Portfolio Selection with Uncertain Exit Time: A Robust CVaR Approach

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Abstract

In this paper we explore the portfolio selection problem involving an uncertain time of eventual exit. To deal with this uncertainty, the worst-case CVaR methodology is adopted in the case where no or only partial information on the exit time is available, and the corresponding problems are integrated into linear programs which can be efficiently solved. Moreover, we present a method for specifying the uncertain information on the distribution of the exit time associated with exogenous and endogenous incentives. Numerical experiments with real market data and Monte Carlo simulation show the usefulness of the proposed model.

Key words: Robust CVaR, Robust Portfolio Selection, Uncertain Exit Time
JEL Classification: C1, G11

1 Introduction

In Markowitz's paper (1952), as well as his book published seven years later (Markowitz, 1959), he suggests that investors should decide the allocation of their investment on the basis of a trade-off between risk and return based on mean-variance analysis. The mean-variance framework is so intuitive and

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so strong that it has been continually applied to different areas within finance and risk management. Indeed, numerous innovations within finance have either been an application of the concept of mean-variance analysis or an extension of the methodology to alternative portfolio risk measures (see Fabozzi, Gupta, and Markowitz, 2002 for current applications). Conditional Value-at-Risk (CVaR) is currently one of the popular risk measures suggested by theoreticians and market practitioners.

As a measure of downside risk, CVaR exhibits some attractive properties. First, it can deal with the asymmetric distribution of asset return better than mean-variance analysis, especially for assets with returns that are heavy-tailed. Second, minimizing CVaR usually results in solving a convex programming problem, such as a linear programming problem, which allows the decision maker to deal with a large scale portfolio problem efficiently (Rockafellar and Uryasev, 2000, 2002). Finally, Artzner et al. (1999) demonstrate that CVaR is a coherent measure of risk [1] , which has been widely accepted as a benchmark to evaluate risk measures.

All the above analysis, however, is based on the assumption that the investment horizon of an investor is pre-specified, either finite or infinite, and that any investor operates the buy-hold strategy until the explicit exit moment. In fact, as well as taking on asset risk, typically an investor faces the exit time risk because he never acknowledges the time of his eventual exit upon entering the market. Generally speaking, there are many exogenous and endogenous factors that can drive the exit strategy of an investor. For example, the investor's sudden consumption is an important reason for exiting the market. In addition, due to the price movement of risky assets, the optimal exercise strategy for American options usually causes the investor to terminate his portfolio. In short, it is quite reasonable for an investor to take into account the uncertainty of his eventual exit time when constructing a portfolio. However, portfolio choice when the investor faces an uncertain exit time—more specifically, how to model the uncertainty of the eventual exit—is a difficult problem to deal with because one must capture the distribution of the asset returns under an uncertain exit time.

Research on portfolio selection with uncertain investment horizon has been limited in the literature, though Merton (1971) addresses a dynamic optimal portfolio selection problem for an investor retiring at an uncertain time. Similar work in a discrete case can be traced back to Yaari (1965) and Hakansson (1969). More recently, Karatzas and Wang (2001) consider an optimal dynamic investment problem which assumes that markets are complete and the eventual exit is a completely endogenous factor—a stopping time of asset price

[1] Pflug (2000) and Acerbi and Tasche (2002) discuss the coherence of CVaR exclusively.

filtration. Liu and Loewenstein (2002) consider the case where the exit time in portfolio selection follows an explicit exponential distribution. Blanchet-Scalliet, El Karoui and Martellini (2005) and Blanchet-Scalliet et al. (2005) investigate the pricing problems associated with an uncertain time-horizon. Martellini and Urošević (2006) first propose the concept of exit time risk and show that the mean-variance efficient frontier in the case where the exit time is independent of the portfolio performance (exogenous exit) coincides with the traditional mean-variance efficient frontier (fixed exit time), and conversely, when the exit time is dependent on portfolio performance (endogenous exit), the set of mean-variance efficient portfolio may rely on the exit time distribution.

In the past decade, some researchers, particularly those specializing in the field of optimization, have paid considerable attention to a type of mathematical programming under uncertainty—robust optimization—which is used to solve an optimization problem involving uncertain parameters. [2] With respect to portfolio management, Lobo and Boyd (2000) are among the first to apply worst-case analysis to robust portfolio selection. [3] Costa and Paiva (2002), as well as Goldfarb and Iyengar (2003) and Erdoğan et al. (2006), study robust portfolio optimization in the mean-variance framework in detail. El Ghaoui et al. (2003) investigate robust portfolio selection using worst-case Value-at-Risk. Zhu and Fukushima (2005) further consider the worst-case CVaR (WCVaR) in the case where only partial information on the underlying probability distribution of returns is given. [4] Although the models developed by these researchers sought to tackle the one-period investment problem with certain time of eventual exit, we believe that they can be similarly applied to the situation where there is an uncertain investment horizon. It is easily imaginable that the uncertainty of risk factors results partly from the uncertainty of eventual exit, while a robust strategy of portfolio selection can well incorporate and assimilate such uncertainty.

This work is greatly motivated by Martellini and Urošević (2006) and Zhu and Fukushima (2005), among others mentioned above. In contrast to the approach developed by Martellini and Urošević (2006) to select a portfolio with uncertain exit time using mean-variance formulation, we propose a worst-case CVaR approach, which is formally defined and applied to robust portfolio man-

[2] Robustness is only a concept or a strategy, which has different meanings in the literature. Some researchers look at robustness as controlled sensitivity to uncertain data from statistical perspective, see for example Mulvey, Vanderbei and Zenios (1995), while others discuss robustness in the "worst case" context. In this paper, we consider robustness in the latter sense.

[3] It should be noted that the robust portfolio management in Mulvey, Vanderbei and Zenios (1995) is different from that in the sense of Lobo and Boyd (2000).

[4] For a complete discussion of robust portfolio management and the associated solution methods, see Fabozzi et al. (2007).

agement in the recent work of Zhu and Fukushima (2005). We show that it can be implemented as an alternative approach to remove or alleviate those difficulties of traditional portfolio selection methodologies, such as mean-variance and CVaR strategies.

There are three original contributions we make in this paper. Firstly, considering the inconveniences and complexity of portfolio modeling when the exit is uncertain, we propose the worst-case CVaR strategy as an effective alternative under this situation. The widely accepted risk measure CVaR and the powerful robust optimization methodology are integrated to generate at least sub-optimal solutions; this makes the model interesting to risk and asset managers that are primarily concerned in controlling large losses but would like to exploit opportunities. Secondly, in contrast to Martellini and Urošević (2006), the estimation of the exit time distribution is explicitly addressed, and exogenous and endogenous factors that drive the exit are simultaneously incorporated into our formulation. Finally, we propose an algorithm to ascertain the bounds of the endogenous exit probability in the case where the exit time distribution is partially, or even completely, unknown.

The remainder of this paper is organized as follows. Section 2 provides background information for CVaR and worst-case CVaR that will be used in later sections. In Section 3, we analyze the properties of return involving asset price risk and exit time risk, and discuss the difficulty of implementing the CVaR approach for the uncertain exit time problem. Section 4 formulates the portfolio selection problem with no or partial information on the eventual exit time by means of the worst-case CVaR strategy. In Section 5, we present a unified model that relates the specification of information on the exit time to the exogenous and endogenous incentives. In Section 6, we show some numerical experiments with real market data and Monte Carlo simulation. Finally, some concluding remarks are given in Section 7.

2 CVaR and Worst-Case CVaR

In this section, we formally define CVaR and worst-case CVaR, and present some theoretical results. First, following Rockafellar and Uryasev (2000) as well as Zhu and Fukushima (2005), let $f(\mathbf{x}, \mathbf{y})$ denote the loss of a portfolio with decision vector $\mathbf{x} \in \mathcal{X} \subseteq \Re^n$ and random vector $\mathbf{y} \in \Re^N$ that represents the value of underlying risk factors at maturity of the investment horizon T. Suppose $\mathrm{E}(|f(\mathbf{x}, \mathbf{y})|) < +\infty$ for each $\mathbf{x} \in \mathcal{X}$. For simplicity of presentation, we assume that $\mathbf{y} \in \Re^N$ has a continuous density function $p(\mathbf{y})$. By way of Rockafellar and Uryasev (2002) and Zhu and Fukushima (2005), all the results can be applied to the case where $p(\mathbf{y})$ follows a discontinuous distribution. For the purpose of clarity, we may denote a random variable and the related deterministic variable/constant as the same symbol since they can be distinguished clearly by context.

For a given portfolio $\mathbf{x} \in \mathcal{X}$, the probability of the loss not exceeding a threshold α is given by

$$\Psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y}) d\mathbf{y}.$$

Given a confidence level β, the VaR associated with the portfolio $\mathbf{x}$ is defined as

$$\text{VaR}_\beta(\mathbf{x}) = \min\{\alpha \in \Re : \Psi(\mathbf{x}, \alpha) \geq \beta\}.$$

The corresponding CVaR is defined as the conditional expectation of the loss of the portfolio exceeding or equal to VaR, i.e.,

$$\text{CVaR}_\beta(\mathbf{x}) = \frac{1}{1-\beta} \int_{f(\mathbf{x},\mathbf{y}) \geq \text{VaR}_\beta(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y}.$$

Rockafellar and Uryasev (2000, 2002) prove that CVaR has an equivalent definition as follows:

$$\text{CVaR}_\beta(\mathbf{x}) = \min_{\alpha \in \Re} F_\beta(\mathbf{x}, \alpha), \tag{1}$$

where $F_\beta(\mathbf{x}, \alpha)$ is expressed as

$$F_\beta(\mathbf{x}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mathbf{y} \in \Re^N} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ p(\mathbf{y}) d\mathbf{y},$$

where $[\cdot]^+$ is defined as $[z]^+ = \max\{0, z\}$ for any $z \in \Re$.

Thus, minimizing CVaR over $\mathbf{x} \in \mathcal{X}$ is equivalent to minimizing $F_\beta(\mathbf{x}, \alpha)$ over $(\mathbf{x}, \alpha) \in \mathcal{X} \times \Re$, i.e.,

$$\min_{\mathbf{x} \in \mathcal{X}} \text{CVaR}_\beta(\mathbf{x}) = \min_{(\mathbf{x},\alpha) \in \mathcal{X} \times \Re} F_\beta(\mathbf{x}, \alpha).$$

If $\mathcal{X}$ is a convex set in $\Re^n$, and the function $f(\mathbf{x}, \mathbf{y})$ is convex with respect to $\mathbf{x}$, then the problem is a convex programming problem.

The remaining task in optimizing a portfolio using the CVaR approach is to achieve the precise knowledge of the distribution of random vector $\mathbf{y}$ with a given explicit investment horizon. More specifically, the investor should know the density function $p(\mathbf{y})$ of the random vector $\mathbf{y}$ at maturity of the investment horizon. However, in many cases, the distribution cannot be precisely specified. Here, we relax the requirement and assume that the density function is only known to belong to a certain set $\mathscr{P}$ of distributions, i.e., $p(\cdot) \in \mathscr{P}$. As a special case, we will discuss this issue arising from the uncertainty of exit time in the next section.

