Evaluation of Firm’s Loss Due to Incomplete Information in Real Investment Decision

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Abstract

We investigate the effect of incomplete information in a model where a start-up with a unique idea and technology pioneers a new market but will eventually be expelled from the market by a large firm’s subsequent entry. We evaluate the start-up’s loss due to incomplete information about the large firm’s behavior. We clarify conditions under which the start-up needs more information about the large firm. The proposed method of evaluating the loss due to incomplete information could also be applied to other real options models involving several firms.

Keywords: Applied Probability; Investment analysis; Real options; Incomplete information; Leader-follower game

1 Introduction

In recent years, the real options approach to investment has become the mainstream. In the real options approach, a firm that faces an irreversible investment generating uncertain profit in future is considered to have an option to make the investment. Then, in order to maximize the expected profit, the firm must invest when the NPV (net present value) of the investment becomes greater than the opportunity cost of investing (i.e., the value of the option to delay the investment). The real options approach to investment has provided

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a new insight into a firm’s real investment decision which tended to rely on managerial experiences and intuitions, and it has gradually come to be applied to investment in the real world since the proposal by Dixit (1989) and McDonald and Siegel (1986).

Although the early literature treated an investment decision of a monopolist, more recent studies have investigated how a firm’s investment decision is affected by its rival firms’ behaviors because a chance to make real investment, unlike financial options, can usually be shared by several firms in the same industry. One of the earliest results in the strategic real options approach has been obtained by Grenadier (1996) who provided the symmetric equilibrium strategy for firms. Weeds (2002) has derived equilibrium strategies in two players who attempt to preempt a single patent from the other, and Huisman and Kort (2003) have investigated a two player real options game in the context of the adoption of new technology.

While the above studies assume complete information about the competitors, Lambrecht and Perraudin (2003) consider a model involving incomplete information about the competitors’ investment costs. Hsu and Lambrecht (2003) introduce asymmetric and incomplete information in real options in the context of a patent race. Bernardo and Chowdhry (2002) and Décamps et al. (2005) have also incorporated incomplete information in real investment problems from another perspective. By using the filtering theory, they have investigated models in which a firm has incomplete information about parameters of its own profit flow rather than the competitors’ behavior.

The effect of incomplete information is practically significant, since how accurately a firm can estimate the behaviors of rival firms has a crucial effect on whether or not its real investment succeeds. Previous studies such as Hsu and Lambrecht (2003) and Lambrecht and Perraudin (2003) dealing with incomplete information pay primary attention to how incomplete information affects the equilibrium strategy. However, there remains the following natural question: How great loss does a firm suffer with incomplete information compared with that in the case of complete information? Answers to this question will unveil a risk of a firm using the real options approach in the real world and also suggest how a firm should act under incomplete information.

In this paper, we answer the above question in a model with a start-up who pioneers a new market by a unique idea and technology and a large firm that will eventually take
over the market from the start-up. We evaluate the start-up’s loss due to incomplete information about the large firm. Then, we clarify conditions under which the start-up needs more information about the large firm. In particular, we show that in some cases the real options strategy under incomplete information gives less expected payoff to the start-up than the zero-NPV strategy (i.e., investing when the NPV of the investment becomes positive) under the same incomplete information.

Our results imply that in some cases a firm using the real options approach to investment has a great risk of incorrect conjectures about the behaviors of its competitors. Although we consider the simple model involving two firms for the purpose of concentrating our attention on the loss due to incomplete information, the proposed method of evaluating the loss due to incomplete information could also be applied to other real options models involving several firms.

This paper is organized as follows. After the model is introduced in Section 2, Section 3 gives the start-up’s value function and optimal strategy under complete information. Section 4 describes our main theoretical results, which show the start-up’s strategy under incomplete information, its expected payoff, and the loss due to incomplete information. Section 5 gives numerical examples, and Section 6 concludes the paper.

2 Model

This section introduces the model treated in this paper. We consider the start-up’s problem of determining the timing of entering the new market which may be taken over by the large firm eventually. In this problem, we will discuss how incomplete information about the large firm affects the expected payoff of the start-up. Throughout this paper, we assume that both stochastic process and random variable are defined on the filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\). The model is described as follows:

**Profit flows and investment costs of the two firms:** The start-up can receive a profit flow \(D_1(1,0)Y(t)\) in the new market by paying an indivisible investment cost \(I_1\), but the flow will be reduced to \(D_1(1,1)Y(t)\) after the large firm’s entry to the market. Here, \((1,0)\) (resp. \((1,1)\)) denotes the situation in which only the start-up (resp. both firms) is active in the market. Quantities \(I_1, D_1(1,1)\) and \(D_1(1,0)\) are constants such that
$I_1 > 0$ and $0 \leq D_1(1,1) < D_1(1,0)$, and $Y(t)$ is the state of the market satisfying the following geometric Brownian motion:

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dB(t) \quad (t > 0), \quad Y(0) = y,$$

where $\mu \geq 0$, $\sigma > 0$ and $y > 0$ are given constants, and $B(t)$ denotes the one-dimensional $\mathcal{F}_t$ standard Brownian motion. In contrast, the large firm does not notice the existence of the potential market until the start-up’s investment. The large firm can obtain a profit flow $D_1(1,0)Y(t)$ in the market by paying an indivisible investment cost $I_2$ after the start-up’s investment. Here, $I_2$ and $D_2(1,1)$ are positive constants. The adapted process $Y(t)$ means the observable state of the market at time $t$, and it causes an exogenous change in the firms’ profit flows in the market. In contrast, $D_i(\cdot, \cdot) \ (i = 1, 2)$ represent endogenous changes due to the firms’ entrance in the market.

**The large firm’s investment decision:** The large firm does not notice the opportunity to preempt the market until the date $\tau_1$ on which the start-up invests. Then, with discount rate $\rho (\rho > \mu)$, the large firm optimizes its investment time $\tau_2$ by solving the following optimal stopping problem:

$$\sup_{\tau_2 \geq \tau_1} E[\int_{\tau_2}^{\tau_1} e^{-\rho t} D_2(1,1)Y(t)dt - e^{-\rho \tau_2} I_2],$$

where $\tau_2$ is any $\mathcal{F}_t$ stopping time that satisfies $\tau_2 \geq \tau_1$. Let us call $Q_i = D_i(1,1)/I_i \ (i = 1, 2)$ the qualities of the start-up and the large firm, respectively. A firm’s quality is determined by such as the firm’s idea and technology. Let $\tau_2^q$ denote an optimal stopping time of problem (2) with $Q_2 = D_2(1,1)/I_2$ replaced by a general constant $q \ (q > 0)$.

