# A Class of Gap Functions for Quasi-Variational Inequality Problems[1]

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August 15, 2006

This paper is dedicated to Professor Alex Rubinov on the occasion of his 65th birthday.

**Abstract.** We present a class of gap functions for the quasi-variational inequality problem (QVIP). We show the equivalence between the optimization reformulation with the gap function and the original QVIP. We also give conditions under which the gap function is continuous and directionally differentiable.

**Keywords.** Quasi-variational inequality problem, optimization reformulation, gap function.

**2000 Mathematics Subject Classification.** 65K10, 90C30

## 1 Introduction

Given a vector-valued function $F : \Re^n \to \Re^n$ and a set-valued function $S : \Re^n \rightrightarrows \Re^n$, the finite-dimensional *quasi-variational inequality problem* (QVIP) is to find a vector $x \in \Re^n$ such that $x \in S(x)$ and

$$\langle F(x), y - x \rangle \geq 0 \qquad \forall y \in S(x), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\Re^n$. Throughout the paper, we make the following assumptions:

**(A1)** $F$ is continuously differentiable on $\Re^n$.

**(A2)** $S(x)$ is nonempty, closed and convex for each $x \in \Re^n$.

These assumptions are standard, except the nonemptyness of $S(x)$ for all $x \in \Re^n$. In fact, in many practical applications, it may happen that $S(x) = \emptyset$ for some $x$. Nevertheless we assume this somewhat strong assumption to simplify our arguments.

[1]This work was supported in part by a Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science.

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Let $X \subseteq \Re^n$ be defined by

$$X = \{x \in \Re^n \,|\, x \in S(x)\}. \tag{2}$$

This is called a feasible set of QVIP (1) and plays a crucial role in the optimization reformulation of the QVIP developed in this paper.

If the set-valued function $S$ is constant, i.e., $S(x) = \hat{S} \subseteq \Re^n$ for all $x \in \Re^n$, then QVIP (1) reduces to the variational inequality problem (VIP), which is to find a vector $x \in \hat{S}$ such that

$$\langle F(x), y - x\rangle \geq 0 \qquad \forall y \in \hat{S}. \tag{3}$$

The VIP is one of the most fundamental equilibrium problems and has a number of important applications in economics, engineering, operations research, and so on. There has been an extensive study on the theory and algorithms for the VIP [6]. In particular, merit functions such as the gap function [1] and the regularized gap function [7] have been invented as a powerful tool in dealing with the VIP by way of its equivalent optimization reformulation.

Compared with the VIP, the literatures on the QVIP are not so many [4, 18, 11]. However, since the QVIP can be used to formulate the generalized Nash game in which not only each player's payoff function but also his/her strategy set depend on the other players strategies, the QVIP has recently attracted growing attention in relation to game theory [2, 9, 14]. The gap function and its modifications have been studied by Giannessi [8] for the QVIP. In this paper, we present a class of gap functions for the QVIP and show the equivalence between the resulting optimization problem and the original QVIP. The underlying idea is an extension of that given in [16], which is to modify the constraints as well as the objective function of the optimization problem that defines the gap function. By those modifications, the gap function possesses some favorable properties regarding differentiability and becomes more amenable to computation particularly when the constraints are defined by nonlinear inequalities.

## 2 Gap Functions

As a direct extension of the gap function for the VIP [1], the *gap function* $f_0 : \Re^n \to \Re \cup \{+\infty\}$ for QVIP (1) is defined as

$$f_0(x) = -\inf\{\langle F(x), y - x\rangle \,|\, y \in S(x)\}. \tag{4}$$

Since $S(x) \neq \emptyset$ for each $x \in \Re^n$, we have $f_0(x) > -\infty$ everywhere. However, it can happen that $f_0(x) = +\infty$ for some $x$. We can easily show the following fact.