Now, we turn to the following definition of worst-case CVaR.[5] We adopt the definition by Zhu and Fukushima (2005): *Given a confidence level β, the worst-case* CVaR (WCVaR) *for a given portfolio $\mathbf{x} \in \mathcal{X}$ with respect to $\mathscr{P}$ is defined as*

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \sup_{p(\cdot)\in\mathscr{P}} \mathrm{CVaR}_\beta(\mathbf{x}). \tag{2}$$

Then by (1), it is clear that

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \sup_{p(\cdot)\in\mathscr{P}} \min_{\alpha\in\Re} F_\beta(\mathbf{x}, \alpha).$$

Thus, minimizing the worst-case CVaR over $\mathbf{x} \in \mathcal{X}$ is equivalent to the following min-sup-min problem:

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{p(\cdot)\in\mathscr{P}} \min_{\alpha\in\Re} F_\beta(\mathbf{x}, \alpha). \tag{3}$$

Zhu and Fukushima (2005) extensively investigate (3) for several concrete structures of $\mathscr{P}$ and reformulate it in a tractable form that can be efficiently solved.

3 Asset Return under Uncertain Exit Time

In this section we consider the specification of asset return associated with asset price risk and exit time risk since an investor may exit the market at any moment before the maturity of his investment horizon. In particular, we discuss the difficulty that lies in CVaR optimization.

[5] As a worst-case risk measure, WCVaR remains coherent in the sense of Artzner et al. (1999).

One of the essential tasks in portfolio management is to set criteria for computing the returns of risky assets available and further specify the joint distribution of those returns. Let the initial time of investment be zero, and the asset price at exit time τ be V_τ. Then the return from time 0 to time τ is defined as

$$y_\tau = \frac{V_\tau - V_0}{V_0}.$$

We follow Martellini and Urošević (2006) and identify the uncertainty of the return from two sources. The first type of uncertainty is the asset price risk, which is due to the irregular fluctuation of the asset price for a given realization of τ, for example, geometric Brownian motion. The second type of risk is called the exit time risk, which derives from the uncertainty of eventual exit time of the investor. More accurately speaking, the exit time risk is caused by the uncertain distribution of the return at different exit times, since the joint distribution of risky assets possesses a time-varying feature. Of course, it should be noted that exiting is an individual action, so it does not change the return structure of the portfolio because the price of the portfolio is determined by the total market. On the contrary, the price movement of the portfolio is a crucial factor driving the exit of the investor.

In the case where the exit time is uncertain, finding a proper way of specifying the distribution of asset returns is obviously a difficult thing. However, according to the discussion of the last paragraph, we can decompose the specification into two steps by first specifying the conditional (on time) distribution of returns and then determining the distribution of exit time.

Before giving the general result, we first consider a simple example consisting of one asset with uncertain exit time. Suppose that the investment horizon is time period $[0, T]$. We assume that the exit time τ follows a truncated exponential distribution with exit intensity ς. This is related to the jump of a Poisson process, which will be further explained in Section 5. Therefore, the exit distribution function $G(t)$ at time t can be written as

$$G(t) = \begin{cases} 1 - e^{-\varsigma t}, \; 0 < t < T, \\ 1, \qquad\qquad\quad t = T. \end{cases}$$

For simplicity, we assume that there are m tradable moments for the investor in the investment horizon, and that at every tradable moment t_i ($i = 1, \cdots, m, t_{i-1} < t_i, t_0 = 0, t_m = T$) the investor can choose to exit or not. Hence, the probability of exiting at t_i is

$$g(t_i) = \Pr(\tau = t_i) = G(t_i) - G(t_{i-1}) = \begin{cases} 1 - e^{-\varsigma t_1}, & i = 1, \\ e^{-\varsigma t_{i-1}} - e^{-\varsigma t_i}, & i = 2, \cdots, m-1, \\ e^{-\varsigma t_{m-1}}, & i = m. \end{cases} \quad (4)$$

In accordance with the general assumption of geometric Brownian motion of the asset price, we assume that the density function of return, $p_t(y)$, conditional on exit time $\tau = t$ is normally distributed with mean μt and variance $\sigma^2 t$. By the conditional probability formula, it is easy to get the unconditional density function $p(y)$ as

$$p(y) = \sum_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2 t_i}} e^{-\frac{(y-\mu t_i)^2}{2\sigma^2 t_i}} g(t_i).$$

The general formula of the unconditional density function is shown in the following proposition whose proof is straightforward from the conditional probability formula.

Proposition 1 *Let $g(\cdot)$ be the density function of exit time τ and $p_t(\cdot)$ be the conditional (on exit time t) density function of the asset returns. Then the corresponding unconditional density function is given by*

$$p(\cdot) = \int_0^T p_t(\cdot) g(t) dt.$$

In particular, if the exit time τ follows a discrete distribution on time $\{t_1, t_2, \cdots, t_m\}$ with $\Pr(\tau = t_i) = \lambda_i$, $\sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, i = 1, \cdots, m$, then we have

$$p(\cdot) = \sum_{i=1}^{m} \lambda_i p_i(\cdot), \quad (5)$$

where we denote $p_i(\cdot)$ as $p_{t_i}(\cdot)$ in the discrete case throughout for brevity.

By Proposition 1, we get the following optimization problem of portfolio selection via minimizing CVaR in accordance with Rockafellar and Uryasev (2000, 2002):

$$\min_{(\mathbf{x},\alpha)\in\mathcal{X}\times\Re} \alpha + \frac{1}{1-\beta} \int_{\mathbf{y}\in\Re^N} [f(\mathbf{x},\mathbf{y}) - \alpha]^+ p(\mathbf{y}) d\mathbf{y}, \quad (6)$$

where

$$p(\mathbf{y}) = \int_0^T p_t(\mathbf{y}) g(t) dt, \quad (7)$$

and $\mathcal{X}$ is specified by a set of constraints including budget constraint, target return constraint, regulation constraint, and so on. In the case where τ follows a discrete distribution, $p(\cdot)$ specified by (7) should be replaced by (5).

Critically, one of the difficulties that lie in solving problem (6) is the specification of density function $p(\mathbf{y})$ since precisely determining $g(t)$ is obviously a hard thing, though $p_t(\mathbf{y})$ may be easily estimated via the historical data. So, we may explore an alternative technique to model the case of uncertain exit time. As a matter of fact, if it is hard to obtain the precise distribution of the exit time, an intuitive indirect approach is to monitor and optimize the most adverse case of exit so that the resulting portfolio is still preferable with uncertain exit time. This is the so called *worst-case analysis* extensively used in system control. It will be seen that the problem resulting from the "uncertainty" of the uncertain (or stochastic) time of eventual exit can be naturally formulated within the framework of Zhu and Fukushima (2005).

4 Robust Formulation with Uncertain Exit Time

In this section, the assumption in model (6) that the probability distribution of the exit time τ is precisely known is relaxed. We assume that the density function of the exit time is only known to belong to a certain set which covers all the possible exit scenarios, and formulate the portfolio selection problem by means of the worst-case CVaR strategy.

From a practical point of view, we deal with a discrete version of the probability distribution of τ to develop the model. [6] The reason is not only that this treatment will result in a tractable model, but also that it meets the general purpose since we usually approximate the continuous distribution via discretization sampling.

Due to the uncertainty of the distribution of asset returns resulting from the exit time, we replace the CVaR criterion by the worst-case CVaR criterion and reformulate (6) as the following problem:

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[6] Mathematically, we may consider the exit time following a general distribution when building the model. On the other hand, we are concerned with the rationality and applicability of our approach in practice because many incentives that may drive an exit are discrete, such as noneconomic factors (death, divorce), changes in taxes, changes in regulations or market structure, and changes in the institution's liability structure. More importantly, all the exits (transactions) are accomplished discretely in reality. Thus, it is fair to say that the assumption of discrete exit is the rule, not the exception.

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{p(\cdot)\in\mathscr{P}_{\mathrm{M}}} \min_{\alpha\in\Re} \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in\Re^N} [f(\mathbf{x},\mathbf{y})-\alpha]^+ p(\mathbf{y})d\mathbf{y}, \tag{8}$$

where the set $\mathscr{P}_{\mathrm{M}}$ represents all the densities of the possible probability distributions of asset returns, and is defined as

$$\mathscr{P}_{\mathrm{M}} = \left\{\sum_{i=1}^{m} \lambda_i p_i(\cdot) : (\lambda_1, \cdots, \lambda_m)' \in \Omega\right\} \tag{9}$$

with Ω being a compact set satisfying the probability measure such that

$$\Omega \subseteq \left\{(\lambda_1, \cdots, \lambda_m)' : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, i = 1, \cdots, m\right\}. \tag{10}$$

Define

$$F_\beta^i(\mathbf{x},\alpha) = \alpha + \frac{1}{1-\beta}\int_{\mathbf{y}\in\Re^N} [f(\mathbf{x},\mathbf{y})-\alpha]^+ p_i(\mathbf{y})d\mathbf{y}. \tag{11}$$

Then we have the following theorem whose proof can be found in Zhu and Fukushima (2005) (to ensure the paper is self-contained, we provide the proof in Appendix):

Theorem 1 *Let Ω be defined in (10), then for each $\mathbf{x}$, $\mathrm{WCVaR}_\beta(\mathbf{x})$ with respect to $\mathscr{P}_{\mathrm{M}}$ defined in (9) is equivalently given by*

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{\alpha\in\Re} \max_{\boldsymbol{\lambda}\in\Omega} \sum_{i=1}^{m} \lambda_i F_\beta^i(\mathbf{x},\alpha).$$

Theorem 1 unveils the fact that, for fixed $\mathbf{x}$, the computation of WCVaR amounts to solving a min-max problem, which is easy to deal with because the objective function is convex in α and concave in $\boldsymbol{\lambda}$. It should also be noted that while Zhu and Fukushima (2005) denote Ω as a general set, here Ω is supposed to be equipped with a sigma-algebra over which a probability measure is assigned, the actual probability representing the frequencies with which the "exit" takes place.

Now define

$$F_\beta^\Omega(\mathbf{x},\alpha) = \max_{\boldsymbol{\lambda}\in\Omega} \sum_{i=1}^{m} \lambda_i F_\beta^i(\mathbf{x},\alpha).$$

By Theorem 1, the following corollary is obtained immediately.

Corollary 1 *Minimizing* $\mathrm{WCVaR}_\beta(\mathbf{x})$ *over* $\mathcal{X}$ *can be achieved by minimizing* $F_\beta^\Omega(\mathbf{x}, \alpha)$ *over* $\mathcal{X} \times \Re$*, i.e.,*

$$\min_{\mathbf{x}\in\mathcal{X}} \mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{(\mathbf{x},\alpha)\in\mathcal{X}\times\Re} F_\beta^\Omega(\mathbf{x}, \alpha). \tag{12}$$

More specifically, if $(\mathbf{x}^*, \alpha^*)$ *attains the right-hand side minimum in (12), then* $\mathbf{x}^*$ *attains the left-hand side minimum. Conversely, if* $\mathbf{x}^*$ *attains the left-hand side minimum, then* $(\mathbf{x}^*, \alpha^*)$ *attains the right-hand side minimum, where* α^* *is the minimizer of* $F_\beta^\Omega(\mathbf{x}^*, \alpha)$*.*

Up to this point, we have transformed the problem of selecting a portfolio with uncertain exit time into a robust optimization problem in the sense of worst-case analysis, which requires further reformulation before it can be efficiently solved. Theorem 1 together with Corollary 1 will serve as a basis for the tractable reformulation.