**The start-up’s investment decision:** Since the start-up does not have complete information about the quality of the large firm, the start-up determines its investment time $\tau_1$ assuming that the quality of the large firm obeys a random variable $X$ independent of filtration $\{\mathcal{F}_t\}$. Then, the start-up believes that its expected payoff of investing at $\tau_1$ is equal to

$$E\left[\int_{\tau_1}^{\tau_2} e^{-\rho t} D_1(1,0)Y(t)dt + \int_{\tau_2}^{\tau_2^X} e^{-\rho t} D_1(1,1)Y(t)dt - e^{-\rho \tau_1} I_1\right],$$

where $\tau_2^X$ represents a random variable which takes a value $\tau_2^X(\omega)(\omega)$ for $\omega \in \Omega$ (note that $\tau_2^X$ also depends on $\tau_1$). The start-up finds its investment time $\tau_1$ by solving the following
optimal stopping problem:
\[
\sup_{\tau_1} \mathbb{E} \left[ \int_{\tau_1}^{\tau_1^*} e^{-\rho t} D_1(1,0) Y(t) \, dt + \int_{\tau_1^*}^{\infty} e^{-\rho t} D_1(1,1) Y(t) \, dt - e^{-\rho \tau_1^*} I_1 \right],
\]
where \( \tau_1 \) is any \( \mathcal{F}_t \) stopping time. Let \( V(y) \) (recall \( y = Y(0) \)) and \( \tau_1^* \) denote the value function and an optimal stopping time of problem (4), respectively. An optimal stopping time \( \tau_1^* \) is expressed in a form independent of the initial point \( y \), as will be shown in Sections 3 and 4. Let \( V(y; q) \) and \( \tau_1^q \) be the value function and an optimal stopping time of a special problem (4) with \( X \) replaced by a constant \( q (> 0) \), respectively. Note that if the start-up knows the real value \( Q_2 \) of the quality of the large firm (i.e., in the case of complete information), the start-up invests at \( \tau_1^{Q_2} \) and its expected payoff becomes equal to \( V(y; Q_2) \).

**Remark 2.1** For simplicity, this paper treats the two player leader-follower game as mentioned above, but similar results can be obtained in a more practical setting that permits several followers, by assuming that the followers make joint investment. There is a possibility that the followers make joint investment even if they are non-cooperative. For details, see Huisman (2001).

Dixit and Pindyck (1994) and Huisman (2001) have investigated a preemption model in which both firms attempt to become a leader assuming complete information and \( D_1(\cdot, \cdot) = D_2(\cdot, \cdot) \). Unlike their model, the model studied in this paper is a leader-follower game. In the remainder of this paper, we assume \( 0 = D_1(1,1) < D_2(1,1) \) to concentrate on the effect of incomplete information. Our model describes the following tendency which has been observed in practice frequently: A start-up with a unique idea and technology has an advantage of pioneering a new market, but it has the weakness of being taken over by a large firm when both firms compete in the market. For simplicity, we will denote \( D_1 = D_1(1,0) \) and \( D_2 = D_2(1,1) \) unless they cause confusion.

### 3 Case of complete information

This section derives the value function \( V(y; q) \) and an optimal stopping time \( \tau_1^q \) of the start-up who believes that the quality of the large firm is a constant \( q (> 0) \) (i.e., \( X \equiv q \) in problem (4)). They can be derived in the same fashion as in the case of complete
information treated in Dixit and Pindyck (1994). In an optimal stopping problem with discount rate $\rho$ and state process $Y(t)$ following (1), solutions of the quadratic equation

$$\frac{\sigma^2 \beta (\beta - 1)}{2} + \mu \beta - \rho = 0,$$

that is,

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1,$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < 0,$$

play an important role. We can easily check the inequalities in (6) and (7) by taking into consideration that the left-hand side of (5) takes negative values at $\beta = 0$ and $\beta = 1$ by $0 \leq \mu < \rho$.

Furthermore, we introduce the following notation:

$$y_M(q) = \frac{\beta_1 (\rho - \mu)}{(\beta_1 - 1)q} \quad (q > 0),$$

$$p(\beta, q_1, q_2) = \left(\frac{1}{\beta}\right)^{\frac{1}{\beta_1}} - \frac{q_2}{q_1} \quad (\beta > 1, q_1, q_2 > 0).$$

Note that an optimal stopping time for the monopolist with quality $q$ is given by $\inf\{t \geq 0 \mid Y(t) \geq y_M(q)\}$ (e.g., see Dixit and Pindyck (1994)).

Under the assumption that the start-up has already entered the market at time $\tau_1$, the large firm’s problem becomes a problem for a monopolist. Therefore, an optimal stopping time $\tau_2^q$ of problem (2) with $Q_2$ replaced by $q$ is expressed as follows:

$$\tau_2^q = \inf\{t \geq \tau_1 \mid Y(t) \geq y_M(q)\}.$$

Then, we can derive the start-up’s value function $V(y; q)$ and investment time $\tau_1$ in the case of $X \equiv q$ and $Y(0) = y$ in problem (4).

**Proposition 3.1** The start-up’s value function $V(y; q)$ and optimal stopping time $\tau_1^q$ are given as follows: If

$$p(\beta_1, Q_1, q) > 0,$$

then

$$V(y; q) = \begin{cases} A(q)y^{\beta_1} & (0 < y < y_M(Q_1)) \\ \frac{D_1y}{\rho - \mu} - I_1 - \frac{D_1y_M(q)^{-\beta_1 + 1}y^{\beta_2}}{\rho - \mu} & (y_M(Q_1) \leq y < y_U(q)) \\ B(q)y^{\beta_2} & (y \geq y_U(q)), \end{cases}$$


Remark 3.1  Until the quality of the large firm exceeds the solution of \( p(\beta_1, Q_1, q) = 0 \), inequality (11) holds, and \( y_U(q) \) and \( V(y; q) \) monotonically decrease with \( q \). On the contrary, we have \( y_U(q) \rightarrow +\infty \) and \( \tau_1^q \rightarrow \inf\{t \geq 0 \mid Y(t) \geq y_M(Q_1)\} \) as \( q \rightarrow +0 \); this means that the stopping time \( \tau_1^q \) tends to an optimal stopping time of a monopolist.

Remark 3.2  By taking \( q = Q_2 \) in Proposition 3.1, we can obtain the expected payoff \( V(y; Q_2) \) of the start-up who has complete information about the quality of the large firm.

In Proposition 3.1, \( y_U(q) \) means a threshold at which the start-up’s expected payoff until the entry of the large firm with quality \( q \) exceeds the value of the start-up’s option to delay its investment. Proposition 3.1 suggests that if the initial state \( Y(0) = y \) is larger than \( y_U(q) \), then the start-up should delay its investment until the state \( Y(t) \) drops to the threshold \( y_U(q) \). It is possible that \( Y(t) \) decreases from the initial point \( y \) to the threshold \( y_U(q) \) even with a positive drift \( \mu \) in (1), because \( Y(t) \) has a positive volatility \( \sigma \) in (1). Even if the start-up makes immediate investment in the case of \( y > y_U(q) \), the large firm is quite likely to enter the market before the start-up gains enough income.

Inequality (11) can be interpreted as a prerequisite condition for the start-up’s investment for the reason mentioned below. In Dixit and Pindyck (1994) and Huisman (2001),
since it is assumed that $D_1(1,1) > 0$, the expected payoff of the start-up’s immediate investment for a very large $y$ becomes always positive, even if the large firm enters the market as soon as the start-up invests. On the other hand, in our model, since it is assumed that $D_1(1,1) = 0$, the start-up’s expected payoff never becomes positive for any time $t$ and any value of $Y(t)$, unless (11) holds (for details see Appendix A).