**Theorem 1** *For each $x \in X$, we have $f_0(x) \geq 0$. Moreover, $x$ solves QVIP $(1)$ if and only if $f_0(x) = 0$ and $x \in X$.*

**Proof.** If $x \in X$, then we have $x \in S(x)$. In view of the definition (4) of $f_0$, we deduce that $f_0(x) \geq 0$. The second part of the proposition is immediate from the first part. $\square$

Theorem 1 indicates that QVIP $(1)$ can be reformulated as the problem of minimizing the function $f_0$ over $X$. If an optimal solution of the minimization problem has the zero objective value, then it is a solution of QVIP $(1)$. As mentioned above, however, the gap function $f_0$ may take the value $+\infty$ somewhere. Moreover, it may not be so easy to evaluate the function value $f_0(x)$ even if it is finite, unless the set $S(x)$ has a simple structure such as polyhedral convexity.

To construct a more tractable reformulation of the QVIP, we introduce the function

$$f(x) = -\inf\{\varphi(x,y) \,|\, y \in \Gamma(x)\}, \tag{5}$$

where the function $\varphi : \Re^n \times \Re^n \to \Re$ and the set-valued function $\Gamma : \Re^n \rightrightarrows \Re^n$ satisfy the following conditions (B1)–(B3) and (C1)–(C4), respectively:

**(B1)** For each fixed $x \in \Re^n$, $\varphi(x,\cdot)$ is strictly convex and everywhere differentiable.

**(B2)** $\varphi(x,x) = 0$ for each $x \in \Re^n$.

**(B3)** $\nabla_y \varphi(x,x) = F(x)$ for each $x \in \Re^n$, where $\nabla_y$ denotes the partial derivative with respect to the second argument.

**(C1)** For each $x \in \Re^n$, $\Gamma(x)$ is closed and convex.

**(C2)** $S(x) \subseteq \Gamma(x)$ for each $x \in \Re^n$.

**(C3)** $X = \{x \in \Re^n \,|\, x \in \Gamma(x)\}$.

**(C4)** $\mathscr{T}_S(x) = \mathscr{T}_\Gamma(x)$ for each $x \in X$, where $\mathscr{T}_S(x)$ and $\mathscr{T}_\Gamma(x)$ denote the tangent cones of $S(x)$ and $\Gamma(x)$, respectively, at point $x$.

With regard to these conditions, some remarks are in order.

Conditions (B1)–(B3) suggest that the function $\varphi(x,\cdot)$ is in some sense a regularization of the linear function $y \mapsto \langle F(x), y-x\rangle$ involved in the definition (4) of the gap function $f_0$. In particular, if we define the function $\varphi$ by

$$\varphi(x,y) = \langle F(x), y-x\rangle + \tfrac{1}{2}\langle y-x, G(y-x)\rangle \tag{6}$$

with a positive definite symmetric matrix $G$, then conditions (B1)–(B3) are all satisfied.

Conditions (C1)–(C4) presume that the set $\Gamma(x)$ is a kind of approximation to $S(x)$, which is constructed with some information on the point $x$ under consideration. To get a more specific idea, let us suppose that the set $S(x)$ is defined by

$$S(x) = \{y \in \Re^n \,|\, g_i(x,y) \le 0,\ i = 1,\dots,m\}, \tag{7}$$

where $g_i : \Re^n \times \Re^n \to \Re$, $i = 1,\dots,m$, are functions such that $g_i(x,\cdot)$ are convex and differentiable for each fixed $x$. Then by definition, we have

$$X = \{x \in \Re^n \,|\, g_i(x,x) \le 0,\ i = 1,\dots,m\}. \tag{8}$$

Define a polyhedral approximation of $S(x)$ by

$$\Gamma(x) = \{y \in \Re^n \,|\, g_i(x,x) + \langle \nabla_y g_i(x,x), y - x\rangle \le 0,\ i = 1,\dots,m\}. \tag{9}$$

Clearly, (C1) holds. By the convexity of $g_i(x,\cdot)$, it is easy to see that (C2) holds. To show (C3), let us define $X' = \{x \in \Re^n \,|\, x \in \Gamma(x)\}$. By (C2), $x \in S(x)$ implies $x \in \Gamma(x)$, i.e., $X \subseteq X'$. On the other hand, if $x \in \Gamma(x)$, then $g_i(x,x) \le 0$, $i = 1,\dots,m$, i.e., $x \in S(x)$. This implies $X' \subseteq X$. Consequently, we have $X = X'$, that is, (C3) holds. To examine condition (C4), choose $x \in X$ arbitrarily. First note that $x \in S(x) \subseteq \Gamma(x)$. Moreover, $x \in S(x)$ implies $g_i(x,x) \le 0$, $i = 1,\dots,m$. Without loss of generality, suppose $g_i(x,x) = 0$, $i = 1,\dots,m'$, and $g_i(x,x) < 0$, $i = m'+1,\dots,m$ for some $m' \in \{0,1,\dots,m\}$. Then, under a suitable constraint qualification, the tangent cone of the set $S(x)$ at point $x$ is given by