4.1 WCVaR formulation with no information on exit time

In this subsection, assuming that there is no information available on the exit time, we discuss the worst-case CVaR strategy for the robust portfolio selection problem.

If there is no available information on exiting, the distribution of the exit time can only be represented in general as

$$\Omega_A = \left\{ (\lambda_1, \cdots, \lambda_m)' : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, i = 1, \cdots, m \right\}.$$

Then by Zhu and Fukushima (2005), Corollary 1 reduces to

$$\min_{\mathbf{x}\in\mathcal{X}} \mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{(\mathbf{x},\alpha)\in\mathcal{X}\times\Re} \max_{i\in\mathcal{L}} F_\beta^i(\mathbf{x}, \alpha) \tag{13}$$

where

$$\mathcal{L} = \{1, 2, \cdots, m\}. \tag{14}$$

Given the worst-case expected target return μ, it can be easily verified that

$$\min_{p(\cdot)\in\mathscr{P}_\mathrm{M}} \int_{\mathbf{y}\in\Re^N} -f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y} = \min_{\boldsymbol{\lambda}\in\Omega_A} \left\{ \sum_{i=1}^{m} \lambda_i \int_{\mathbf{y}\in\Re^N} -f(\mathbf{x}, \mathbf{y}) p_i(\mathbf{y}) d\mathbf{y} \right\} \geq \mu$$

is equivalent to

$$\int_{\mathbf{y}\in\Re^N} -f(\mathbf{x},\mathbf{y})p_i(\mathbf{y})d\mathbf{y} \geq \mu, i = 1,\cdots,m.$$

Thus, the feasible set of asset positions that satisfy the budget constraint, target return constraint, and regulation constraint can be explicitly formulated as

$$\mathcal{X}_A = \left\{\mathbf{x} : \mathbf{e}'\mathbf{x} = 1, \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}, \int_{\mathbf{y}\in\Re^N} -f(\mathbf{x},\mathbf{y})p_i(\mathbf{y})d\mathbf{y} \geq \mu, i = 1,\cdots,m\right\} \quad (15)$$

where $\mathbf{e}$ denotes the vector of ones, $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ are the lower and upper regulation bounds on the portfolio positions satisfying $\underline{\mathbf{x}} \geq \mathbf{0}$ and $\overline{\mathbf{x}} \leq \mathbf{e}$.

The difficulty in computing (11) lies in the calculation of the integral of the multivariate and non-smooth function. In this paper, we adopt approximation via sampling method (Rockafellar and Uryasev, 2000) as follows:

$$F^i_\beta(\mathbf{x},\alpha) \approx \alpha + \frac{1}{1-\beta}\sum_{k=1}^{S^i} \pi^i_k[f(\mathbf{x},\mathbf{y}^i_{[k]}) - \alpha]^+, \quad i = 1,\cdots,m,$$

where S^i denotes the number of samples with respect to the i-th distribution scenario $p_i(\cdot)$, $\mathbf{y}^i_{[k]}$ denotes the k-th sample of $p_i(\cdot)$, and π^i_k denotes the corresponding probability of $\mathbf{y}^i_{[k]}$ (we use the subscript $[k]$ to distinguish a vector from a scalar).

Now, from (13) we are in a position to establish the following proposition:

Proposition 2 *Let $\boldsymbol{\pi}^i = (\pi^i_1,\cdots,\pi^i_{S^i})'$ and $l = \sum_{i=1}^m S^i$. Then, by introducing an auxiliary vector $\mathbf{u} = (\mathbf{u}^1;\mathbf{u}^2;\cdots;\mathbf{u}^m) \in \Re^l$, the optimization problem (8) with $\Omega = \Omega_A$ can be approximated by the following minimization problem with variables $(\mathbf{x},\mathbf{u},\alpha,\theta) \in \Re^n \times \Re^l \times \Re \times \Re$,*

$$\begin{aligned} &\min\ \theta \\ &\text{s.t. } \mathbf{x} \in \mathcal{X}_A, \\ &\qquad \alpha + \frac{1}{1-\beta}(\boldsymbol{\pi}^i)'\mathbf{u}^i \leq \theta, \\ &\qquad u^i_k \geq f(\mathbf{x},\mathbf{y}^i_{[k]}) - \alpha, \\ &\qquad u^i_k \geq 0, \quad k = 1,\cdots,S^i, \quad i = 1,\cdots,m. \end{aligned} \quad (16)$$

Apparently, with sampling technique, the "min-sup-min" optimization problem (8) reduces to a general optimization problem. If $f(\mathbf{x},\mathbf{y})$ is a convex function with respect to $\mathbf{x}$, then problem (16) is a convex program. Furthermore, if $f(\mathbf{x}, \mathbf{y})$ is a linear function with respect to $\mathbf{x}$, then this problem is a linear program, and therefore can be efficiently solved. Note that in the special case where $m = 1$ (the exit time is fixed without any uncertainty), problem (16) reduces to the ordinary CVaR minimization problem.

Suppose there exist n risky assets for an investor to construct portfolios. Let random vector $\mathbf{y} = (y_1, \cdots, y_n)' \in \Re^n$ denote uncertain returns of the n risky assets, and $\mathbf{x} = (x_1, \cdots, x_n)'$ denote the amount of the portfolio to be invested into the n risky assets. Then the loss function is defined as

$$f(\mathbf{x}, \mathbf{y}) = -\mathbf{x}'\mathbf{y}.$$

By definition, the portfolio return is the negative of the loss, i.e., $\mathbf{x}'\mathbf{y}$. Thus the constraints $\int_{\mathbf{y} \in \Re^N} -f(\mathbf{x}, \mathbf{y}) p_i(\mathbf{y}) d\mathbf{y} \geq \mu$ $(i = 1, \cdots, m)$ can be written as

$$\mathbf{x}'\bar{\mathbf{y}}^i \geq \mu, \quad i = 1, \cdots, m,$$

where $\bar{\mathbf{y}}^i$ denotes the expectation of $\mathbf{y}$ with respect to the distribution $p_i(\cdot)$.

Together with (15) and (16), the robust portfolio selection problem with uncertain exit time can then be cast as the following linear program with variables $(\mathbf{x}, \mathbf{u}, \alpha, \theta) \in \Re^n \times \Re^l \times \Re \times \Re$,

$$\begin{aligned} \min\ & \theta \\ \text{s.t. } & \mathbf{e}'\mathbf{x} = 1, \\ & \underline{\mathbf{x}} \leq \mathbf{x} \leq \bar{\mathbf{x}}, \\ & \mathbf{x}'\bar{\mathbf{y}}^i \geq \mu, \\ & \alpha + \frac{1}{1-\beta} (\boldsymbol{\pi}^i)' \mathbf{u}^i \leq \theta, \\ & u_k^i \geq -\mathbf{x}'\mathbf{y}_{[k]}^i - \alpha, \\ & u_k^i \geq 0, \quad k = 1, \cdots, S^i, \quad i = 1, \cdots, m. \end{aligned} \tag{17}$$

4.2 WCVaR formulation with partial information on exit time

In this subsection we consider the portfolio selection problem by means of the worst-case CVaR strategy in the case where partial information on exiting is available.

Suppose Ω in (10) is given as a component-wise bounded set such that

$$\Omega_B = \left\{ \boldsymbol{\lambda} : \mathbf{e}'\boldsymbol{\lambda} = 1, \ \underline{\boldsymbol{\lambda}} \le \boldsymbol{\lambda} \le \overline{\boldsymbol{\lambda}} \right\}, \tag{18}$$

where $\underline{\boldsymbol{\lambda}}$ and $\overline{\boldsymbol{\lambda}}$ are two given constant vectors. The condition $\mathbf{e}'\boldsymbol{\lambda} = 1$ ensures $\boldsymbol{\lambda}$ to be a probability distribution, and the non-negativity constraint $\boldsymbol{\lambda} \ge 0$ is included in the bound constraints $\underline{\boldsymbol{\lambda}} \le \boldsymbol{\lambda} \le \overline{\boldsymbol{\lambda}}$. Since Ω_B can be easily specified and reformulated in a tractable way, it is one of the most often used uncertain sets in robust optimization formulation.

Denote

$$\boldsymbol{\pi} \cdot \mathbf{u} = \begin{bmatrix} (\boldsymbol{\pi}^1)'\mathbf{u}^1 \\ \vdots \\ (\boldsymbol{\pi}^m)'\mathbf{u}^m \end{bmatrix}.$$

By Corollary 1, we have the counterpart of problem (16) that minimizing $\text{WCVaR}_\beta(\mathbf{x})$ over $\mathcal{X}$ can be achieved by solving the following optimization problem with decision variables $(\mathbf{x}, \mathbf{u}, \alpha, \theta) \in \Re^n \times \Re^l \times \Re \times \Re$, i.e.,

$$\begin{aligned} &\min \ \theta \\ &\ \text{s.t.} \ \mathbf{x} \in \mathcal{X}, \\ &\qquad \max_{\boldsymbol{\lambda} \in \Omega_B} \boldsymbol{\lambda}'(\mathbf{e}\alpha + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \mathbf{u}) \le \theta, \\ &\qquad u_k^i \ge f(\mathbf{x}, \mathbf{y}_{[k]}^i) - \alpha, \\ &\qquad u_k^i \ge 0, \quad k = 1, \cdots, S^i, \quad i = 1, \cdots, m. \end{aligned} \tag{19}$$

In the sequel, we reformulate (19) into a more tractable one. For brevity, we denote

$$\mathbf{v} = \mathbf{e}\alpha + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \mathbf{u}.$$

Consider the following linear program:

$$\begin{aligned} &\max_{\boldsymbol{\lambda} \in \Re^m} \ \boldsymbol{\lambda}'\mathbf{v} \\ &\ \text{s.t.} \ \ \mathbf{e}'\boldsymbol{\lambda} = 1, \\ &\qquad \underline{\boldsymbol{\lambda}} \le \boldsymbol{\lambda} \le \overline{\boldsymbol{\lambda}}. \end{aligned} \tag{20}$$

We obtain the corresponding dual program as follows:[7]

$$
\begin{array}{lll}
\min\limits_{(z,\boldsymbol{\xi},\boldsymbol{\omega})\in\Re\times\Re^m\times\Re^m} & z+\overline{\boldsymbol{\lambda}}'\boldsymbol{\xi}+\underline{\boldsymbol{\lambda}}'\boldsymbol{\omega} & \\
\text{s.t.} & \mathbf{e}z+\boldsymbol{\xi}+\boldsymbol{\omega}=\mathbf{v}, & \\
 & \boldsymbol{\xi}\geq 0, \boldsymbol{\omega}\leq 0. &
\end{array} \tag{21}
$$

