NAME:

DEPARTMENT:

OR Advanced, Part 1, Final Report

- ✓ Answer the questions in Japanese or English.
- ✓ The final report corresponds to 70% of the grade of the "nonlinear optimization" part, and the two small reports correspond to 30% (each one 15%) of the total grade.
- ✓ Submit this report on November 28's class.

Exercise 1. [15 points] Let $x^* \in \mathbb{R}^n$ be a local minimizer of the following nonlinear programming problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i = 1, \dots, m, \end{array} \tag{P1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are differentiable. Assume that x^* satisfies the linear independence constraint qualification (LICQ), and let $\lambda^* \in \mathbb{R}^m$ be its corresponding Lagrange multiplier that satisfies the KKT conditions of (P1). Prove that λ^* is unique.

Exercise 2. Consider the following quadratic optimization problem:

$$\begin{aligned} \min \quad & \frac{1}{2}x^\top Q x + r^\top x \\ \text{s.t.} \quad & Ax \le b, \end{aligned} \tag{P2}$$

where $Q \in \mathbb{R}^{n\times n}$, $r \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given and $x \in \mathbb{R}^n$ is the decision variable. Assume also that Q is symmetric and positive definite.

(a) [5 points] Prove that the objective function of (P2) is convex.

(b) [15 points] Prove that the dual of (P2) is also a quadratic problem of the form

$$\begin{aligned} \max \quad & \lambda^\top \tilde{Q} \lambda + \tilde{r}^\top \lambda + \tilde{s} \\ \text{s.t.} \quad & \lambda \in \mathbb{R}^m_+. \end{aligned}$$

Write the formulas of $\tilde{Q}$, $\tilde{r}$ and $\tilde{s}$ explicitly, using only Q, r, A and b.

Exercise 3. [15 points] Consider the following nonlinear optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq 0, \\ & h(x) = 0, \end{array} \tag{P3}$$

with $f\colon \mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $h\colon \mathbb{R}^n \to \mathbb{R}^p$. We say that $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p$ is a *saddle point* of the Lagrangian function L if the following inequalities hold:

$$L(x^*, \lambda, \mu) \leq L(x^*, \lambda^*, \mu^*) \leq L(x, \lambda^*, \mu^*) \quad \text{for all } (x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p.$$

Assume that f and g are convex and that h is affine. Suppose also that the Slater constraint qualification holds. Prove that $x^* \in \mathbb{R}^n$ is an optimal solution of problem (P3) if, and only if, there exists $(\lambda^*, \mu^*) \in \mathbb{R}^m_+ \times \mathbb{R}^p$ such that (x^*, λ^*, μ^*) is a saddle point of L.

Exercise 4. Consider the following optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \le 0, \quad i = 1, \dots, m, \end{aligned} \tag{P4}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$, $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are differentiable and convex. Let $\phi_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m$, be increasing, differentiable and convex functions.

(a) [10 points] Prove that the function $q\colon \mathbb{R}^n \to \mathbb{R}$ defined below is convex:

$$q(x) := f(x) + \sum_{i=1}^{m} \phi_i\big(g_i(x)\big).$$

(b) [10 points] Assume that $\bar{x} \in \mathbb{R}^n$ is a minimizer of q. Find a feasible point for the dual of (P4) by using $\bar{x}$, justifying your answer. Write out the corresponding lower bound on (P4)'s optimal value.