Robust Nash Equilibria and Second-Order Cone Complementarity Problems*

Shunsuke Hayashi, Nobuo Yamashita, Masao Fukushima†

Department of Applied Mathematics and Physics,
Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

Abstract In this paper we consider a bimatrix game in which the players can neither evaluate their cost functions exactly nor estimate their opponents' strategies accurately. To formulate such a game, we introduce the concept of robust Nash equilibrium that results from robust optimization by each player, and prove its existence under some mild conditions. Moreover, we show that a robust Nash equilibrium in the bimatrix game can be characterized as a solution of a second-order cone complementarity problem (SOCCP). Some numerical results are presented to illustrate the behavior of robust Nash equilibria.

Keywords: bimatrix game; robust Nash equilibrium; second-order cone complementarity problem

1 Introduction

We consider a bimatrix game where two players attempt to minimize their own costs. Let $y \in \Re^n$ and $z \in \Re^m$ denote strategies of Players 1 and 2, respectively. Moreover, let Player 1's cost function be given by $f_1(y,z) := y^T Az$ with cost matrix $A \in \Re^{n\times m}$, and Player 2's cost function be given by $f_2(y,z) := y^T Bz$ with cost matrix $B \in \Re^{n\times m}$. We suppose that the two players choose their strategies y and z from the nonempty closed convex sets $S_1 \subseteq \Re^n$ and $S_2 \subseteq \Re^m$, respectively. Then, the players determine their strategies by solving the following minimization problems with the opponents' strategies fixed:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad y^T Az \quad \text{subject to} \;\; y \in S_1, \\ &\underset{z}{\text{minimize}} \quad y^T Bz \quad \text{subject to} \;\; z \in S_2. \end{aligned} \tag{1}$$

A point $(\overline{y}, \overline{z})$ satisfying $\overline{y} \in \operatorname{argmin}_{y\in S_1} y^T A\overline{z}$ and $\overline{z} \in \operatorname{argmin}_{z\in S_2} \overline{y}^T Bz$ is called a Nash equilibrium [2]. Since the minimization problems (1) are convex, the problem of finding a Nash equilibrium can be formulated as a variational inequality problem (VIP) [12]. Moreover, if S_1 and S_2 are given by $S_1 = \{y \in \Re^n \mid g_i(y) \le 0,\ i = 1,\dots,N\}$ and $S_2 = \{z \in \Re^m \mid h_j(z) \le 0, j = 1,\dots,M\}$ with some convex functions $g_i : \Re^n \to \Re$ and $h_j : \Re^m \to \Re$, respectively, then the VIP is further reformulated as a mixed complementarity problem (MCP), which is also called a box-constrained variational problem. Recently, MCP has been extensively studied and many efficient algorithms have been developed for solving it [7, 8, 25].

The concept of Nash equilibrium is premised on the accurate estimation of opponent's strategy and the exact evaluation of player's own cost function. Thus Nash equilibrium may hardly represent the

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†*E-mail address*: {hayashi, nobuo, fuku}@amp.i.kyoto-u.ac.jp

actual situation when those operations are subject to errors. To deal with such situations, we introduce the concept of *robust Nash equilibrium*, which is parallel to that of robust optimization [3, 4, 5, 11].

In the field of game theory, there has been much study on games with incomplete information and the robustness of equilibria. Harsanyi [14, 15, 16] defines a game with incomplete information as a game where each player's payoff function is given in a stochastic manner with its probability distribution. This is one of the most popular formulations of games with incomplete information. Kajii and Morris [20] adopt Harsanyi's formulation to define a concept of robust equilibria to incomplete information. Especially, they show that games with strict equilibria do not necessarily have robust equilibria, and the unique correlated equilibrium of a game is robust. Moreover, they introduce the notion of p-dominance, and show that a p-dominant equilibrium is robust under an appropriate assumption. Ui [26] considers the robustness of equilibria of potential games, which is a class of games involving Monderer and Shapley's potential function [22]. He shows that the action profile uniquely maximizing a potential function is robust. Recently, Morris and Ui [24] have unified the above discussions for p-dominant equilibria and equilibria of potential games. They define a generalized potential function that contains Monderer and Shapley's potential function [22] and Morris's characteristic potential function [23], and show that an action maximizing the generalized potential function is a robust equilibrium to incomplete information.

In the above-mentioned references, the "robustness" means that an equilibrium is stable with respect to estimation errors. On the other hand, the robust Nash equilibrium introduced in this paper is an equilibrium that results from robust optimization [3, 4, 5, 11] by each player. More precisely, our formulation is premised on the conditions (I) – (III) in the next section. Indeed, these conditions are applicable to several actual problems such as dynamic economic systems based on duopolistic competition with random disturbances, traffic equilibrium problems with incomplete information on travel costs, etc. We note that the concept of robustness in this paper is different from those considered in [20, 24, 26].