In relation to (19), let us consider the following minimization problem in $(\mathbf{x}, \mathbf{u}, z, \boldsymbol{\xi}, \boldsymbol{\omega}, \alpha,$
$\theta) \in \Re^n \times \Re^l \times \Re \times \Re^m \times \Re^m \times \Re \times \Re$:

$$
\begin{array}{ll}
\min \theta & \\
\text{s.t. } \mathbf{x} \in \mathcal{X}, & \\
\quad z+\overline{\boldsymbol{\lambda}}'\boldsymbol{\xi}+\underline{\boldsymbol{\lambda}}'\boldsymbol{\omega} \leq \theta, & \\
\quad \mathbf{e}z+\boldsymbol{\xi}+\boldsymbol{\omega}=\mathbf{e}\alpha+\dfrac{1}{1-\beta}\boldsymbol{\pi}\cdot\mathbf{u}, & \\
\quad \boldsymbol{\xi}\geq 0, \boldsymbol{\omega}\leq 0, & \\
\quad u_k^i \geq f(\mathbf{x}, \mathbf{y}_{[k]}^i) - \alpha, & \\
\quad u_k^i \geq 0, \quad k=1,\cdots,S^i, \quad i=1,\cdots,m. &
\end{array} \tag{22}
$$

Proposition 3 *If* $(\mathbf{x}^*, \mathbf{u}^*, z^*, \boldsymbol{\xi}^*, \boldsymbol{\omega}^*, \alpha^*, \theta^*)$ *solves (22), then* $(\mathbf{x}^*, \mathbf{u}^*, \alpha^*, \theta^*)$ *solves (19). Conversely, if* $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{u}}^*, \tilde{\alpha}^*, \tilde{\theta}^*)$ *solves (19), then* $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{u}}^*, \tilde{z}^*, \tilde{\boldsymbol{\xi}}^*, \tilde{\boldsymbol{\omega}}^*,$ $\tilde{\alpha}^*, \tilde{\theta}^*)$ *solves (22), where* $(\tilde{z}^*, \tilde{\boldsymbol{\xi}}^*, \tilde{\boldsymbol{\omega}}^*)$ *is an optimal solution to (21) with* $\mathbf{v} = \mathbf{e}\tilde{\alpha}^* + \frac{1}{1-\beta}\boldsymbol{\pi}\cdot\tilde{\mathbf{u}}^*$.

The proof of Proposition 3 is provided in the appendix. Proposition 3 shows that solving problem (19) derived from the min-max formulation can be substituted by solving a more tractable formulation (22). Moreover, if $f(\mathbf{x}, \mathbf{y})$ is linear with respect to $\mathbf{x}$ and $\mathcal{X}$ is a convex polyhedron, then the problem can actually be reduced to a linear programming problem, as shown below.

In the special case where $\underline{\boldsymbol{\lambda}} = 0$ and $\overline{\boldsymbol{\lambda}} = \mathbf{e}$, (22) reduces to the minimization problem (16). Moreover, if $m = 1$, (22) reduces to the ordinary CVaR minimization problem.

Recall that the return of the portfolio position $\mathbf{x}$ is given by $\mathbf{x}'\mathbf{y}$. Here, the constraint on the worst-case target return is specified by

[7] In the Appendix, we provide a concise review on the primal linear program and the dual linear program. Interested readers may also refer to Vanderbei (1996) and references therein for details.

$$\min_{p(\cdot)\in\mathscr{P}_{\mathrm{M}}} \int_{\mathbf{y}\in\Re^N} \mathbf{x}'\mathbf{y}p(\mathbf{y})d\mathbf{y} = \min_{\boldsymbol{\lambda}\in\Omega_B} \left\{ \sum_{i=1}^{m} \lambda_i \int_{\mathbf{y}\in\Re^N} \mathbf{x}'\mathbf{y}p_i(\mathbf{y})d\mathbf{y} \right\} \geq \mu,$$

which can be simply expressed as

$$\min_{\boldsymbol{\lambda}\in\Omega_B} \sum_{i=1}^{m} \lambda_i \left(\mathbf{x}'\bar{\mathbf{y}}^i \right) \geq \mu.$$

Denote the matrix constructed by the expected asset returns conditional on m time points as

$$\bar{Y} = \begin{bmatrix} (\bar{\mathbf{y}}^1)' \\ \vdots \\ (\bar{\mathbf{y}}^m)' \end{bmatrix}.$$

Then, by (18), the feasible set of asset positions $\mathcal{X}$ is given as

$$\mathcal{X}_B = \left\{ \mathbf{x} : \ \mathbf{e}'\mathbf{x} = 1, \ \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}, \ \min_{\left\{\boldsymbol{\lambda}: \ \mathbf{e}'\boldsymbol{\lambda}=1, \ \underline{\boldsymbol{\lambda}}\leq\boldsymbol{\lambda}\leq\overline{\boldsymbol{\lambda}}\right\}} (\bar{Y}\mathbf{x})'\boldsymbol{\lambda} \geq \mu \right\}. \tag{23}$$

The dual problem of the linear program involved in (23), i.e.,

$$\begin{aligned} \min_{\boldsymbol{\lambda}\in\Re^m} \ & (\bar{Y}\mathbf{x})'\boldsymbol{\lambda} \\ \text{s.t.} \ & \mathbf{e}'\boldsymbol{\lambda} = 1, \\ & \underline{\boldsymbol{\lambda}} \leq \boldsymbol{\lambda} \leq \overline{\boldsymbol{\lambda}}, \end{aligned} \tag{24}$$

is expressed as

$$\begin{aligned} \max_{(\delta,\boldsymbol{\rho},\boldsymbol{\nu})\in\Re\times\Re^m\times\Re^m} \quad & \delta + \overline{\boldsymbol{\lambda}}'\boldsymbol{\rho} + \underline{\boldsymbol{\lambda}}'\boldsymbol{\nu} \\ \text{s.t.} \quad & \mathbf{e}\delta + \boldsymbol{\rho} + \boldsymbol{\nu} = \bar{Y}\mathbf{x}, \\ & \boldsymbol{\rho} \leq 0, \ \boldsymbol{\nu} \geq 0. \end{aligned} \tag{25}$$

By the duality theory of linear programming, the optimal objective value of (25) gives a the lower bound of problem (24). Moreover, if both the primal problem (24) and the dual problem (25) have optimal solutions, then the duality gap is zero. Therefore, it can be easily verified that $\mathcal{X}_B$ coincides with the following set Φ^B, which is expressed as

$$\Phi^B = \left\{ \mathbf{x} : \exists(\delta, \boldsymbol{\rho}, \boldsymbol{\nu}) \text{ satisfying } \begin{array}{l} \mathbf{e}'\mathbf{x} = 1,\ \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}},\ \mathbf{e}\delta + \boldsymbol{\rho} + \boldsymbol{\nu} = \bar{Y}\mathbf{x}, \\ \boldsymbol{\rho} \leq 0,\ \boldsymbol{\nu} \geq 0,\ \delta + \overline{\boldsymbol{\lambda}}'\boldsymbol{\rho} + \underline{\boldsymbol{\lambda}}'\boldsymbol{\nu} \geq \mu \end{array} \right\}.$$

Thus, by (22), the robust portfolio selection problem with partial information on uncertain exit time specified by (18) can be formulated as the following linear program with variables $(\mathbf{x}, \mathbf{u}, z, \boldsymbol{\xi}, \boldsymbol{\omega}, \alpha, \theta, \delta, \boldsymbol{\rho}, \boldsymbol{\nu}) \in \Re^n \times \Re^l \times \Re \times \Re^m \times \Re^m \times \Re \times \Re \times \Re \times \Re^m \times \Re^m$:

$$\begin{aligned} \min\ & \theta \\ \text{s.t. } & \mathbf{e}'\mathbf{x} = 1, \\ & \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}, \\ & \delta + \overline{\boldsymbol{\lambda}}'\boldsymbol{\rho} + \underline{\boldsymbol{\lambda}}'\boldsymbol{\nu} \geq \mu, \\ & \mathbf{e}\delta + \boldsymbol{\rho} + \boldsymbol{\nu} = \bar{Y}\mathbf{x}, \\ & \boldsymbol{\rho} \leq 0,\ \boldsymbol{\nu} \geq 0, \\ & z + \overline{\boldsymbol{\lambda}}'\boldsymbol{\xi} + \underline{\boldsymbol{\lambda}}'\boldsymbol{\omega} \leq \theta, \\ & \mathbf{e}z + \boldsymbol{\xi} + \boldsymbol{\omega} = \mathbf{e}\alpha + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \mathbf{u}, \\ & \boldsymbol{\xi} \geq 0, \boldsymbol{\omega} \leq 0, \\ & u_k^i \geq -\mathbf{x}'\mathbf{y}_{[k]}^i - \alpha, \\ & u_k^i \geq 0, \quad k = 1, \cdots, S^i, \quad i = 1, \cdots, m. \end{aligned} \tag{26}$$

5 Specification of Information on Distribution of Exit

In this section, we relate the specification of information on the exit time to the incentives of exogenous and endogenous factors which drive the investor to terminate his portfolio.

We begin with a discussion of the classification of the eventual exit time. According to the relation between the exit time and the asset prices, we may consider two types of exit: *exogenous* and *endogenous* exit times. An exit is an exogenous exit if the investor exits the market regardless of price fluctuation of any asset in his portfolio, such as the time of order execution, the time of the investor's death, and the time of sudden purchasing or selling a house, etc. (Yaari, 1965; Hakansson, 1969; Merton, 1971; Richard, 1975). On the other hand, an exit is an endogenous exit if the exit of the investor heavily depends on the price behavior of the assets in his portfolio, such as the exit depicted as the disposition effect in behavioral finance (Shefrin and Statman, 1985; Odean, 1998) or the optimal exercise time for an American option (Hull, 1999). In practice, however, it is a difficult task for an investor to predict the type of his eventual exit, either exogenous or endogenous. On the contrary, the time of his exit may depend not only on the exogenous accidental events, but also on the price fluctuation of his portfolio, though this will make the treatment more complex. More specifically, the density function is not only dependent on time τ, but also dependent on the price V_τ. To the best of our knowledge, research under such a setting does not exist, though Martellini and Urošević (2006) have explored mean-variance analysis of exogenous and endogenous exit times separately.

We now consider to specify the bound on the distribution probability $\boldsymbol{\lambda}$ of the exit time τ. For each $i = 1, \cdots, m$, denote

$$E_i^{exo} = \{\text{Exit at time } t_i \text{ driven by exogenous factors}\},$$
$$E_i^{end} = \{\text{Exit at time } t_i \text{ driven by endogenous factors}\}.$$

Since the exogenous factors and endogenous factors that drive the exit are independent, the exit probability λ_i at time t_i $(i = 1, \cdots, m)$ can be decomposed into two distinct parts as

$$\lambda_i = \Pr\{\tau = t_i\} = \Pr\left\{E_i^{exo} \bigcup E_i^{end}\right\} = \Pr\{E_i^{exo}\} + \Pr\left\{E_i^{end}\right\}.$$

This provides us a great convenience in specifying the information on the distribution of the uncertain exit time.