$$\mathscr{T}_S(x) = \{d \in \Re^n \,|\, \langle \nabla_y g_i(x,x), d\rangle \le 0,\ i = 1,\dots,m'\}. \tag{10}$$

On the other hand, since $x \in \Gamma(x)$ and the active index set in $\Gamma(x)$ coincides with that in $S(x)$, the tangent cone of the polyhedral convex set $\Gamma(x)$ at $x$ is also represented as the right-hand side of (10). Thus (C4) holds.

It is worth mentioning that if the function $\varphi$ is given by (6) and the set-valued function $S$ and its approximation $\Gamma$ are given by (7) and (9), respectively, then the minimization problem on the right-hand side of (5) becomes a strictly convex quadratic programming problem. The latter problem is known to have a unique solution, which can be computed by using an efficient solution method.

In the rest of this section, we investigate the properties of the gap function $f$ under conditions (B1)–(B3) and (C1)–(C4). We also assume that the infimum on the right-hand side of (5) is attained at some point, which is unique by (B1) and (C1), and is denoted $y(x)$.

The following theorem is a counterpart of Theorem 1.

**Theorem 2** *For each $x \in X$, we have $f(x) \geq 0$. Moreover, $x$ solves QVIP* (1) *if and only if $f(x) = 0$ and $x \in X$.*

**Proof.** If $x \in X$, then we have $x \in \Gamma(x)$ by (C3). From (B2) and the definition (5) of $f$, we deduce that $f(x) \geq 0$.

To prove the second half of the theorem, we first suppose that $x$ solves QVIP (1). Then we have $x \in X$, which implies $x \in \Gamma(x)$ as noted above. Since $y(x)$ minimizes the function $\varphi(x, \cdot)$ over the closed convex set $\Gamma(x)$, it satisfies the first-order optimality condition

$$\langle \nabla_y \varphi(x, y(x)), y - y(x) \rangle \geq 0 \qquad \forall y \in \Gamma(x).$$

In particular, since $x \in \Gamma(x)$, we have

$$\langle \nabla_y \varphi(x, y(x)), x - y(x) \rangle \geq 0. \tag{11}$$

On the other hand, since $S(x)$ is convex, $x \in S(x) \subseteq \Gamma(x)$, and, by (C4), $\mathscr{T}_S(x) = \mathscr{T}_\Gamma(x)$ holds, we have

$$\begin{aligned} \langle F(x), y - x \rangle \geq 0 \quad \forall y \in S(x) \quad &\Longleftrightarrow \quad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in x + \mathscr{T}_S(x) \\ &\Longleftrightarrow \quad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in x + \mathscr{T}_\Gamma(x) \\ &\Longleftrightarrow \quad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \Gamma(x). \end{aligned}$$

Hence, $y(x) \in \Gamma(x)$ yields

$$\langle F(x), y(x) - x \rangle \geq 0. \tag{12}$$

It then follows from (11), (12) and (B3) that

$$\langle \nabla_y \varphi(x, y(x)) - \nabla_y \varphi(x, x), y(x) - x \rangle \leq 0. \tag{13}$$

However, since $\varphi(x, \cdot)$ is strictly convex, $\nabla_y \varphi(x, \cdot)$ is a strictly monotone mapping. Therefore, (13) holds only if $x = y(x)$. Hence we have

$$f(x) = -\varphi(x, y(x)) = -\varphi(x, x) = 0,$$

where the last equality follows from (B2).

