Master's Thesis

Conjugate gradient methods for optimization problems on symplectic Stiefel manifold

Guidance

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Abstract

The symplectic Stiefel manifold $\mathrm{Sp}(2p, 2n)$ with positive integers p and n satisfying $p \leq n$ is a Riemannian manifold consisting of the entire $2n \times 2p$ matrices satisfying symplectic constraints. This manifold is a generalization of the symplectic group since it reduces to the group when $n = p$. Optimization problems on this manifold have applications in various fields, such as quantum physics, numerical linear algebra, and model order reduction of dynamical systems. For example, any positive definite symmetric matrices can be diagonalized by a symplectic matrix, and its diagonal components are called symplectic eigenvalues. This is one of the important applications in numerical linear algebra.

In this paper, we propose new conjugate gradient methods on symplectic Stiefel manifolds and compare them with the steepest descent method proposed in the existing studies through numerical experiments. The Riemannian conjugate gradient methods, i.e., the conjugate gradient methods on Riemannian manifolds, are first-order methods that are generalizations of nonlinear conjugate gradient methods in Euclidean spaces. As with Euclidean conjugate gradient methods, Riemannian conjugate gradient methods are also known to be effective for mid- to large-scale problems. While the theoretical basis of the Riemannian conjugate gradient methods has already been established, special treatment is required to address a specific manifold since the methods utilize some mappings, such as a retraction and vector transport on the manifold.

For the proposed conjugate gradient methods on the symplectic Stiefel manifolds, a retraction called the Cayley retraction is used to update a current point with a given search direction and step length. A mapping from a tangent space to another tangent space of the manifold is also required at each iteration to transport the previous search direction to the tangent space at the current point. On the one hand, for the case of the symplectic group, the inverse of the Cayley retraction is an appropriate choice of such a mapping, and a specific formula for computing it is derived. This approach gives a theoretically rigorous optimization method since its global convergence can be guaranteed. On the other hand, for the case of the general symplectic Stiefel manifold, deriving such a formula is challenging. Hence, as an alternative approach, the orthogonal projection-based vector transport is implemented for this case. In both cases, numerical experiments demonstrate that the proposed methods outperform the existing methods and turn out to be efficient.