Denote for each $i = 1, \cdots, m$

$$\lambda_i^{exo} = \Pr\{E_i^{exo}\} \text{ and } \lambda_i^{end} = \Pr\left\{E_i^{end}\right\}.$$

If the bounds of λ_i^{exo} and λ_i^{end} $(i = 1, \cdots, m-1)$ are determined respectively, then the bound of λ_i $(i = 1, \cdots, m-1)$ can be calculated via the addition operation of interval numbers in the following manner:

$$[a, b] + [c, d] = [a + c, b + d].$$

Notice that $\lambda_m = 1 - \left(\sum_{i=1}^{m-1} \lambda_i\right)$, the bound of λ_m can be calculated via the subtraction operation of interval numbers defined by

$$[a, b] - [c, d] = [a - d, b - c].$$

Recall that $0 \leq \lambda_i \leq 1$ should never be neglected to construct a reasonable bound.

5.1 Exogenous exit

Generally, an uncertain sudden exit driven by an exogenous factor can be modeled as the jump of a Poisson process. Although it may involve many different exogenous reasons, exogenous exit can be well captured by the jump of a Poisson process since it is well known that the sum of independent Poisson processes remains a Poisson process.

By the fact that the distribution of waiting time of the first jump of Poisson process with intensity ς follows the exponential distribution with parameter ς, we conclude that the exogenous exit probability is given by (4).

Because of the lack of data, it is difficult to estimate the exact exit intensity driven by many of the exogenous factors. However, we can conservatively choose a certain interval that may cover all of the possible exit intensities ς, i.e., $\varsigma \in [\underline{\varsigma}, \overline{\varsigma}]$, where $\overline{\varsigma} > \underline{\varsigma} > 0$. Consequently, the upper and the lower bounds of λ_i^{exo} $(i = 1, \cdots, m-1)$ are given by

$$\underline{\lambda}_i^{exo} = \min_{\varsigma \in [\underline{\varsigma}, \overline{\varsigma}]} \left\{ e^{-\varsigma t_{i-1}} - e^{-\varsigma t_i} \right\}, \tag{27}$$

$$\overline{\lambda}_i^{exo} = \max_{\varsigma \in [\underline{\varsigma}, \overline{\varsigma}]} \left\{ e^{-\varsigma t_{i-1}} - e^{-\varsigma t_i} \right\}, \tag{28}$$

where $t_0 = 0$.

For λ_1^{exo}, it is easy to see that

$$\underline{\lambda}_1^{exo} = 1 - e^{-\underline{\varsigma} t_1} \text{ and } \overline{\lambda}_1^{exo} = 1 - e^{-\overline{\varsigma} t_1}.$$

For $i = 2, \cdots, m-1$, denote

$$g_i(\varsigma) = e^{-\varsigma t_{i-1}} - e^{-\varsigma t_i}.$$

Solving the equation

$$g_i'(\varsigma) = t_i e^{-\varsigma t_i} - t_{i-1} e^{-\varsigma t_{i-1}} = 0,$$

we get the unique root

$$\varsigma^* = \frac{\ln(t_i) - \ln(t_{i-1})}{t_i - t_{i-1}}.$$

Notice that $0 < g_i(\varsigma) < 1$ for any $\varsigma \in [\underline{\varsigma}, \overline{\varsigma}]$. By (27) and (28), simple calculus yields

$$\underline{\lambda}_i^{exo} = \begin{cases} \min\left\{g_i(\underline{\varsigma}), g_i(\overline{\varsigma}), g_i(\varsigma^*)\right\}, \text{ if } \varsigma^* \in [\underline{\varsigma}, \overline{\varsigma}], \\ \min\left\{g_i(\underline{\varsigma}), g_i(\overline{\varsigma})\right\}, \qquad \text{else,} \end{cases} \tag{29}$$

and

$$\overline{\lambda}_i^{exo} = \begin{cases} \max\left\{g_i(\underline{\varsigma}), g_i(\overline{\varsigma}), g_i(\varsigma^*)\right\}, \text{ if } \varsigma^* \in [\underline{\varsigma}, \overline{\varsigma}], \\ \max\left\{g_i(\underline{\varsigma}), g_i(\overline{\varsigma})\right\}, \qquad \text{else.} \end{cases} \tag{30}$$

5.2 Endogenous exit

Generally speaking, there are two cases providing an investor incentives to terminate his portfolio, a large drawdown or a large appreciation. In the presence of a large drawdown, the investor may exit the market to reduce his loss. On the contrary, the investor may also terminate his portfolio when faced with a large appreciation, since he may believe the portfolio reached its near-term maximum value. But, in portfolio optimization problems, without choosing a portfolio position first, how can one predict the probability of drawdown or appreciation precisely? Thus the difficulty in modeling precisely the endogenous exit is naturally embedded in the portfolio selection problem.

Fortunately, in the worst-case CVaR framework, we do not necessarily require the precise value of λ_i^{end}, but an interval covering all of the possible endogenous exit intensities, which makes the problem relaxed and hence much tractable. In view of this point, the remaining task is to ascertain the upper and lower bounds of λ_i^{end}.

For simplicity, we assume here that the investor exits the market if and only if the portfolio return rises above a high-water threshold γ. Recall that $\mathbf{y}^i$ denotes the vector of uncertain returns at time t_i, where y_j^i represents the return of asset j, and that $\mathbf{e}'\mathbf{x} = 1$ and $\mathbf{x} \geq 0$. For $i = 1$, since

$$\min_j\{y_j^1\} \leq \mathbf{x}'\mathbf{y}^1 \leq \max_j\{y_j^1\},$$

we have

$$\Pr\left\{\min_j\{y_j^1\} \geq \gamma\right\} \leq \Pr\left\{\mathbf{x}'\mathbf{y}^1 \geq \gamma\right\} \leq \Pr\left\{\max_j\{y_j^1\} \geq \gamma\right\}.$$

Hence, a lower bound of endogenous exit probability at time t_1 is given by

$$\underline{\lambda}_1^{end} = \Pr\left\{\min_j\{y_j^1\} \geq \gamma\right\},$$

and a upper bound is given by

$$\overline{\lambda}_1^{end} = \Pr\left\{\max_j\{y_j^1\} \geq \gamma\right\}.$$

Similarly, for $i = 2, \cdots, m-1$, since

$$\min_j\{y_j^k\} \leq \mathbf{x}'\mathbf{y}^k \leq \max_j\{y_j^k\}, \quad k = 2, \cdots, i,$$

we obtain

$$\begin{aligned}\underline{\lambda}_i^{end} &= \Pr\left\{\min_j\{y_j^i\} \geq \gamma,\ \max_j\{y_j^k\} < \gamma,\ k = 1, \cdots, i-1\right\}\\ &\leq \Pr\left\{\mathbf{x}'\mathbf{y}^i \geq \gamma,\ \mathbf{x}'\mathbf{y}^k < \gamma,\ k = 1, \cdots, i-1\right\}\\ &\leq \Pr\left\{\max_j\{y_j^i\} \geq \gamma,\ \min_j\{y_j^k\} < \gamma,\ k = 1, \cdots, i-1\right\} = \overline{\lambda}_i^{end}.\end{aligned}$$

Based on the lower and upper bounds of λ_i^{end} specified above, we can easily get the portfolio decision $\tilde{\mathbf{x}}^{(0)}$ by solving model (26). However, it should be mentioned that the bounds of the endogenous exit probability in this case are not tight enough in practice because they depend on the extreme scenarios of individual risky asset. But, given the portfolio $\tilde{\mathbf{x}}^{(0)}$, the probability of endogenous exit can be precisely predicted. Thus we can refine the portfolio decision with this new information via iteration. More specifically, we perform the portfolio selection procedure in the following steps:

Step 1: For fixed γ and μ, find an optimal portfolio $\tilde{\mathbf{x}}^{(0)}$ by solving model (26) where the probability bounds of endogenous exit are derived from real historical market data or Monte Carlo simulation using the above approach.

Step 2: For each $i = 1, \cdots, m-1$, estimate the actual exit probabilities of the endogenous incentives with $\tilde{\mathbf{x}}^{(j-1)}$ as

$$(\lambda_i^{end})^{(j-1)} = \begin{cases} \Pr\left\{(\tilde{\mathbf{x}}^{(j-1)})'\mathbf{y}^1 \geq \gamma\right\}, & i = 1,\\ \Pr\left\{(\tilde{\mathbf{x}}^{(j-1)})'\mathbf{y}^i \geq \gamma,\ (\tilde{\mathbf{x}}^{(j-1)})'\mathbf{y}^k < \gamma,\ k = 1, \cdots, i-1\right\}, & i = 2, \cdots, m-1.\end{cases}$$

Step 3: For each $i = 1, \cdots, m-1$, compute the bounds of exit probability as

$$[\underline{\lambda}_i^{(j-1)}, \overline{\lambda}_i^{(j-1)}] = [\underline{\lambda}_i^{exo} + (\lambda_i^{end})^{(j-1)}, \overline{\lambda}_i^{exo} + (\lambda_i^{end})^{(j-1)}].$$

Step 4: Find the optimal portfolio $\tilde{\mathbf{x}}^{(j)}$ by solving model (26) using the bounds of exit probability obtained from **Step** 3.

Step 5: Compute the distance between portfolios as

$$d(\tilde{\mathbf{x}}^{(j)}, \tilde{\mathbf{x}}^{(j-1)}) = \frac{1}{n}\sum_{k=1}^{n} |x_k^{(j)} - x_k^{(j-1)}| \text{ or } |\mathrm{WCVaR}_\beta(\tilde{\mathbf{x}}^{(j)}) - \mathrm{WCVaR}_\beta(\tilde{\mathbf{x}}^{(j-1)})|.$$

If $d(\tilde{\mathbf{x}}^{(j)}, \tilde{\mathbf{x}}^{(j-1)}) \leq \epsilon$ (we set $\epsilon = 0.05$ in our numerical experiments), $\tilde{\mathbf{x}}^{(j)}$ is an approximate optimal portfolio, and $(\lambda_i^{end})^{(j)}$ $(i = 1, \cdots, m-1)$ is the endogenous exit probability; terminate. Otherwise, go to **Step** 2 with $j := j+1$.

More generally, we can further consider that γ is time-varying, which will make the specification of the information more practical. Other endogenous exit factors arising from some different portfolio operational strategies can be also modeled in their own manners.

6 Empirical Applications

In this section, we demonstrate how the proposed model can be implemented in practice and compare the portfolio performance from this model to the traditional procedures commonly used in the analysis of real market data and simulated data. Real market data experiments investigate what would result if an investor employed our approach compared with the traditional approaches, while the controlled experiments with simulated data are performed to study the applicability and the implications of our approach. We use MatLab6p5 and SeDuMi1.05 (Sturm, 2001) for solving our linear programming problems on PC with Intel Pentium 4 CPU 3.00GHz, 1.5GB RAM. All problems were successfully solved within 9 seconds.

6.1 Real market data simulation analysis

In this subsection we consider a portfolio consisting of 10 stocks from Tokyo Stock Exchange and present some numerical experiments in the case of no or partial information available on the exit time with the worst-case CVaR formulations. To construct the portfolio, we collected the historical data of daily closing prices of these stocks from January 4, 1994 to December 30, 2004, which includes 2,711 samples. Suppose that the investment horizon is three days, and that the investor may terminate his portfolio at the end of the first two days. More specifically, there are three possible exit moments during the investment horizon, i.e., $m = 3$. It should be mentioned that we assume the investment horizon $T = 1$. Then the first and second possible exit time is 1/3 and 2/3, respectively.