To prove the converse, let us suppose that $x \in X$ and $f(x) = 0$. By the definition (5) of the gap function, $f(x) = 0$ implies

$$\varphi(x, y) \geq 0 \qquad \forall y \in \Gamma(x). \tag{14}$$

Since $\varphi(x, x) = 0$ by (B2) and $x \in \Gamma(x)$ by (C3), (14) implies that $x = y(x)$, which is the unique minimizer of $\varphi(x, \cdot)$ over $\Gamma(x)$. Then, by the first-order optimality condition for this minimization problem, we have

$$\langle \nabla_y \varphi(x, x), y - x \rangle \geq 0 \qquad \forall y \in x + \mathscr{T}_\Gamma(x). \tag{15}$$

In a similar manner to the above, (C4) can be used to show that (15) is equivalent to

$$\langle \nabla_y \varphi(x,x), y-x \rangle \geq 0 \qquad \forall y \in S(x).$$

In view of (B3), this means that $x$ is a solution of QVIP (1). □

Theorem 2 indicates that QVIP (1) can be reformulated as the following minimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X. \end{array} \tag{16}$$

If we can find an optimal solution with zero objective value, it must solve QVIP (1).

Recall that a set-valued function $\Phi : \Re^n \rightrightarrows \Re^n$ is said to be closed if the graph $\{(x,y) \,|\, y \in \Phi(x)\}$ is a closed subset of $\Re^n \times \Re^n$. The following proposition follows from some well-known results in sensitivity analysis of parametric optimization.

**Proposition 1** *If $S : \Re^n \rightrightarrows \Re^n$ is closed, then the set $X$ defined by* (2) *is closed.*

**Proof.** Obvious. □

**Proposition 2** *Let $x \in \Re^n$ be fixed and suppose that* (i) *$\varphi : \Re^n \times \Re^n \to \Re$ is continuous,* (ii) *$\Gamma : \Re^n \rightrightarrows \Re^n$ is closed,* (iii) *there exists a compact set $C \subset \Re^n$ and a real number $\beta \in \Re$ such that for every $x'$ in a neighborhood of $x$, the set $\{y \in \Gamma(x') \,|\, \varphi(x',y) \leq \beta\}$ is nonempty and contained in $C$, and* (iv) *for any neighborhood $\mathscr{U}$ of the set $\Gamma(x)$, there exists a neighborhood $\mathscr{V}$ of $x$ such that $\mathscr{U} \cap \Gamma(x) \neq \emptyset$ for all $x \in \mathscr{V}$. Then the function $f$ defined by* (5) *is continuous at $x$. Moreover, if the above conditions hold at every $x \in \Re^n$, then $f$ is continuous on $\Re^n$.*

**Proof.** See [3, Proposition 4.4]. □

Under the conditions given in the above propositions, (16) is a problem of minimizing a continuous function over a closed set in $\Re^n$. Let us examine those conditions in the case where $\varphi$, $S$ and $\Gamma$ are given by (6), (7) and (9), respectively. Clearly, $\varphi$ is continuous if $F$ is continuous. The set-valued functions $S$ and $\Gamma$ are closed whenever the functions $g_i$, $i = 1, \ldots, m$ are continuously differentiable. Moreover, condition (iii) in Proposition 2 holds since the matrix $G$ in (6) is assumed to be positive definite. Recall that we have assumed $S(x) \neq \emptyset$ for all $x$, which together with (C2) implies $\Gamma(x) \neq \emptyset$ for all $x$. Hence (iii) actually holds for any $x$. Finally, it can be shown that (iv) holds under Slater's condition, i.e., there exists some $x^0$ such that $g_i(x, x^0) < 0$, $i = 1, \ldots, m$.

Unfortunately, the function $f$ is in general nondifferentiable. Nevertheless we can show that it is directionally differentiable under suitable assumptions. In the remainder of this section, we restrict ourselves to the case where $\varphi$, $S$ and $\Gamma$ are given by (6), (7) and (9), respectively. We assume that $F$ is continuously differentiable and $g_i$, $i=1,\dots,m$ are twice continuously differentiable. Moreover, we suppose that the above-mentioned Slater's condition holds for any $x$. Then we can show the following theorem on the directional differentiability of the gap function.

