.1 into (A) $k_1 \leq k_2$ and (B) $m \geq \frac{2q_{11}}{pr_0}$. In Experiment 1, the assumptions (A) and (B) are satisfied. In Experiment 2, the assumption (A) is not satisfied. In Experiment 3, the assumption (B) is not satisfied.

We show the result in Table 2. The function $V_r[S(s,r;m)]$ is the variance of $S(s,r_1;m) := x_1^1 + x_1^2$, the variable part of Asset 1's equilibrium price $S_1 = \frac{\sum_{i'=1}^{2}\tilde{S}_{i'}}{\sum_{t=1}^{2}\bar{p}^t}(x_1^1+x_1^2)$, with respect to $r := r_1/r_0$, and the variable s is the standard deviation of 2 class investors' risk preferences. In Experiment 1, $V_r[S(s,r;m)]$ monotonically decreases with respect

Table 1: Parameter values in Experiments 1-3

	Ex.1	Ex.2	Ex.3
k_1	100	1	100
k_2	1	100	1
q_{11}	1	1	40

to s, at the same time, in Experiments 2 and 3, $V_r[S(s,r;m)]$ does not. Therefore, we confirmed that sufficient conditions (35) and (36) under which the variance of assets' market prices monotonically decreases with respect to the variance of 2 class investors' risk preferences seems correct.

Table 2: Values of $V_r[S(s,r;m)]$ at each s in Experiments 1-3

s	0	5	10	15	19
Ex.1	$\mathbf{3.94 \times 10^{-2}}$	3.83×10^{-2}	3.65×10^{-2}	3.27×10^{-2}	2.53×10^{-2}
Ex.2	3.94×10^{-2}	4.00×10^{-2}	4.037×10^{-2}	$\mathbf{4.045 \times 10^{-2}}$	3.67×10^{-2}
Ex.3	3.62×10^{-3}	3.84×10^{-3}	4.02×10^{-3}	4.19×10^{-3}	$\mathbf{4.34 \times 10^{-3}}$

6.2 Relaxation possibility of assumptions in Theorem 5.1

In this subsection, consider if we can relax assumptions in sufficient conditions (35) and (36). By Assumption 4.1, (21), and Theorem 5.1, the following assumptions leave room for relaxation.

- Class 1's investment capital is equal to Class 2's investment capital (i.e. $\bar{p}^1 = \bar{p}^2$).
- The random variable r_2 is constant r_0.
- Consider only Asset 1's variance (risk) (i.e. the components $q_{12} = q_{21} = q_{22} = 0$ of the 2×2 matrix Q).
- The kind of assets $N = 2$ and the number of class investors $T = 2$.
- The average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$) m is constant.

First, under the assumptions the kind of assets $N = 2$, the number of class investors $T = 2$, and that m, the average of 2 class investors' risk preferences, is constant, we conduct 8 experiments. In each experiment, the following parameters are common.

- The average of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$) $m = 20$.
- The components of the 2×2 matrix Q, $q_{12} = q_{21} = 0$.

- 2 assets' true values $\tilde{S}_1 = \tilde{S}_2 = 1$.

Moreover, we set other parameters as Table 3, and solve problem (34) under formula (20) and Assumption 4.1 (i)-(iii). Then, regarding assets' rates of return R_1 and R_2, as Experiments 1-3, we divide the interval $[-0.5,\ 1.5]$ into 100 parts and suppose that the 100 numbers, $-0.48, -0.46, \cdots, 1.5$, randomly appear without overlap in the 100 samples of each of R_1 and R_2.

Table 3: Parameter values in Experiments 4-11

	Ex.4	Ex.5	Ex.6	Ex.7	Ex.8	Ex.9	Ex.10	Ex.11
k_1	100	1	100	1	100	1	100	1
k_2	1	100	1	100	1	100	1	100
$\bar{p}^1$	100	10	100	200	100	10	100	200
$\bar{p}^2$	10	100	200	100	10	100	200	100
q_{11}	1	1	1	1	2	2	2	2
q_{22}	0	0	0	0	1	1	1	1
R_1	$[-0.5,\ 1.5]$ uniform distribution							
R_2	0	0	0	0	$[-0.5,\ 1.5]$ uniform distribution			

In Experiments 4-7, we suppose the assumption $r_2 = r_0$ and relax the assumption $\bar{p}^1 = \bar{p}^2$. In Experiment 4, $k_1 > k_2$ and $\bar{p}^1 > \bar{p}^2$. In Experiment 5, $k_1 < k_2$ and $\bar{p}^1 < \bar{p}^2$. In Experiment 6, $k_1 > k_2$ and $\bar{p}^1 < \bar{p}^2$. In Experiment 7, $k_1 < k_2$ and $\bar{p}^1 > \bar{p}^2$. In Experiments 8-11, we relax the assumptions $r_2 = r_0$ and $\bar{p}^1 = \bar{p}^2$. Experiments 8-11 are similar to Experiments 4-7 respectively, except for the random variable r_2 randomly varies.