In this example, we assume that the samples of future returns are generated by the historical returns. To improve the precision of the calculation, we multiply the returns by 100, i.e.,

$$\begin{aligned} y_t^1 &= \frac{V_t - V_{t-1}}{V_{t-1}} \times 100, \quad t = 2, 3, 4, \cdots, \\ y_t^2 &= \frac{V_t - V_{t-2}}{V_{t-2}} \times 100, \quad t = 3, 5, 7, \cdots, \\ y_t^3 &= \frac{V_t - V_{t-3}}{V_{t-3}} \times 100, \quad t = 4, 7, 10, \cdots, \end{aligned}$$

which means $S^1 = 2,710, S^2 = 1,305$, and $S^3 = 903$. Table 1 list the expected values and covariance of daily returns of the 10 risky assets.

On the other hand, we set $\beta = 0.95$, $\underline{\mathbf{x}} = 0$, and $\overline{\mathbf{x}} = \mathbf{e}$, which implies that short positions are prohibitive. Numerical experiments for the ordinary and the robust portfolio optimization problems are performed via the linear programming models (17) and (26). The former employs the ordinary CVaR as the risk measure, which assumes that the investor terminates his portfolio at maturity, while the latter uses the worst-case CVaR in the presence of no or partial information is given. In the computation of the ordinary portfolio optimization problem, we set $m = 1$ and $S = S^3 = 903$, i.e., only the samples at maturity are used in the model to compute the CVaR.

To proceed further, we need to ascertain the lower and upper bounds of the exit probabilities at each possible exiting moment. As for the exogenous incentives, without loss of generality we set $\underline{\varsigma} = 0.6$ and $\overline{\varsigma} = 1$, which we will pay particular attention to the Monte Carlo analysis later. Hence, we obtain the exogenous bounds of the first two exiting moments as $[\underline{\lambda}_1^{exo}, \ \overline{\lambda}_1^{exo}] = [0.1813, \ 0.2835]$ and $[\underline{\lambda}_2^{exo}, \ \overline{\lambda}_2^{exo}] = [0.1484, \ 0.2021]$.

As the determination of endogenous exit probabilities, we compute them in two steps. In the first step, we simulate the bounds of the endogenous incentives with historical data based on the analysis of Section 5.2. Suppose $\gamma = 5\%$, i.e., the investor may terminate his portfolio if the return of the portfolio is greater than or equal to 5%, then $[\underline{\lambda}_1^{end},\ \overline{\lambda}_1^{end}] = [0,\ 0.1561]$ and $[\underline{\lambda}_2^{end},\ \overline{\lambda}_2^{end}] = [0,\ 0.2775]$. Thus, we get $[\underline{\lambda}_1,\ \overline{\lambda}_1] = [0.1813,\ 0.2835] + [0,\ 0.1561] = [0.1813, 0.4396]$ and $[\underline{\lambda}_2,\ \overline{\lambda}_2] = [0.1484,\ 0.2021] + [0,\ 0.2775] = [0.1484, 0.4796]$, as shown in Table 2 (column *), which lists the concrete bounds of the exit probability at each exit moment.

To explain our method, we assume that the worst-case expected return of the portfolio is $\mu = 0.00025$, then the second step is to get the precise probability of endogenous exit via iterations based on the bounds obtained in the first step. Table 3 shows the concrete process of iterations and its corresponding information. It is of interest that after 2 iterations, we can obtain the precise endogenous probability λ_i^{end} at each possible exit moment. At the same time, we exhibit the optimal portfolio positions after each iteration. As expected, the optimal portfolio resulting from the second iteration changes marginally from that of the first iteration, i.e., the distance $d(\tilde{\mathbf{x}}^{(2)}, \tilde{\mathbf{x}}^{(1)})$ between $\tilde{\mathbf{x}}^{(2)}$ and $\tilde{\mathbf{x}}^{(1)}$ is 0.00016. In fact, it is natural that the value of the worst-case CVaR decreases as the number of iteration increases, because the uncertainty resulting from the endogenous exit is reduced gradually. From Table 2, we can also make a comparison between the bounds with and without iterations. Obviously, we get a more tight bound of exit probability which only involves the uncertainty of the exogenous incentives.

To compare the performances of the ordinary portfolio optimization problem and the robust portfolio optimization problem with uncertain exit time for various values of the required minimal expected/worst-case expected return μ, Table 4 shows the expected values and the CVaRs at the 0.95 confidence level of the corresponding portfolios. It should be mentioned that the ordinary optimal portfolio is obtained by solving model (17) with $m = 1$ and $S = S^3 = 903$. It can also be obtained by solving model (26) with $m = 1$, $\underline{\boldsymbol{\lambda}} = 0$, and $\overline{\boldsymbol{\lambda}} = \mathbf{e}$. The robust optimal portfolio with no or partial information on exit can be obtained by solving (17) and (26) directly. Hence, we can compute the actual expected returns and corresponding CVaRs of the ordinary and robust optimal portfolios when the investor exits the market at different moments. It is obvious that the larger the required minimal expected/worst-case expected returns, the larger the associated risk. For the same value of μ, the risk of the robust optimal portfolio strategy appears to be larger than that of the ordinary optimal portfolio strategy, especially for the worst-case CVaRs with no information. However, higher risk is compensated by higher return. In fact, the larger value of the worst-case CVaR does not necessarily imply higher risk than that of the ordinary CVaR policy, which is only because the investor considers more uncertainty of future extreme scenarios and hence takes a conservative strategy. For the robust CVaR formulation with no exit information, the worst-case expected return can be guaranteed whenever the investor terminates his portfolio. While for the robust CVaR model with partial information of exit, if we define unit risk-return ratio as $L =$ (actual return)/risk, it will be easy to show that this robust strategy is much preferred generally to the ordinary CVaR model (see $\mu = 0.0002, 0.00025, 0.00030$ and 0.00040). For example, if $\mu = 0.00025$ and the exit takes place at the first day, then L(Ordinary CVaR) = 0.000049, while L(Robust CVaR) = 0.000057. It should be noted that such advantage is also possessed by the robust CVaR strategy with no information. There is another interesting phenomenon that the optimal portfolio resulting from the ordinary CVaR is infeasible in general for the both robust formulations except some small μ, which also implies the advantage of the worst-case CVaR for the uncertain exit time problem.

Figure 1 graphs the optimal portfolio positions with the ordinary and the robust CVaR strategies with $\mu = 0.00025$. Obviously, the portfolio with the robust strategy is different from that of the ordinary CVaR policy. There are several implications that can be drawn from this figure as well as Table 1. First, the robust CVaR strategy is more diversified than the ordinary CVaR strategy. In this example, the optimal portfolio of the ordinary CVaR consists of 7 stocks, while both the robust CVaR models with no or partial information have 8 stocks. Second, the ordinary CVaR may give up some higher return assets, such as the 1st and the 5th stocks (the daily expected returns and variances of the two assets are (0.00033, 0.000344) and (0.00062403, 0.00085535), respectively). However, typically an investor may pay particular attention to those assets with higher returns although they have higher volatilities. Moveover, as the worst-case expected return is guaranteed, the investor has no reason to refuse the higher return assets. Finally, the 2nd, 7th and 10th stocks are the three assets that are most likely to be selected by any investor. Conversely, the 3rd stock is the most controversial. Actually, its expected return approaches zero (0.000006) though it has the smallest risk among the 10 stocks (0.00025213).

6.2 Monte Carlo simulation analysis

In this part, we first perform a Monte Carlo simulation analysis to explore the appropriate times of possible exit in a given investment horizon T, i.e., how to determine an appropriate value of m. After that we discuss the sensitivity of the worst-case CVaR with respect to the bounds of exogenous exit probability. Thus some key implications that may help to successfully perform our methodologies in practice will be obtained.

We take the example given by Alexander and Baptista (2002), where the investor seeks to determine how to allocate his wealth among different asset classes. The portfolio is to be constructed by six classes of assets: Four involving U.S. securities (large stocks, small stocks, corporate bonds, and real estate investment trusts (REITs)), and two involving foreign securities (stocks in developed markets and stocks in emerging markets). The following indices are used to measure the rates of return on these classes: The S&P 500 index (large stocks), the Russell 2000 index (small stocks), the Merrill Lynch U.S. corporate bond index, the index for all publicly traded REITs provided by the National Association of Real Estate Investment Trusts, the Morgan Stanley Capital International (MSCI) EAFE index (stocks in developed markets), and the MSCI EM index (stocks in emerging markets). Table 5 exhibits the annual return means, variances, and covariances associated with the six indices from the data for the period of 1989-1999. Despite the preponderance of evidence that asset return distributions are not normal, for simplicity, we assume in this example that the rates of return of these risky securities have a multivariate normal distribution.

In this example, we set $\beta = 0.95$, $\underline{\mathbf{x}} = 0$, $\bar{\mathbf{x}} = \mathbf{e}$, $\gamma = 0.25$, and $\mu = 0.018$. Assuming the investment horizon is one year, we explore the likely possible times of exit before maturity, i.e., we try to find an appropriate value of m which can approximate all the possible exit scenarios. Table 6 exhibits the sensitivity of the worst-case CVaRs with respect to the times of possible exit with $\underline{\varsigma} = 0.5$ and $\bar{\varsigma} = 12$. The first column is the times of possible exit in the whole investment horizon, and the second column is the corresponding moments at which the events of exit take place. For example, the fact that the number of possible exit is 5 means that the investor may terminate his portfolio at five different moments in his investment horizon, which correspond to the ends of the 2nd, 4th, 6th, 9th, and 12th months, respectively. The third and the last columns are the values of the worst-case CVaRs with no or partial information (WCVaR (I) and WCVaR (II)). Assume that the annual returns $\mathbf{y}^T \sim \mathcal{N}(\bar{\mathbf{y}}, \Sigma)$ and the returns of the tth month $\mathbf{y}^t \sim \mathcal{N}(\frac{t}{12}\bar{\mathbf{y}}, \frac{t}{12}\Sigma)$, where $\mathcal{N}(\cdot,\cdot)$ denotes the multivariate normal distribution. Then, we can adopt the Monte Carlo approach to simulate the return evolution of these risky assets. Obviously, as the times of exit increase, the value of the worst-case CVaR increases. This is because an increase of m increases the complexity of the set of possible exit moments, and hence gives rise to a larger CVaR. However, when $m > 5$, the increase of the worst-case CVaR is marginal, unlike the variations of $m < 5$. To some extent, we can safely conclude that the most appropriate times of possible exits in this example is 5. Indeed, there is a tradeoff between the value of m and the computational complexity. Due to not taking fully account of all the possible exit scenarios, smaller m may give rise to modelling risk, while larger m may cause computational risk because of the higher complexity.