**Theorem 3** *Let the assumptions stated prior to the theorem be satisfied. Then the gap function $f$ is directionally differentiable everywhere, and the directional derivative of $f$ at $x$ along direction $d$ is given by*

$$\begin{aligned} f'(x;d) &= \min_{\lambda\in\Lambda(x)} \big\{ \langle F(x)-(\nabla F(x)-G)(y(x)-x), d\rangle \\ &\quad - \sum_{i=1}^{m}\lambda_i \langle \nabla_x g_i(x,x) + (\nabla_{xy} g_i(x,x) + \nabla_{yy} g_i(x,x))(y(x)-x), d\rangle \big\}, \end{aligned} \tag{17}$$

*where $\Lambda(x)$ is a subset of $\Re^m$ defined by*

$$\begin{aligned} \Lambda(x) &= \big\{ \lambda\in\Re^m \mid F(x)+G(y(x)-x)+\textstyle\sum_{i=1}^{m}\lambda_i\nabla_y g_i(x,x)=0,\ \ \lambda\ge 0, \\ &\qquad \lambda_i(g_i(x,x)+\langle\nabla_y g_i(x,x), y(x)-x\rangle)=0,\ i=1,\dots,m \big\}. \end{aligned} \tag{18}$$

*Moreover, if $\Lambda(x)$ is a singleton (which is particularly true when $\nabla_y g_i(x,x)$, $i\in\mathscr{I}(x) := \{i \mid g_i(x,x)+\langle\nabla_y g_i(x,x), y(x)-x\rangle = 0\}$ are linearly independent), then $f$ is differentiable at $x$ and the gradient of $f$ at $x$ is given by*

$$\begin{aligned} \nabla f(x) &= F(x)-(\nabla F(x)-G)(y(x)-x) \\ &\quad - \sum_{i=1}^{m}\lambda_i \big(\nabla_x g_i(x,x) + (\nabla_{xy} g_i(x,x)+\nabla_{yy} g_i(x,x))(y(x)-x)\big). \end{aligned}$$

**Proof.** To simplify the notation, denote $h_i(x,y) = g_i(x,x)+\langle\nabla_y g_i(x,x), y-x\rangle$, $i=1,\dots,m$ and $\hat{y} = y(x)$. Then, by [10, Theorem 2], the directional derivative of $f$ at $x$ along $d$ exists and is given by

$$f'(x;d) = \min_{\lambda\in\bar{\Lambda}(x)} \big\{ -\langle\nabla_x\varphi(x,\hat{y}), d\rangle - \sum_{i=1}^{m}\lambda_i\langle\nabla_x h_i(x,\hat{y}), d\rangle \big\}, \tag{19}$$

where $\bar{\Lambda}(x)$ is the set of optimal solutions of the dual problem of the convex programming problem $\min_y\{\varphi(x,y) \mid h_i(x,y)\le 0,\ i=1,\dots,m\}$. By the duality theory in convex programming, the set $\bar{\Lambda}(x)$ is nothing but the set $\Lambda(x)$ defined by (18). Moreover, by direct calculation, the minimand in (19) can be rewritten as that in (17). The proof is complete. $\square$

## 3 Concluding Remarks

Since the set $X$ is given by (8), $X$ is convex if the functions $\hat{g}_i$, $i=1,\dots,m$ defined by $\hat{g}_i(x) = g_i(x,x)$ are convex. By the directional differentiability of $f$ shown in Theorem 3, the first-order necessary condition of optimality for problem (16) can be stated as

$$f'(x; y - x) \geq 0 \qquad \forall y \in X. \tag{20}$$

If one want to solve QVIP (1) by way of the optimization reformulation (16), it is required to obtain its global optimal solution. However, since the function $f$ is in general non-convex, it is not easy to find a global minimizer of $f$ on the feasible set $X$. Therefore it would be desirable if any point satisfying the stationarity condition (20) becomes a global optimal solution. For VIP (3), it is possible to give conditions, typically the strict monotonicity of $F$, under which any stationary point solves the VIP [7, 16]. However, we have been unable to give such a simple condition for the QVIP. It would be an interesting subject of future research to develop an optimization reformulation of the QVIP that possesses more desirable properties.

From the viewpoint of application, it is certainly important and interesting to study how generalized Nash games can effectively be dealt with by means of a reformulation approach that uses gap functions, or more generally, merit functions. Relations to other approaches such as Nikaido-Isoda type function methods would also be worth investigation [15, 12, 5, 17, 13].

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