Let us examine how the prerequisite condition (11) is affected by the values of parameters $\mu, \rho$ and $\sigma$. First observe from (6) that $\partial \beta_1/\partial \sigma < 0, \lim_{\sigma \to -\infty} \beta_1 = 1, \lim_{\sigma \to +0} \beta_1 = \rho/\mu > 1, \partial \beta_1/\partial \mu < 0$ and $\partial \beta_1/\partial \rho > 0$ (also see Dixit and Pindyck (1994)). Since $p(\beta, Q_1, q)$ is monotonically increasing for $\beta > 1$ by (9), the prerequisite condition (11) becomes more restrictive as the expected return $\mu$ and the volatility $\sigma$ (resp. the discount rate $\rho$) of the market increase (resp. decrease). Moreover, we note that $p(\beta, Q_1, q) \to 1/e - q/Q_1$ as $\beta \downarrow 1$ and $p(\beta, Q_1, q) \to 1 - q/Q_1$ as $\beta \uparrow +\infty$ by (9). Thus, when $q/Q_1 < 1/e$, the prerequisite condition (11) always holds regardless of $\mu, \rho$ and $\sigma$, but when $q/Q_1 \geq 1$, the prerequisite condition never holds.

The next section describes our main results, which evaluate the start-up’s loss due to incomplete information about the quality of the large firm.

4 Loss due to incomplete information

This section evaluates the start-up’s loss due to incomplete information about the quality of the large firm by the following procedure:

1. Derive the start-up’s value function $V(y)$ and its optimal stopping time $\tau_1^*$ of problem (4) which the start-up believes.

2. Derive the expected payoff $\hat{V}(y)$ which can be obtained by the start-up who invests at time $\tau_1^*$ derived in Step 1.

3. Compute $W(y) = V(y; Q_2) - \hat{V}(y)$, which is the difference between the expected payoff of the start-up who has complete information about the quality of the large firm and the expected payoff of the start-up who invests at time $\tau_1^*$ under incomplete information.
The quantity $W(y)$ computed in Step 3 is regarded as the loss due to incomplete information. In this paper, we use the above method to evaluate the loss which the firm suffers with incomplete information. The proposed method may also be applied to other real options models involving several firms. The loss due to incomplete information is identified as the value of information about the rival firm, and hence it tells us whether the firm should conduct a further survey on the rival firm or not. Subsection 4.1, Subsection 4.2 and Subsection 4.3 describe Step 1, Step 2 and Step 3, respectively.

### 4.1 Start-up’s strategy under incomplete information

The start-up determines its investment time believing that the quality of the large firm obeys a random variable $X$ independent of $\{F_t\}$. Here we assume that $X > 0$ and $E[X^{\beta_1-1}] < +\infty$. Expectation (3) with $\tau_1 = 0$ becomes

$$g(y) = \frac{D_1 y}{\rho - \mu} - \frac{D_1 E\left[y_M(X) \vee y\right]^{\beta_1+1}}{\rho - \mu} y^{\beta_1} - I_1 \quad (y > 0),$$

where $a \vee b$ means $\max\{a, b\}$ (for details see Appendix B). Generally, it is hard to derive an explicit form of the value function $V(y)$ and an optimal stopping time of problem (4). However, we can show that problem (4) is reduced to the problem with $q = \tilde{Q}_2$ in Section 3, where $\tilde{Q}_2 = E[X^{\beta_1-1}]^{1/(\beta_1-1)}$, provided that the following condition holds:

**Condition (a):** The inequality $g(y) \leq V(y; \tilde{Q}_2)$ holds for all $y > 0$.

In relation to (16), we define

$$h(y) = \frac{D_1 y}{\rho - \mu} - \frac{D_1 E\left[y_M(X)\right]^{\beta_1+1}}{\rho - \mu} y^{\beta_1} - I_1 \quad (y > 0).$$

From the definitions of $g(y), h(y), \tilde{Q}_2$ and Proposition 3.1, it immediately follows that

$$h(y) \leq g(y) \quad (y > 0),$$

$$h(y) = V(y; \tilde{Q}_2) \quad (y_M(Q_1) < y < y_U(\tilde{Q}_2), \quad p(\beta_1, Q_1, \tilde{Q}_2) > 0).$$

By using this property, we can show the following proposition, which is the key result to evaluating the loss due to incomplete information.

**Proposition 4.1** Assume that Condition (a) holds. Then, the value function $V(y)$ and an optimal stopping time of problem (4) which the start-up believes are given as $V(y) =$
$V(y; \tilde{Q}_2)$ and $\tau^*_1 = \tau_1^{\tilde{Q}_2}$ for all $y > 0$, respectively, where $V(y; \tilde{Q}_2)$ and $\tau^*_1$ are given in Proposition 3.1.

(Proof) See Appendix B.

**Remark 4.1** In Section 5, we will observe that Condition (a) is likely to hold when the support of $X$ is not wide. In particular, we can easily show that Condition (a) certainly holds whenever $X$ is a constant.

**Remark 4.2** Figure 1 illustrates the function $V(y) = V(y; \tilde{Q}_2)$ together with the functions $g(y)$ and $h(y)$. In particular, it shows that $V(y) = V(y; \tilde{Q}_2) = g(y) = h(y)$ holds for $y \in [y_M(Q_1), y_U(\tilde{Q}_2)]$.

By Proposition 4.2, the start-up who regards the quality of the large firm as the random variable $X$ invests at the same timing as the start-up who regards the quality of the large firm as the constant $\tilde{Q}_2$, provided Condition (a) holds. However, this is not always true in general case. In the rest of the paper, we will restrict our attention to the case where Condition (a) is satisfied.

![Figure 1: $g(y), h(y)$ and $V(y) = V(y; \tilde{Q}_2)$.](image)
4.2 The expected payoff of the start-up

We derive the expected payoff $\tilde{V}(y)$ of the start-up who invests at time $\tau_1^*$ given in Proposition 4.1. Since the large firm actually has quality $Q_2$, its real investment time is equal to

$$\tau_2^{Q_2} = \inf\{t \geq \tau_1^* \mid Y(t) \geq y_M(Q_2)\} \quad (20)$$

by (10). Then, the expected payoff $\tilde{V}(y)$ becomes

$$\tilde{V}(y) = E\left[\int_{\tau_1^*}^{\tau_2^{Q_2}} e^{-\rho t} D_1 Y(t) dt - e^{-\rho \tau_1^*} I_1\right]. \quad (21)$$

We can show the following proposition by computing expectation (21).