The last problem we will tackle here is to perform a sensitivity analysis for the worst-case CVaR with respect to the lower and upper bounds of the exogenous exit probability. Figure 2 plots $\underline{\varsigma}$-WCVaR and $\bar{\varsigma}$-WCVaR curves. The left panel shows that as $\underline{\varsigma}$ increases, the worst-case CVaR decreases where the upper bound $\bar{\varsigma}$ is fixed as 12 or 30. For example, when $\underline{\varsigma}$ varies from 1 to 10, the worst-case CVaR decreases from 0.0589 to 0.0501 ($\overline{\varsigma} = 12$) or from 0.060 to 0.0529 ($\overline{\varsigma} = 30$). This is natural because as $\underline{\varsigma}$ increases, the interval of $[\underline{\varsigma}, \overline{\varsigma}]$ shrinks, which reduces the uncertainty of the exogenous incentives. Consequently, the risk is reduced. On the contrary, as $\overline{\varsigma}$ increases with fixed $\underline{\varsigma}$ ($\underline{\varsigma} = 1$ and 5 in the right panel), the interval of $[\underline{\varsigma}, \overline{\varsigma}]$ expands, which leads to an increase of the worst-case CVaR. However, the variations of the worst-case CVaR resulting from $\overline{\varsigma}$ are less significant than those from the changes of $\underline{\varsigma}$. Actually, the worst-case CVaR varies from 0.0589 to 0.0601 as $\overline{\varsigma}$ increase from 10 to 40 with fixed $\underline{\varsigma} = 1$. After that, the value of the worst-case CVaR does not change any more. That is, an investor should pay more attention to the lower bound rather than the upper bound of the probability of exogenous exit when constructing a portfolio.

7 Conclusion

This paper develops modelling of the uncertainty of eventual exit time in portfolio management. In practice, in addition to the asset price risk, an investor typically faces an exit time risk because he never acknowledges the time of his eventual exit upon entering the market. Considering the inconvenience and complexity of portfolio modelling in the case where the exit time is uncertain, we propose a worst-case CVaR approach as an effective alternative, which is formally defined and applied to robust portfolio management in the recent work of Zhu and Fukushima (2005). The proposed model can accommodate the case where no or partial information on exit is available. In addition, a unifying model incorporating exogenous and endogenous factors is proposed to deal with this uncertainty. At the same time, how to ascertain the bounds of the exit probability at each possible exit moment is explored explicitly.

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Appendix

Proof of Theorem 1

For given $\mathbf{x} \in \mathcal{X}$, define

$$\begin{aligned} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) &= \alpha + \frac{1}{1-\beta} \int_{\mathbf{y} \in \Re^N} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ \left[\sum_{i=1}^m \lambda_i p_i(\mathbf{y}) \right] d\mathbf{y} \\ &= \sum_{i=1}^m \lambda_i F_\beta^i(\mathbf{x}, \alpha), \end{aligned} \tag{31}$$

where $\boldsymbol{\lambda} \in \Omega$. $H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda})$ is convex in α (see Rockafellar and Uryasev 2000, 2002) and affine (concave) in $\boldsymbol{\lambda}$. It is easy to see that $\min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda})$ is a continuous function with respect to $\boldsymbol{\lambda}$. By (1), (2), (9), and the fact that Ω is compact, we can write

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \Re} \sum_{i=1}^m \lambda_i F_\beta^i(\mathbf{x}, \alpha). \tag{32}$$

For each i and fixed $\mathbf{x}$, the optimal solution set of $\min_{\alpha \in \Re} F_\beta^i(\mathbf{x}, \alpha)$ is a nonempty, closed and bounded interval (see Rockafellar and Uryasev 2000, 2002). Thus we can denote

$$[\underline{\alpha}_i^*, \overline{\alpha}_i^*] \triangleq \operatorname*{argmin}_{\alpha \in \Re} F_\beta^i(\mathbf{x}, \alpha), \quad i = 1, \cdots, m.$$

Suppose $g_1(t)$ and $g_2(t)$ are two convex functions defined on $\Re$, and the nonempty, closed and bounded intervals $\left[\underline{t}_1^*, \overline{t}_1^*\right]$, $\left[\underline{t}_2^*, \overline{t}_2^*\right]$ are the sets of minima of these two functions, respectively. It can be easily verified that for any $\beta_1 \geq 0$ and $\beta_2 \geq 0$ such that $\beta_1 + \beta_2 > 0$, $\beta_1 g_1(t) + \beta_2 g_2(t)$ is convex too, and the set of minima of $\beta_1 g_1(t) + \beta_2 g_2(t)$ must lie in the nonempty, closed and bounded interval $\left[\min\{\underline{t}_1^*, \underline{t}_2^*\}, \max\{\overline{t}_1^*, \overline{t}_2^*\}\right]$. From this fact and (31), we get

$$\operatorname*{argmin}_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) \subseteq \mathcal{A}, \quad \forall \boldsymbol{\lambda} \in \Omega,$$

where $\mathcal{A}$ is the nonempty, closed and bounded interval given by

$$\mathcal{A} \triangleq \left[\min_{i \in \mathcal{L}} \underline{\alpha}_i^*, \max_{i \in \mathcal{L}} \overline{\alpha}_i^* \right],$$

where $\mathcal{L}$ has the same meaning as (14). This implies

$$\min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \min_{\alpha \in \mathcal{A}} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}).$$

Therefore, by min-max theory,[8] we have

$$\max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \mathcal{A}} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \min_{\alpha \in \mathcal{A}} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}). \quad (33)$$

It is obvious that

$$\min_{\alpha \in \mathcal{A}} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) \geq \inf_{\alpha \in \Re} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}). \quad (34)$$

By (33), (34) and the well known result on the min-max inequality

$$\inf_{\alpha \in \Re} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) \geq \max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}),$$

we immediately get

$$\max_{\boldsymbol{\lambda} \in \Omega} \min_{\alpha \in \Re} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \min_{\alpha \in \Re} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}).$$

It then follows from (32), along with (31) that

$$\mathrm{WCVaR}_\beta(\mathbf{x}) = \min_{\alpha \in \Re} \max_{\boldsymbol{\lambda} \in \Omega} H_\beta(\mathbf{x}, \alpha, \boldsymbol{\lambda}) = \min_{\alpha \in \Re} \max_{\boldsymbol{\lambda} \in \Omega} \sum_{i=1}^{m} \lambda_i F_\beta^i(\mathbf{x}, \alpha).$$

This completes the proof. □

Proof of Proposition 3

Let $(\mathbf{x}^*, \mathbf{u}^*, z^*, \boldsymbol{\xi}^*, \boldsymbol{\omega}^*, \alpha^*, \theta^*)$ be an optimal solution to (22), and set $\mathbf{v} = \mathbf{v}^* = \mathbf{e}\alpha^* + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \mathbf{u}^*$. By the weak duality theorem of linear programming, we have from (20) and (21) that

$$\max_{\boldsymbol{\lambda} \in \Omega_B} \boldsymbol{\lambda}'\mathbf{v}^* = \max_{\{\boldsymbol{\lambda}:\ \mathbf{e}'\boldsymbol{\lambda}=1, \underline{\boldsymbol{\lambda}} \leq \boldsymbol{\lambda} \leq \overline{\boldsymbol{\lambda}}\}} \boldsymbol{\lambda}'\mathbf{v}^* \leq z^* + \overline{\boldsymbol{\lambda}}'\boldsymbol{\xi}^* + \underline{\boldsymbol{\lambda}}'\boldsymbol{\omega}^* \leq \theta^*,$$

[8] Minimax theory states that suppose $\mathcal{X} \subseteq \mathcal{R}^n$ and $\mathcal{Y} \subseteq \mathcal{R}^N$ are two nonempty compact convex sets, and the function $\psi(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ for fixed $\mathbf{y}$, and concave in $\boldsymbol{y}$ for fixed $\mathbf{x}$. Then we have the equality $\min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{y} \in \mathcal{Y}} \psi(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{y} \in \mathcal{Y}} \min_{\mathbf{x} \in \mathcal{X}} \psi(\mathbf{x}, \mathbf{y})$.

where the last inequality follows from the second constraint of (22). This, together with other constraints in (22), implies that $(\mathbf{x}^*, \mathbf{u}^*, \alpha^*, \theta^*)$ is feasible for problem (19). In the following, we prove that it is even optimal to (19).

Suppose to the contrary that $(\mathbf{x}^*, \mathbf{u}^*, \alpha^*, \theta^*)$ is not optimal to (19), i.e., there exists an optimal solution $(\bar{\mathbf{x}}^*, \bar{\mathbf{u}}^*, \bar{\alpha}^*, \bar{\theta}^*)$ such that

$$\bar{\theta}^* < \theta^*.$$

Let $(\bar{z}^*, \bar{\boldsymbol{\xi}}^*, \bar{\boldsymbol{\omega}}^*)$ be an optimal solution to (21) with $\mathbf{v} = \mathbf{e}\bar{\alpha}^* + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \bar{\mathbf{u}}^*$. Since zero duality gap can be guaranteed by the strong duality theorem of linear programming, we have

$$\begin{aligned}
&\bar{z}^* + \overline{\boldsymbol{\lambda}}'\bar{\boldsymbol{\xi}}^* + \underline{\boldsymbol{\lambda}}'\bar{\boldsymbol{\omega}}^* \\
=& \max_{\{\boldsymbol{\lambda}:\ \mathbf{e}'\boldsymbol{\lambda}=1, \underline{\boldsymbol{\lambda}} \le \boldsymbol{\lambda} \le \overline{\boldsymbol{\lambda}}\}} \boldsymbol{\lambda}'(\mathbf{e}\bar{\alpha}^* + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \bar{\mathbf{u}}^*) \\
=& \max_{\boldsymbol{\lambda} \in \Omega_B} \boldsymbol{\lambda}'(\mathbf{e}\bar{\alpha}^* + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \bar{\mathbf{u}}^*) \\
\le& \bar{\theta}^*.
\end{aligned}$$

This, together with other constraints in (19) and (21), implies that $(\bar{\mathbf{x}}^*, \bar{\mathbf{u}}^*, \bar{z}^*, \bar{\boldsymbol{\xi}}^*, \bar{\boldsymbol{\omega}}^*, \bar{\alpha}^*, \bar{\theta}^*)$ is feasible to problem (22). This contradicts the fact that $(\mathbf{x}^*, \mathbf{u}^*, z^*, \boldsymbol{\xi}^*, \boldsymbol{\omega}^*, \alpha^*, \theta^*)$ is an optimal solution to (22) since $\bar{\theta}^* < \theta^*$. Therefore, $(\mathbf{x}^*, \mathbf{u}^*, \alpha^*, \theta^*)$ is an optimal solution to (19).