We show the result in Table 4. It shows that, if, regarding the number of investors, $k_1 > k_2$ (Experiments 4, 6, 8, and 10), then the function $V_r[S(s, r; m)]$ monotonically decreases with respect to s regardless of the magnitude relation between class's investment capitals $\bar{p}^1$ and $\bar{p}^2$. Therefore, even if we do not suppose the following 3 assumptions.

- Class 1's investment capital is equal to Class 2's investment capital (i.e. $\bar{p}^1 = \bar{p}^2$).
- The random variable r_2 is constant r_0.
- Consider only Asset 1's variance (risk) (i.e. the components $q_{12} = q_{21} = q_{22} = 0$ of the 2×2 matrix Q).

then the condition $k_1 \geq k_2$ may be a sufficient condition under which diversity yields stability.

Next, under the assumptions the kind of assets $N = 2$ and the number of class investors $T = 2$, we relax the assumption m, the average of 2 class investors' risk preferences, is constant. More precisely, at this time, we fix the variance s^2 and vary the average m.

Table 4: Values of $V_{\boldsymbol{r}}[S(s, \boldsymbol{r}; m)]$ at each s in Experiments 4-11

s	0	5	10	15	19
Ex.4	**107**	101	91	69	19
Ex.5	107	110	113	115	**116**
Ex.6	**812**	792	760	710	647
Ex.7	812	824	**826**	811	652
Ex.8	**209**	200	185	150	57
Ex.9	209	215	219	222	**224**
Ex.10	**1582**	1546	1488	1390	1245
Ex.11	1582	1603	**1610**	1593	1346

We conduct 4 experiments. In each experiment, the following parameters are common.

- The variance of 2 class investors' risk preferences ($\alpha_1 \leq \alpha_2$) $s^2 = 100$.
- The components $q_{12} = q_{21} = 0$ of the 2×2 matrix Q.
- 2 assets' true values $\tilde{S}_1 = \tilde{S}_2 = 1$.

Moreover, we set other parameters as Table 5, and solve problem (34) under formula (20) and Assumption 4.1 (i)-(iii). Then, regarding assets' rates of return R_1 and R_2, as the above experiments, we divide the interval $[-0.5,\ 1.5]$ into 100 parts and suppose that the 100 numbers, $-0.48, -0.46, \cdots, 1.5$, randomly appear without overlap in the 100 samples of each of R_1 and R_2.

Table 5: Parameter values in Experiments 12-15

	Ex.12	Ex.13	Ex.14	Ex.15
k_1	100	1	100	1
k_2	1	100	1	100
$\bar{p}^1$	100	10	100	200
$\bar{p}^2$	10	100	200	100
q_{11}	2	2	2	2
q_{22}	1	1	1	1
R_1	$[-0.5,\ 1.5]$ uniform distribution			
R_2	$[-0.5,\ 1.5]$ uniform distribution			

Each of Experiments 12-15 is similar to each of Experiments 8-11 except for fixing s^2 and changing m.

We show the result in Table 6. It shows that the function $V_r[S(s,r;m)]$ monotonically increases with respect to m regardless of the magnitude relation between k_1 and k_2, the number of investors in each of the 2 classes, and the magnitude relation between $\bar{p}^1$ and $\bar{p}^2$, class's investment capitals. Therefore, we may be able to relax the assumption that m is constant into is not increasing.

Table 6: Values of $V_{\boldsymbol{r}}[S(m,\boldsymbol{r};s)]$ at each s in Experiments 12-15

m	15	45	75	105	135
Ex.12	150	224	233	237	**239**
Ex.13	214	231	236	238	**240**
Ex.14	1368	1681	1741	1765	**1779**
Ex.15	1529	1720	1756	1774	**1785**

Next, under the assumptions that m, the average of each class investors' risk preferences, is constant, we relax the assumption the kind of assets $N = 2$ and the number of class investors $T = 2$. In the following, we define $S_1(s,\boldsymbol{r};m) := x_1^1 + x_1^2 + x_1^3$.