**Proposition 4.2** Assume that Condition (a) holds. Then the expected payoff $\tilde{V}(y)$ of the start-up who invests at $\tau_1^*$ is given as follows. If $p(\beta_1, Q_1, \tilde{Q}_2) > 0$, then

$$\tilde{V}(y) = \begin{cases} \tilde{A}(Q_2) y^{\beta_1} & (0 < y < y_M(Q_1)) \\ \frac{D_1 y}{\rho - \mu} - \frac{D_1 (y \lor y_M(Q_2))^{\beta_1 + 1} y^{\beta_1}}{\rho - \mu} - I_1 & (y_M(Q_1) \leq y < y_U(\tilde{Q}_2)) \\ \tilde{B}(\tilde{Q}_2) y^{\beta_2} & (y \geq y_U(\tilde{Q}_2)) \end{cases} \quad (22)$$

where $y_M(\cdot)$ is defined by (8), $y_U(\cdot)$ is the unique solution of equation (14), and $\tilde{A}(\cdot)$ and $\tilde{B}(\cdot)$ are given by

$$\tilde{A}(q) = y_M(Q_1)^{-\beta_1} \left( \frac{D_1 y_M(Q_1)}{\rho - \mu} - \frac{D_1 (y_M(Q_1) \lor y_M(q))^{\beta_1 + 1} y_M(Q_1)^{\beta_1}}{\rho - \mu} - I_1 \right),$$

$$\tilde{B}(q) = y_U(q)^{-\beta_2} \left( \frac{D_1 y_U(q)}{\rho - \mu} - \frac{D_1 (y_U(q) \lor y_M(Q_2))^{\beta_1 + 1} y_U(q)^{\beta_1}}{\rho - \mu} - I_1 \right). \quad (23)$$

If $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$, then $\tilde{V}(y) = 0$ for all $y > 0$.

**(Proof)** See Appendix C.

**Remark 4.3** Propositions 3.1, 4.1 and 4.2 ensure that, under Condition (a), we have

$$\tilde{V}(y) = V(y; Q_2) = V(y; \tilde{Q}_2) = V(y)$$

whenever $\tilde{Q}_2 = Q_2$.

4.3 The start-up’s loss due to incomplete information

Finally we evaluate the start-up’s loss $W(y) = V(y; Q_2) - \tilde{V}(y)$ due to incomplete information about the quality of the large firm. The loss $W(y)$ varies according to the relation
between $\tilde{Q}_2$ and $Q_2$. Note that $y_M(\cdot)$ is monotonically decreasing by (8).

**Case 1: $\tilde{Q}_2 < Q_2$**  
The start-up underestimates the quality of the large firm, and the inequality $y_M(\tilde{Q}_2) > y_M(Q_2)$ holds with respect to the threshold.

**Case 2: $\tilde{Q}_2 = Q_2$**  
The start-up correctly estimates the quality of the large firm, and the equality $y_M(\tilde{Q}_2) = y_M(Q_2)$ holds with respect to the threshold.

**Case 3: $\tilde{Q}_2 > Q_2$**  
The start-up overestimates the quality of the large firm, and the inequality $y_M(\tilde{Q}_2) < y_M(Q_2)$ holds with respect to the threshold.

**Proposition 4.3**  
Assume that Condition (a) holds. The start-up’s loss $W(y)$ due to incomplete information is given as follows.

**Case 1: $\tilde{Q}_2 < Q_2$**

**Case 1.1:** $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$  
$W(y) = 0$ for all $y > 0$.

**Case 1.2:** $p(\beta_1, Q_1, \tilde{Q}_2) > 0$ and $p(\beta_1, Q_1, Q_2) \leq 0$  
$W(y) = -\hat{V}(y)$ for all $y > 0$.

**Case 1.3:** $p(\beta_1, Q_1, Q_2) > 0$

$$W(y) = \begin{cases} 
0 & (0 < y < y_U(Q_2)) \\
B(Q_2)y^{\beta_2} - \frac{D_1y}{\rho - \mu} + \frac{D_1(y \lor y_M(Q_2))^{-\beta_1+1}y^{\beta_1}}{\rho - \mu} + I_1(y_U(Q_2) \leq y < y_U(\tilde{Q}_2)) \\
\left(B(Q_2) - \tilde{B}(\tilde{Q}_2)\right)y^{\beta_2} & (y \geq y_U(\tilde{Q}_2))
\end{cases}$$

**Case 2: $\tilde{Q}_2 = Q_2$**  
$W(y) = 0$ for all $y > 0$.

**Case 3: $\tilde{Q}_2 > Q_2$**

**Case 3.1:** $p(\beta_1, Q_1, Q_2) \leq 0$  
$W(y) = 0$ for all $y > 0$.

**Case 3.2:** $p(\beta_1, Q_1, Q_2) > 0$ and $p(\beta_1, Q_1, \tilde{Q}_2) \leq 0$  
$W(y) = V(y; Q_2)$ for all $y > 0$.

**Case 3.3:** $p(\beta_1, Q_1, \tilde{Q}_2) > 0$

$$W(y) = \begin{cases} 
0 & (0 < y < y_U(\tilde{Q}_2)) \\
\frac{D_1y}{\rho - \mu} - I_1 - \frac{D_1y_M(Q_2)^{-\beta_1+1}y^{\beta_1}}{\rho - \mu} - \tilde{B}(\tilde{Q}_2) & (y_U(\tilde{Q}_2) \leq y < y_U(Q_2)) \\
\left(B(Q_2) - \tilde{B}(\tilde{Q}_2)\right)y^{\beta_2} & (y \geq y_U(Q_2))
\end{cases}$$

Here, $y_U(\cdot)$ is the unique solution of equation (14), and $B(\cdot)$ and $\tilde{B}(\cdot)$ are defined by (15) and (23), respectively.

**Proof**  
In Case 1 (i.e., $\tilde{Q}_2 < Q_2$), we have $p(\beta_1, Q_1, Q_2) < p(\beta_1, Q_1, \tilde{Q}_2)$ by (9). Thus, we can easily compute $W(y) = V(y; Q_2) - \hat{V}(y)$ from Propositions 3.1 and 4.2 for each
of Cases 1.1, 1.2, and 1.3. In Case 2 (i.e., $\tilde{Q}_2 = Q_2$), we have $\tilde{V}(y) = V(y; Q_2)$ and hence $W(y) = V(y; Q_2) - \tilde{V}(y) = 0$. In Case 3 (i.e., $\tilde{Q}_2 > Q_2$), we have $p(\beta_1, Q_1, \tilde{Q}_2) < p(\beta_1, Q_1, Q_2)$. Then, we can compute $W(y)$ from Propositions 3.1 and 4.2 for each of Cases 3.1, 3.2, and 3.3.

Let us explain the start-up’s investment strategy in each case. Needless to say, in Case 2 the start-up’s strategy becomes optimal as $\tau_1^* = \tau_{1}^{\tilde{Q}_2} = \tau_{1}^{Q_2}$, and hence the start-up suffers no loss $W(y)$ for any initial point $y > 0$. In Cases 1.1 and 3.1, the prerequisite condition for the start-up’s investment does not actually hold (i.e., $p(\beta_1, Q_1, Q_2) \leq 0$), and the start-up never attempts to invest. As a result, in these cases the start-up’s strategy becomes optimal, and the loss $W(y)$ never arises for any $y > 0$.