Conversely, let $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{u}}^*, \tilde{\alpha}^*, \tilde{\theta}^*)$ solve (19) and $(\tilde{z}^*, \tilde{\boldsymbol{\xi}}^*, \tilde{\boldsymbol{\omega}}^*)$ solve (21) with $\mathbf{v} = \mathbf{e}\tilde{\alpha}^* + \frac{1}{1-\beta}\boldsymbol{\pi} \cdot \tilde{\mathbf{u}}^*$. Then $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{u}}^*, \tilde{z}^*, \tilde{\boldsymbol{\xi}}^*, \tilde{\boldsymbol{\omega}}^*, \tilde{\alpha}^*, \tilde{\theta}^*)$ must solve (22). Otherwise, there exists an optimal solution $(\mathbf{x}^*, \mathbf{u}^*, z^*, \boldsymbol{\xi}^*, \boldsymbol{\omega}^*, \alpha^*, \theta^*)$ of (22) such that $\theta^* < \tilde{\theta}^*$. From the first part of the proof, $(\mathbf{x}^*, \mathbf{u}^*, \alpha^*, \theta^*)$ must be an optimal solution of (19), which contradicts the fact that $(\tilde{\mathbf{x}}^*, \tilde{\mathbf{u}}^*, \tilde{\alpha}^*, \tilde{\theta}^*)$ solves (19) since $\theta^* < \tilde{\theta}^*$. The proof is complete. □

Linear Program and its Duality

Let the following linear program be the *primal problem*:

$$\begin{aligned}
\text{(P)} \qquad & \max_{\mathbf{x} \in \Re^n} \mathbf{c}^\top \mathbf{x} \\
& \text{s.t. } A\mathbf{x} \le \mathbf{b}, \\
& \qquad \mathbf{x} \ge \mathbf{0},
\end{aligned}$$

where $\mathbf{c} \in \Re^n, A \in \Re^{m\times n}$, and $\mathbf{b} \in \Re^m$. Then, the associated *dual linear program* is given by

$$\text{(D)} \qquad \min_{\mathbf{y}\in\Re^m} \ \mathbf{b}^\top \mathbf{y} \quad \text{s.t. } A^\top \mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq 0.$$

Thus, we present the following two well-known theorems whose proofs may be found in a standard text book, e.g., Vanderbei (1996).
Weak Duality Theorem: If $\bar{\mathbf{x}}$ is feasible for the primal problem (P) and $\bar{\mathbf{y}}$ is feasible for the dual problem (D), then

$$\mathbf{c}^\top \bar{\mathbf{x}} \leq \mathbf{b}^\top \bar{\mathbf{y}}.$$

Strong Duality Theorem: If the primal problem (P) has an optimal solution $\mathbf{x}^*$, then the dual problem (D) also has an optimal solution $\mathbf{y}^*$ such that

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*.$$

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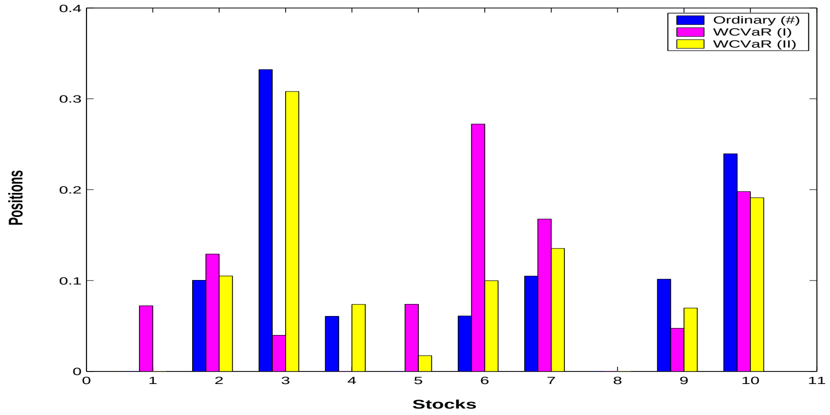

Fig. 1. Optimal portfolio positions with CVaR and robust CVaR strategies ($\mu = 0.00025$).

Notes: (#): Ordinary CVaR exiting at maturity; (I): WCVaR with no information on exit; (II): WCVaR with partial information on exit. $[\underline{\varsigma}, \overline{\varsigma}] = [0.6, 1]$ and $\gamma = 5\%$.

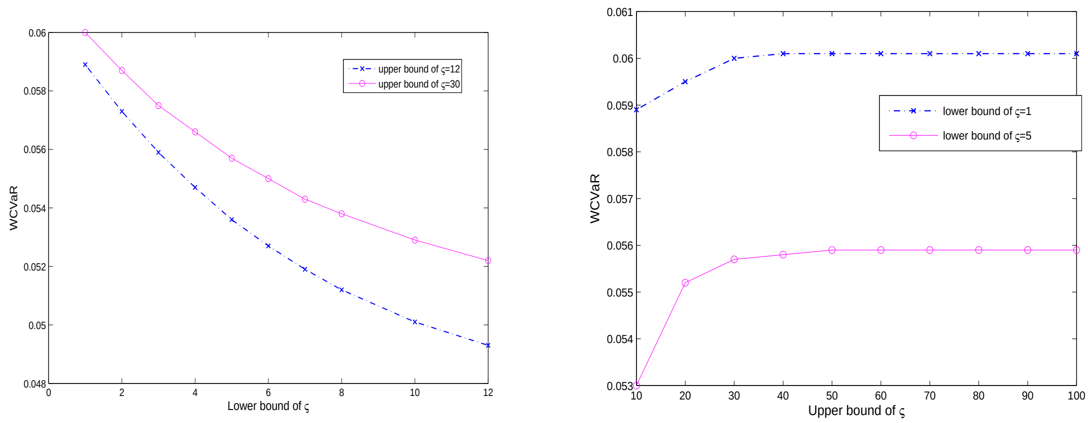

Fig. 2. WCVaR with respect to $\underline{\varsigma}$ and $\overline{\varsigma}$ ($\mu = 0.018$).

Table 1
Summary statistics of the daily returns of risky assets.

Asset	Mean (%)	Covariance (10^{-4}) (1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
(1)	0.0330	6.2403	3.2159	0.7699	1.3127	3.3934	2.4670	1.7294	3.1847	1.2199	1.3818
(2)	0.0254		4.7073	0.6262	1.0800	2.7647	2.1057	1.5289	2.7659	1.3613	1.1087
(3)	0.0006			2.5213	0.6126	0.7224	0.7044	0.5158	0.6261	0.5949	0.6455
(4)	0.0146				4.7663	1.5266	1.6444	1.0569	1.1965	0.9135	0.8293
(5)	0.0344					8.5535	2.7169	2.1450	3.1225	1.4447	1.3433
(6)	0.0340						5.7858	1.7463	2.3009	1.4064	1.2756
(7)	0.0274							5.4695	2.9231	1.2372	0.8940
(8)	0.0274								6.7661	1.4327	1.0968
(9)	0.0102									4.3867	1.2468
(10)	0.0119										3.8311

Notes: (1): Nippon Flour Mills, (2): Nisshin Seifun Group, (3): Ezaki Glico, (4): Teikoku Sen-I, (5): Mitsubishi Rayon, (6): Nippon Oil Corporation, (7): Showa Shell Sekiyu, (8): Daishi Bank, (9): Shizuoka Bank, (10): 16 Bank.

Table 2
Bounds of exit probability.

$\boldsymbol{\lambda}$	Moment1 *	Moment1 **	Moment2 *	Moment2 **	Moment3 *	Moment3 **
$\underline{\boldsymbol{\lambda}}$	0.1813	0.1857	0.1484	0.1639	0.0808	0.4945
$\overline{\boldsymbol{\lambda}}$	0.4396	0.2879	0.4796	0.2176	0.6703	0.6504

* bounds without iterations,

** bounds after removing the uncertainty from endogenous exit with iterations.

Table 3
Specification of endogenous exit probability

Iterations	λ_1^{end}	λ_2^{end}	$\mathrm{WCVaR}_{0.95}$ (%)	Positions of optimal portfolio (1) / (6)	(2) / (7)	(3) / (8)	(4) / (9)	(5) / (10)
0	[0, 0.1561]	[0, 0.2775]	3.7746	0.0123	0.1180	0.2333	0.0529	0.0252
				0.1529	0.1489	0.0000	0.0703	0.1851
1	0.0048	0.0140	3.6252	0.0000	0.1050	0.3081	0.0737	0.0173
				0.0998	0.1353	0.0000	0.0697	0.1912
2	0.0037	0.0133	3.6243	0.0000	0.1048	0.3083	0.0740	0.0171
				0.0997	0.1351	0.0000	0.0696	0.1915

Table 4
Comparison of performances of ordinary and robust optimal portfolios.

μ (%)	$[\underline{\varsigma}, \overline{\varsigma}] = [0.6, 1]$ $\gamma = 5\%$	Mean (%) Moment1	Moment2	Moment3	$\text{CVaR}_{0.95}$ (%) Moment1	Moment2	Moment3
0.015	#	0.0108	0.0174	0.0206	2.5228	3.3405	3.7547
	I	0.0150	0.0246	0.0310	2.6133	3.4446	3.8543
	II	0.0107	0.0172	0.0204	2.4953	3.3203	3.7610
0.020	#	0.0108	0.0174	0.0206	2.5228	3.3405	3.7547
	I	0.0200	0.0336	0.0441	2.8435	3.7136	4.1553
	II	0.0122	0.0200	0.0245	2.5182	3.3347	3.7801
0.025	#	0.0125	0.0204	0.0250	2.5276	3.3446	3.7738
	I	0.0250	0.0428	0.0576	3.2201	4.1366	4.6109
	II	0.0146	0.02246	0.0312	2.5441	3.3765	3.8629
0.030	#	0.0143	0.0238	0.0300	2.5497	3.3805	3.8325
	I	0.0300	0.0522	0.0716	3.6902	4.7106	5.3170
	II	0.0172	0.0291	0.0378	2.6439	3.5085	3.9878
0.040	#	0.0181	0.0307	0.0400	2.7057	3.5734	4.0307
	I	—	—	—	—	—	—
	II	0.0233	0.0383	0.0510	3.0045	3.8968	4.3623

Notes: (#): Ordinary CVaR exiting at maturity; (I): WCVaR with no information on exit; (II): WCVaR with partial information on exit.

Table 5
Summary statistics of the annual returns of risky assets.

Assets	Expected rate of return (%)	Standard deviation (%)	Correlation coefficient (1)	(2)	(3)	(4)	(5)	(6)
(1) Large stocks (U.S.)	18.98	14.16	1.00	0.67	0.63	0.41	0.40	0.00
(2) Small stocks (U.S.)	13.01	18.15		1.00	0.51	0.78	0.42	0.52
(3) Corporate bonds (U.S.)	8.60	7.89			1.00	0.42	0.00	-0.11
(4) Real estate (U.S.)	9.75	19.67				1.00	0.13	0.29
(5) Stocks (dev. markets)	6.59	16.74					1.00	0.69
(6) Stocks (emerg. markets)	8.09	34.91						1.00

Table 6
WCVaR v.s. exit moments ($\beta = 0.95$)

Number	Moments	WCVaR (I)	WCVaR (II)
2	[2, 12]	0.0551	0.0550
3	[2, 4, 12]	0.0587	0.0573
4	[2, 4, 6, 12]	0.0612	0.0586
5	[2, 4, 6, 9, 12]	0.0653	0.0630
6	[2, 4, 6, 9, 10, 12]	0.0653	0.0631
8	[2, 3, 4, 6, 8, 9, 10, 12]	0.0654	0.0631

Notes: WCVaR (I): WCVaR with no information on exit; WCVaR (II): WCVaR with partial information on exit. Moments represents the possible exit time.