We conduct 6 experiments. In each experiment, the following parameters are common.

- The average of 3 class investors' risk preferences ($\alpha_1 \le \alpha_2 \le \alpha_3$) $m = 20$.
- 3 class investors' investment capitals $\bar{p}^1 = \bar{p}^2 = \bar{p}^3 = 100$.
- The 3×3 matrix $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
- 3 assets' true values $\tilde{S}_1 = \tilde{S}_2 = \tilde{S}_3 = 1$.
- 3 assets' rates of return $R_1,\ R_2$ and R_3 accord with $[-0.5,\ 1.5]$ uniform distribution.

Moreover, we set other parameters as Table 7, and solve problem (34) under formula (20) and Assumption 4.1 (ii) and (iii). Then, regarding assets' rates of return R_1, R_2 and R_3, as the above experiments, we divide the interval $[-0.5,\ 1.5]$ into 100 parts and suppose that the 100 numbers, $-0.48, -0.46, \cdots, 1.5$, randomly appear without overlap in the 100 samples of each of R_1, R_2, and R_3.

We show the result in Table 8. It shows that, regarding the number of investors, if k_1 is maximum or $k_1 + k_2$ is comparatively large (Experiments 16-18), then the function $V_{\boldsymbol{r}}[S_1(s,\boldsymbol{r};m)]$ monotonically decreases with respect to s. Although we do not show values, $V_{\boldsymbol{r}}[S_2(s,\boldsymbol{r};m)]$ and $V_{\boldsymbol{r}}[S_3(s,\boldsymbol{r};m)]$, the variance of $S_2(s,\boldsymbol{r};m) := x_2^1 + x_2^2 + x_2^3$ and $S_3(s,\boldsymbol{r};m) := x_3^1 + x_3^2 + x_3^3$ with respect to $\boldsymbol{r}$, respectively, monotonically decreases with respect to s in each of Experiments 16-18. Therefore, even if $N \ge 3$ or $T \ge 3$, it may be a sufficient condition that there are more investors who dislike risk.

Table 7: Parameter values in Experiments 16-21

	Ex.16	Ex.17	Ex.18	Ex.19	Ex.20	Ex.21
k_1	100	100	50	50	1	1
k_2	50	1	100	1	100	50
k_3	1	50	1	100	50	100

Table 8: Values of $V_{\boldsymbol{r}}[S_1(s, \boldsymbol{r}; m)]$ at each s in Experiments 16-21

s	0	5	10	15	19
Ex.16	**1016**	1007	994	974	933
Ex.17	**1016**	1015	1012	991	922
Ex.18	**1016**	1008	996	977	935
Ex.19	1016.2	**1016.3**	1014	994	923
Ex.20	1016	1022	1026	**1027**	1007
Ex.21	1016	1023	1028	**1029**	1007

7 Conclusion

In this paper, we first proposed a non-cooperative game that has a convex quadratic cost function. The cost function has two kinds of parameters: varying at random one and being different on the player one. Moreover, we proved that the unique Nash equilibrium solution exists in the proposed model. Next, we defined stability of players and diversity of Nash equilibria, and mathematically presented sufficient conditions under which diversity of players' criteria yields stability of a Nash equilibrium. After that, as applications of the proposed model, for each of asset equilibrium problems (AEP) and traffic equilibrium problems (TEP), we analyzed the relation between the diversity and the stability. Then, we concluded that, in theory, the diversity often yields the stability. Finally, we conducted numerical experiments for AEPs, and we showed an validity of the presented condition and considered the possibility of relaxation of assumption.

As a future work, we should mathematically present sufficient conditions under which diversity yields stability for more complicated models. The conclusion of this paper does not accomplish the level that we can apply it to the real world because it needs a lot of assumptions, such as the strategy dimension $N = 2$ and the number of class players $T = 2$. However, as described in Section 6.2, the result of numerical experiments shows that we may relax almost all assumptions in this paper. In fact, as we suppose that the strategy dimension $N = 2$ and the number of class players $T = 2$, we may derive a more general conclusion in the same way as this paper if we can overcome a large amount of calculations. However, it may not be easy to relax the assumption the strategy dimension $N = 2$ and the number of class players $T = 2$ because it becomes more complicated to calculate the inverse matrix when the dimension of the matrix becomes more than 6. We should develop analyzing methods that do not need the direct calculating of the inverse matrix.

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