Case 1.2 and Case 3.2 correspond to the case where the start-up attempts to invest although the prerequisite condition does not actually hold and the case where the start-up never attempts to invest although the prerequisite condition actually holds, respectively. Therefore, in both cases, the start-up suffers the loss $W(y)$ for all $y > 0$. We note that $\tilde{V}(y) < V(y; Q_2) = 0$ for all $y > 0$ in Case 1.2.

In Cases 1.3 and 3.3, the prerequisite condition actually holds, and also the start-up attempts to invest. The start-up however makes its investment at $\tau_{1}^{Q_2} = \inf\{t \geq 0 \mid Y(t) \in [y_{M}(Q_1), y_{U}(\tilde{Q}_2)]\}$, though the optimal investment timing $\tau_{1}^{Q_2}$ is given as $\inf\{t \geq 0 \mid Y(t) \in [y_{M}(Q_1), y_{U}(Q_2)]\}$. In Case 1.3, since $y_{U}(\tilde{Q}_2) > y_{U}(Q_2)$, the start-up makes investment earlier than $\tau_{1}^{Q_2}$ and suffers the loss $W(y)$ when $y > y_{U}(Q_2)$; contrarily, in Case 3.3, since $y_{U}(\tilde{Q}_2) < y_{U}(Q_2)$, the start-up makes investment later than $\tau_{1}^{Q_2}$ and suffers the loss $W(y)$ when $y > y_{U}(\tilde{Q}_2)$.

**Corollary 4.1** Assume that Condition (a) holds. Also assume that the random variable $X$ has a support $(0, Q_U]$ for some constant $Q_U$, and that $Q_2$ (the real quality of the large firm) satisfies $Q_2 \in (0, Q_U]$. If conditions $p(\beta_1, Q_1, Q_U) > 0$ and $y \leq y_{U}(Q_U)$ are satisfied, then the start-up suffers no loss $W(y)$ due to incomplete information. Here, $y_{U}(\cdot)$ is defined as the unique solution of equation (14).

The first condition means that it is certain that the quality of the start-up $Q_1$ is sufficiently better than the quality of the large firm $Q_2$. The second condition means that the initial state of the new market $Y(0) = y$ cannot generate great profit immediately. Thus, by Proposition 4.3, more detailed information about the large firm is of little value when the
quality of the start-up is much better than that of the large firm in the new market that cannot generate great profit immediately.

The expected payoff $\tilde{V}(y)$ obtained by the real options strategy $\tau_1^*$ may generate less profit than the expected payoff $\tilde{V}_{NPV}(y)$ obtained by the zero-NPV strategy (which means to invest when the NPV of the investment becomes positive) with the same prospect $X$. To see this, consider the function $g(y)$ defined by (16) and assume that the equation $g(y) = 0$ ($y > 0$) has exactly two solutions denoted $0 < y_L^{NPV} < y_U^{NPV}$ (see Figure 1), which is expected to hold in many cases. Then, the start-up who employs the zero-NPV strategy invests at time $\tau_1^{NPV} = \inf\{t \geq 0 \mid Y(t) \in [y_L^{NPV}, y_U^{NPV}]\}$, although the start-up who uses the real options strategy invests at $\tau_1^* = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\}$. Since $y_L^{NPV} < y_M(Q_1) < y_U(\tilde{Q}_2) < y_U^{NPV}$ (see Figure 1), the zero-NPV timing $\tau_1^{NPV}$ is not later than the real options timing $\tau_1^*$. We define $Q_{NPV}$ as the unique solution of $y_U(q) = y_U^{NPV}$. Taking into consideration that the zero-NPV timing is expressed as $\inf\{t \geq 0 \mid Y(t) \in [y_L^{NPV}, y_U(Q_{NPV})]\}$, we can show the following corollary.

**Corollary 4.2** Assume that Condition (a) holds. Also assume that the equation $g(y) = 0$ ($y > 0$) has exactly two solutions. Then, $\tilde{V}_{NPV}(y) > \tilde{V}(y)$ holds if and only if one of the following three conditions is satisfied in Case 3.3. (i.e., $\tilde{Q}_2 > Q_2$ and $p(\beta_1, Q_1, \tilde{Q}_2) > 0$):

- $Q_{NPV} < Q_2$ and $y > y_U(\tilde{Q}_2)$
- $Q_2 \leq Q_{NPV}$, $B(\tilde{Q}_2) < B(Q_{NPV})$ and $y > y_U(\tilde{Q}_2)$
- $Q_2 \leq Q_{NPV}$, $\tilde{B}(Q_{NPV}) \leq \tilde{B}(\tilde{Q}_2)$ and $y_U(\tilde{Q}_2) < y < y_C$

Here, $y_U(\cdot)$ is the unique solution of equation (14) and $\tilde{B}(\cdot)$ is defined by (23). Moreover, $y_C$ is the unique solution of the equation

$$\frac{D_1}{\rho - \mu} (y \lor y_M(\tilde{Q}_2))^{-\beta_1+1} y^{\beta_1} + \tilde{B}(\tilde{Q}_2) y^{\beta_2} - \frac{D_1}{\rho - \mu} y + I_1 = 0 \quad (y_U(Q_2) < y \leq y_U(Q_{NPV})),$$

which is obtained as the intersection of the graphs of two functions $\tilde{V}_{NPV}(y) = D_1 y / (\rho - \mu) - I_1 - D_1 (y \lor y_M(\tilde{Q}_2))^{-\beta_1+1} y^{\beta_1} / (\rho - \mu)$ and $\tilde{V}(y) = \tilde{B}(\tilde{Q}_2) y^{\beta_2}$.

To conclude this section, let us examine the special case where the mean of the start-up’s prospect $X$ is equal to the real quality of the large firm $Q_2$. In this case, contrary to intuition, the start-up’s strategy does not always become optimal.
Proposition 4.4 Assume that Condition (a) holds and $E[X] = Q_2$. Then,

$$
\tilde{Q}_2 = \begin{cases}
\leq Q_2 & (1 < \beta_1 < 2) \\
= Q_2 & (\beta_1 = 2) \\
\geq Q_2 & (\beta_1 > 2).
\end{cases}
$$

(Proof) Note that

$$
\tilde{Q}_2^{\beta_1-1} = E[X^{\beta_1-1}] = \begin{cases}
\leq E[X]^{\beta_1-1} = Q_2^{\beta_1-1} & (1 < \beta_1 < 2) \\
= Q_2 & (\beta_1 = 2) \\
\geq E[X]^{\beta_1-1} = Q_2^{\beta_1-1} & (\beta_1 > 2),
\end{cases}
$$

(24)

where (24) follows from the Jensen inequality (e.g., see Øksendal (2003)), since the function $x^{\beta_1-1}$ $(x > 0)$ is concave when $1 < \beta_1 < 2$ and it is convex when $\beta_1 > 2$. We can deduce the conclusion, because $x^{\beta_1-1}$ $(x > 0)$ is continuous and monotonically increasing. □

Recall that, from (6), $\partial \beta_1 / \partial \mu < 0, \partial \beta_1 / \partial \sigma < 0$ and $\partial \beta_1 / \partial \rho > 0$. When the expected return $\mu$ and the volatility $\sigma$ of the market are high and the discount rate $\rho$ is low, $\beta_1$ satisfies $1 < \beta_1 < 2$, and hence the start-up takes the strategy in the case of underestimating the large firm’s quality (i.e., $\tilde{Q}_2 \leq Q_2$) by Proposition 4.4. On the other hand, when the expected return and the volatility of the market are low and the discount rate is high, the start-up takes the strategy in the case of overestimating the large firm’s quality by Proposition 4.4. In other words, the start-up in the market with high expected return, high volatility and low discount rate can take a better strategy (i.e., $\tilde{Q}_2$ becomes closer to $Q_2$) when the mean $E[X]$ of the prospect slightly exceeds $Q_2$. In the market with low expected return, low volatility and high discount rate, the contrary holds.

5 Numerical Examples

This section presents some examples in which the start-up’s loss $W(y)$ due to incomplete information is numerically computed. We set the parameters related to the state of the new market $Y(t)$ and the start-up as in Lambrecht and Perraudin (2003), that is, $\mu = 0, \rho = 0.07, \sigma = 0.1, D_1 = 1$, and $I_1 = 4$. Then, by definition, we have the start-up’s quality $Q_1 = D_1/I_1 = 0.25$, and by (6), (7) and (9) we can compute $\beta_1 = 4.2749, \beta_2 =$
\[ y_M(Q_1) = 0.3655, \text{ and } p(\beta_1, q_1, q_2) = 0.6417 - q_2/q_1 > 0. \] We set the quality of the large firm as \( Q_2 = Q_1/2 \), that is, \( Q_2 = 0.125 \). Notice that in this case, by \( p(\beta_1, Q_1, Q_2) = 0.1417 > 0 \), the prerequisite condition actually holds.

First, we computed the value function \( V(y; Q_2) \) of the start-up who has complete information about the quality of the large firm (see Figure 2). Moreover, in order to examine when Condition (a) (i.e., \( g(y) \leq V(y; Q_2) \)) is satisfied, we computed \( g(y) \) together with \( V(y; Q_2) \), for various uniform distributions of the random variable \( X \) that satisfy \( Q_2 = Q_2 \) but have different support widths \( 0.005, 0.01 \) and \( 0.015 \). From Figure 2, we can observe that Condition (a) does not hold when the support of \( X \) is too wide. Since the random variable \( X \) means the start-up’s prospect for the large firm’s quality, this corresponds to the situation where the start-up’s prospect about the quality of the large firm is obscure. On the other hand, it is expected that Condition (a) holds when the start-up’s prospect is decisive.

Next, we computed the start-up’s expected payoff \( \bar{V}(y) \) and loss \( W(y) \) in the case of incomplete information. Figures 3 and 4 illustrate \( \bar{V}(y) \) and \( W(y) \) of the start-up who believes the quality of the large firm follows uniform distributions with various supports. In comparison with the first experiment, here, we employed uniform distributions that have the same support width \( 0.05 \) but different values of \( Q_2 \), in order to calculate the loss in various cases. For instance, \( [0.06, 0.11] \) in Figure 3 shows \( \bar{V}(y) \) in the case where the start-up’s prospect \( X \) follows the uniform distribution with support \( [0.06, 0.11] \).

Since these examples satisfy Condition (a), \( \bar{V}(y) \) and \( W(y) \) can be computed by the formulas given in Propositions 4.2 and 4.3. Table 1 shows quantities \( Q_2, y_M(Q_2), p(\beta_1, Q_1, Q_2) \) and \( y_M(Q_2) \) for each uniform distribution of \( X \), where the top row shows the values in the case of \( Q_2 = Q_2 = 0.125 \). From \( Q_2 \) and \( p(\beta_1, Q_1, Q_2) \) in Table 1, we see that \( \{0.06, 0.11\} \) and \( \{0.08, 0.13\} \) belong to Case 1.3 (i.e., \( Q_2 < Q_2 \) and \( p(\beta_1, Q_1, Q_2) > 0 \)), while \( \{0.1, 0.15\} \) and \( \{0.12, 0.17\} \) belong to Case 3.3 (i.e., \( Q_2 > Q_2 \) and \( p(\beta_1, Q_1, Q_2) > 0 \)). Since \( \{0.14, 0.19\} \) does not satisfy the prerequisite condition (i.e., \( p(\beta_1, Q_1, Q_2) = -0.0239 < 0 \), it corresponds to Case 3.2 (i.e., \( Q_2 > Q_2 \) and \( p(\beta_1, Q_1, Q_2) \leq 0 \)). Moreover, in the case of \( \{0.14, 0.19\} \), we have \( \bar{V}(y) = 0 (y > 0) \) and \( W(y) \) becomes equal to \( V(y; Q_2) \) in Figure 2, and hence they are not shown in Figures 3 and 4. In the case of \( \{0.1, 0.15\} \), the mean of the start-up’s prospect \( X \) agrees with \( Q_2 \), but \( Q_2 = 0.1269 > 0.125 = Q_2 \) and therefore
the start-up suffers the loss when \( y > 0.5131 \) (cf. Proposition 4.4). However, in this case, since the loss is quite small (the maximum loss is just \( W(y) = 0.0082 \) for \( y = 0.523 \)), we do not depict \( W(y) \) in Figure 4.

In Figure 4, it is remarkable that the start-up’s loss in the case of underestimating the quality of the large firm (i.e., Case 1) is greater than that of the overestimation case (i.e., Case 3). This is because, in this example, the value function \( V(y; Q_2) \) is much smaller than the investment cost \( I_1 = 4 \) for all \( y > 0 \). We note that the maximum loss is \( I_1 + V(y; Q_2) \) in Case 1 and is \( V(y; Q_2) \) in Case 3. Therefore, we may not observe such a remarkable phenomenon when \( V(y; Q_2) \) is larger than \( I_1 \) in a wide area of \( y > 0 \).

Finally, we examined how the start-up’s loss \( W(y) \) due to incomplete information varies with the volatility \( \sigma \) of the new market. Figure 5 illustrates the relative loss \( W(y)/V(y; Q_2) \) for \( \sigma = 0.1, 0.15, 0.2 \) and 0.25, where the start-up’s prospect \( X \) is assumed to follow the uniform distribution with support \([0.08, 0.13]\). We attempted to compute the loss for \( \sigma = 0.05 \) and 0.3, but for \( \sigma = 0.05 \) Condition (a) does not hold and for \( \sigma = 0.3 \) we have \( p(\beta_1, \beta_2, Q_2) < 0 \) and \( V(y; Q_2) = 0 \) (recall that the prerequisite condition is restrictive when the volatility \( \sigma \) is high as mentioned in Section 3). In these examples, the difference between \( \hat{Q}_2 \) and the real value \( Q_2 \) is monotonically increasing with respect to \( \sigma \). However we did not observe a remarkable relation between the volatility \( \sigma \) and the relative loss \( W(y)/V(y; Q_2) \). A possible reason for this is that \( \beta_1 \) and \( \beta_2 \) also vary with \( \sigma \).

Table 1: \( \hat{Q}_2; y_M(\hat{Q}_2) \) and \( y_U(\hat{Q}_2) \) for various uniform distributions of \( X \).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \hat{Q}_2 )</th>
<th>( y_M(\hat{Q}_2) )</th>
<th>( p(\beta_1, Q_1, \hat{Q}_2) )</th>
<th>( y_U(\hat{Q}_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>0.125</td>
<td>0.731</td>
<td>0.1417</td>
<td>0.523</td>
</tr>
<tr>
<td>[0.06,0.11]</td>
<td>0.087</td>
<td>0.1053</td>
<td>0.2937</td>
<td>0.8029</td>
</tr>
<tr>
<td>[0.08,0.13]</td>
<td>0.1072</td>
<td>0.8524</td>
<td>0.2129</td>
<td>0.6312</td>
</tr>
<tr>
<td>[0.1,0.15]</td>
<td>0.1269</td>
<td>0.7201</td>
<td>0.1341</td>
<td>0.5131</td>
</tr>
<tr>
<td>[0.12,0.17]</td>
<td>0.1466</td>
<td>0.6233</td>
<td>0.0553</td>
<td>0.4217</td>
</tr>
<tr>
<td>[0.14,0.19]</td>
<td>0.1664</td>
<td>0.5491</td>
<td>-0.0239</td>
<td>N/A</td>
</tr>
</tbody>
</table>
Figure 2: $V(y; Q_2)$ and $g(y)$. 

Figure 3: The start-up’s expected payoff $\tilde{V}(y)$. 

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Figure 4: The start-up’s loss due to incomplete information, i.e., $W(y)$.

Figure 5: $W(y)/V(y; Q_2)$ for various parameters $\sigma$. 
6 Conclusion

This paper has investigated the effect of incomplete information in the model in which a start-up with a unique idea and technology pioneers a new market that will be taken over by a large firm eventually. The main contribution of this paper is to evaluate the start-up’s loss due to incomplete information about the large firm. The proposed method could be applied in other real options models involving several firms. The results obtained in this paper can be summarized as follows.

If the quality of the start-up is much better than that of the large firm and the current state of the market cannot generate great profit immediately, then the start-up requires no further survey on the quality of the large firm.

On the other hand, information about the quality of the large firm is valuable in the market that can readily generate great profit, even if the quality of the start-up is much better than that of the large firm. In this case, it is quite likely that the start-up’s immediate investment does not produce much income for the start-up even before the large firm takes over the market from the start-up.

When it is doubtful that the quality of the start-up overwhelms that of the large firm, information about the quality of the large firm is always valuable regardless of the state of the market. The reason for this is that there is a possibility that the investment in the market is of no value (i.e., the prerequisite condition for the start-up’s investment does not hold), in addition to the same risk as in the previous case, that is, the possibility that the start-up obtains little profit until the large firm’s entry.

Furthermore, under incomplete information, the expected payoff of the start-up investing at the zero-NPV trigger could become greater than that of the start-up obeying the real options approach, and the start-up usually suffers the loss due to incomplete information even if the mean of the start-up’s prospect for the quality of the large firm is equal to the real value.

In the real world, a start-up who has a unique idea and technology but is not competitive in the market may want to sell its idea and technology to a large firm, instead of pioneering the market by itself. Then, the value function which the start-up believes can be interpreted as a reward which the start-up demands for its idea and technology. As revealed in this paper, the value of the investment which the start-up believes under
complete information is generally different from the real value of the investment. Because of this gap, negotiations between the start-up and the large firm may not go smoothly. It remains as an interesting issue of future research to reveal the effect of incomplete information in such a negotiation problem of a firm having an option to sell its idea and technology to the rival firm.

Appendix A  Proof of Proposition 3.1

Taking account of (10), we can compute (3) as follows:

\[
E \left[ \int_{\tau_1}^{\tau_2} e^{-\rho t} D_1 Y(t) dt - e^{-\rho \tau_1} I_1 \right]
= E \left[ e^{-\rho \tau_1} \left( D_1 E^{Y(\tau_1)} \left[ \int_{0}^{\tau_2} e^{-\rho t} Y(t) dt \right] - I_1 \right) \right]
= E \left[ e^{-\rho \tau_1} \left( D_1 E^{Y(\tau_1)} \left[ \int_{0}^{\infty} e^{-\rho t} Y(t) dt - \int_{\tau_2}^{\infty} e^{-\rho t} Y(t) dt \right] - I_1 \right) \right]
= E \left[ e^{-\rho \tau_1} \left( D_1 E^{Y(\tau_1)} \left[ \int_{0}^{\infty} e^{-\rho t} Y(t) dt - e^{-\rho \tau_2} E^{Y(\tau_2)} \left[ \int_{0}^{\infty} e^{-\rho t} Y(t) dt \right] - I_1 \right) \right) \right]
= E \left[ e^{-\rho \tau_1} \left( \frac{D_1 Y(\tau_1)}{\rho - \mu} - \frac{D_1 (Y(\tau_1) \vee y_M(q))^{-\beta_1 + 1} Y(\tau_1)^{\beta_1}}{\rho - \mu} - I_1 \right) \right],
\]

where we use the strong Markov property (e.g. see Øksendal (2003)) of the geometric Brownian motion \(Y(t)\) to deduce (25) and (26), and use the formula of the expectation involving a hitting time (e.g. see Dixit and Pindyck (1994)) to deduce (27). Here, for a random variable \(Z\), \(E^{Y(\tau_i)}[Z]\) denotes a random variable \(G(Y(\tau_i))\), where for \(y' > 0\), \(G(y')\) is defined as an expectation \(E[Z]\) in the case where \(Y(t)\) starts at \(Y(0) = y'\). Thus, problem (4) with \(X\) replaced by \(q\) is equivalent to \(\sup_{\tau_1} E \left[ e^{-\rho \tau_1} f(Y(\tau_1); q) \right], \)

\[
f(y; q) = \frac{D_1 y}{\rho - \mu} - \frac{D_1 (y \vee y_M(q))^{-\beta_1 + 1} y^{\beta_1}}{\rho - \mu} - I_1.
\]

Consider the case where \(f(y; q) \leq 0\) for all \(y > 0\). In this case, the value function and an optimal stopping time are trivially given by \(V(y; q) = 0\) and \(\tau_q \equiv +\infty\), respectively, for all \(y > 0\). Now, let us derive a necessary and sufficient condition for \(f(y; q) \leq 0\) to hold for all \(y > 0\). Since \(f(y; q)\) is concave for \(y \in [0, y_M(q)]\) by \(\beta_1 > 1\) and \(f(y; q) = -I_1\)
holds for $y = 0$ and $y \geq y_M(q)$, $f(y; q)$ ($y > 0$) takes the maximum value at $y = \beta_1^{1/(\beta_1 - 1)}y_M(q)$, which is the unique solution of $\partial f(y; q)/\partial y = 0$ for $y \in [0, y_M(q)]$. Since we have $f(\beta_1^{1/(\beta_1 - 1)}y_M(q); q) = D_1p(\beta_1, Q_1, q)/q$ by (8), (9) and (28), we can deduce that $p(\beta_1, Q_1, q) \leq 0$ is a necessary and sufficient condition for $f(y; q) \leq 0$ to hold for all $y > 0$. Thus, if $\phi(y)$ satisfies the following conditions:

\[
\sigma^2 d^2 \phi(y) + \left(\mu + \frac{\sigma^2}{2}\right) d\phi(y) - \rho \phi(y) \begin{cases} 
\leq 0 & \text{for all } y > 0, \\
= 0 & \text{for all } y \notin [y_M(Q_1), y_U(q)],
\end{cases}
\]

\[
\phi(y) - f(y) \begin{cases} 
\geq 0 & \text{for all } y > 0, \\
= 0 & \text{for all } y \in [y_M(Q_1), y_U(q)],
\end{cases}
\]

\[
\lim_{y \to 0} \phi(y) = 0, \\
\lim_{y \to +\infty} \phi(y) = 0,
\]

then we obtain the value function $V(y; q) = \phi(y)$ and an optimal stopping time $\tau^q_1 = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(q)]\}$ via the relation between optimal stopping and variational inequalities (for details see Øksendal (2003)). Note that the thresholds $y_M(q)$ and $y_U(q)$ are defined so that $\phi(y)$ is continuously differentiable at the thresholds (i.e, value matching and smooth pasting, see also Dixit and Pindyck (1994)). Since we can check all the conditions for $\phi(y)$ by direct calculation, we obtain the proposition. \hfill \Box

**Appendix B  Proof of Proposition 4.1**
Note that
\[
E \left[ \int_{\tau_1}^{\tau_2} e^{-\rho t} D_1(1,0)Y(t)dt + \int_{\tau_2}^{+\infty} e^{-\rho t} D_1(1,1)Y(t)dt - e^{-\rho \tau_1} I_1 \right]
\]
\[
= \int_0^{+\infty} E \left[ \int_{\tau_1}^{\tau_2} e^{-\rho t} D_1(1,0)Y(t)dt + \int_{\tau_2}^{+\infty} e^{-\rho t} D_1(1,1)Y(t)dt - e^{-\rho \tau_1} I_1 \mid X = q \right] d\Psi_X(q)
\]
\[
= \int_0^{+\infty} E \left[ e^{-\rho \tau_1} f(Y(\tau_1); q) \right] d\Psi_X(q)
\]
\[
= E \left[ e^{-\rho \tau_1} g(Y(\tau_1)) \right],
\]
where \(\Psi_X(q)\) denotes the distribution of \(X\), and \(f\) and \(g\) are defined by (28) and (16), respectively. Here, (30) and (32) follow from the independence between \(X\) and \(Y(t)\), and (31) follows from the strong Markov property as in Appendix A.

First, we consider the case where \(g(y) \leq 0\) for all \(y > 0\). In this case, apparently, the value function and an optimal stopping time are given by \(V(y) = 0\) and \(\tau_1^* = +\infty\), respectively, for all \(y > 0\). Since \(h(y) \leq 0\) holds for all \(y > 0\) by (18), \(V(y; \tilde{Q}_2) = 0\) and \(\tau_1^\tilde{Q}_2 = +\infty\) hold for all \(y > 0\). This implies \(V(y) = V(y; \tilde{Q}_2)\) and \(\tau_1^* = \tau_1^\tilde{Q}_2\) for all \(y > 0\).

Next, let us assume that there exists some \(\tilde{y} > 0\) such that \(g(\tilde{y}) > 0\). We have \(V(\tilde{y}; \tilde{Q}_2) > 0\) by Condition (a) (i.e., \(g(y) \leq V(y; \tilde{Q}_2)\) for all \(y > 0\)). Then, we can deduce that \(p(y, Q_1, \tilde{Q}_2) > 0\), taking into consideration that \(V(y; \tilde{Q}_2) = 0\) holds for all \(y > 0\) whenever \(p(y, Q_1, \tilde{Q}_2) \leq 0\) by Proposition 3.1. We have only to check the conditions (29) with \(q\) and \(f\) replaced by \(\tilde{Q}_2\) and \(g\) for \(\phi(y) = V(y; \tilde{Q}_2)\) (i.e., the right-hand side of (12) with \(q\) replaced by \(\tilde{Q}_2\)). The conditions (29) except for the second can be checked directly as in Appendix A. Condition (a) ensures \(\phi(y) - g(y) \geq 0\) for all \(y > 0\). By (18) and (19), for all \(y \in [y_M(Q_1), y_U(\tilde{Q}_2)]\), we have \(\phi(y) - g(y) = h(y) - g(y) \leq 0\), where \(h(y)\) is defined by (17). These imply the second condition. Therefore, we obtain \(V(y) = \phi(y)\) and \(\tau_1^* = \inf\{t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)]\}\) via the relation between optimal stopping and variational inequalities (e.g., see Øksendal (2003)).

\[\square\]

**Appendix C  Proof of Proposition 4.2**

We have only to compute expectation (21). First, we assume \(p(\beta_1, Q_1, \tilde{Q}_2) \leq 0\). In this case, we have \(\tau_1^* = \tau_1^\tilde{Q}_2 = +\infty\) by Propositions 3.1 and 4.1, and hence we have also \(\tau_2^\tilde{Q}_2 =\)
by (20). Thus, \( \tilde{V}(y) = 0 \) holds for all \( y > 0 \). Next, let us assume \( p(\beta_1, Q_1, \tilde{Q}_2) > 0 \). In this case, we have

\[
\tau_1^* = \tau_1^{Q_2} = \inf \{ t \geq 0 \mid Y(t) \in [y_M(Q_1), y_U(\tilde{Q}_2)] \}
\]  

by Propositions 3.1 and 4.1. As in Appendix A, by the strong Markov property, (21) is equal to (27) with \( \tau_1 \) and \( q \) replaced by \( \tau_1^* \) and \( Q_2 \), respectively, that is, \( \tilde{V}(y) = E \left[ e^{-\rho \tau_1^*} f(Y(\tau_1^*); Q_2) \right] \), where \( f \) is defined by (28). Since \( Y(\tau_1^*) \) is a constant such that

\[
Y(\tau_1^*) = \begin{cases} 
 y_M(Q_1) & (0 < y < y_M(Q_1)) \\
 y & (y_M(Q_1) \leq y < y_U(\tilde{Q}_2)) \\
 y_U(\tilde{Q}_2) & (y \geq y_U(\tilde{Q}_2))
\end{cases}
\]  

by (33), we have

\[
\tilde{V}(y) = f(Y(\tau_1^*); Q_2) E \left[ e^{-\rho \tau_1^*} \right].
\]  

Thus, by applying the formula of the expectation involving a hitting time to (35), we obtain the formula of \( \tilde{V}(y) \) given in the proposition. \( \square \)

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**References**