In what follows, we first define a robust Nash equilibrium for a bimatrix game, and discuss its existence. Then, we show that, under certain assumptions, the robust Nash equilibrium problem can be formulated as a second-order cone complementarity problem (SOCCP). The SOCCP is a class of complementarity problems where the complementarity condition is associated with the Cartesian product of second-order cones. Several methods for solving SOCCPs have been proposed recently [6, 9, 13, 17, 18].

Throughout the paper, we use the following notations. For a set X, $\mathcal{P}(X)$ denotes the set of all subsets of X. For a function $f : \Re^n \times \Re^m \to \Re$, $f(\cdot, z) : \Re^n \to \Re$ and $f(y, \cdot) : \Re^m \to \Re$ denote the functions with z and y, respectively, being fixed. $\Re^n_+$ denotes the nonnegative orthant in $\Re^n$, that is, $\Re^n_+ := \{x \in \Re^n \,|\, x \geq 0\}$. For a vector $x \in \Re^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^T x}$. For a matrix $M \in \Re^{n \times m}$, $\|M\|_F$ denotes the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^n \sum_{j=1}^m (M_{ij})^2)^{1/2}$. $I_n \in \Re^{n \times n}$ denotes the identity matrix, and $e_n \in \Re^n$ denotes the vector of ones. For a matrix $M \in \Re^{n \times m}$, M_i^r denotes the i-th row vector and M_i^c denotes the i-th column vector.

2 Robust Nash equilibria and its existence

In this section, we define the robust Nash equilibrium of a bimatrix game, and give sufficient conditions for its existence.

Throughout the paper, we assume that the following three statements hold for each player i ($i = 1, 2$):

(I) Player 1 cannot estimate Player 2's strategy z exactly, but can only estimate that it belongs to a set $Z(z) \subseteq \Re^m$ containing z. Similarly, Player 2 cannot estimate Player 1's strategy y exactly, but can only estimate that it belongs to a set $Y(y) \subseteq \Re^n$ containing y.

(II) Player 1 cannot estimate his/her cost matrix exactly, but can only estimate that it belongs to a nonempty set $D_A \subseteq \Re^{n \times m}$. Player 2 cannot estimate his/her cost matrix exactly, but can only estimate that it belongs to a nonempty set $D_B \subseteq \Re^{n \times m}$.

(III) Each player tries to minimize his/her worst cost under (I) and (II).

Now, we define the robust Nash equilibrium under the above three assumptions. To realize (III), we define functions $\tilde{f}_i : \Re^n \times \Re^m \to \Re$ ($i = 1, 2$) by

$$\begin{aligned} \tilde{f}_1(y, z) &:= \max \left\{ y^T \hat{A} \hat{z} \;\middle|\; \hat{A} \in D_A, \hat{z} \in Z(z) \right\}, \\ \tilde{f}_2(y, z) &:= \max \left\{ \hat{y}^T \hat{B} z \;\middle|\; \hat{B} \in D_B, \hat{y} \in Y(y) \right\}. \end{aligned} \tag{2}$$

The functions $\tilde{f}_1(\cdot, z)$ and $\tilde{f}_2(y, \cdot)$ represent Player 1's and Player 2's worst costs, respectively, under uncertainty as assumed in (I) and (II). Players 1 and 2 then solve the following minimization problems, respectively:

$$\begin{aligned} &\underset{y}{\text{minimize}} \;\; \tilde{f}_1(y, z) \quad \text{subject to} \;\; y \in S_1, \\ &\underset{z}{\text{minimize}} \;\; \tilde{f}_2(y, z) \quad \text{subject to} \;\; z \in S_2. \end{aligned} \tag{3}$$

Now, we are in a position to define the robust Nash equilibrium.

Definition 1 *Let functions $\tilde{f}_1$ and $\tilde{f}_2$ be defined by* (2). *If $\overline{y}^r \in \operatorname{argmin}_{y \in S_1} \tilde{f}_1(y, \overline{z}^r)$ and $\overline{z}^r \in \operatorname{argmin}_{z \in S_2} \tilde{f}_2(\overline{y}^r, z)$, that is, $(\overline{y}^r, \overline{z}^r)$ is a Nash equilibrium of game* (3)*, then $(\overline{y}^r, \overline{z}^r)$ is called a robust Nash equilibrium of game* (1).

Next, we give a condition for the existence of a robust Nash equilibrium of game (1). Note that $Y(\cdot)$ and $Z(\cdot)$ given in (I) can be regarded as set-valued mappings. In what follows, we suppose that $Y(\cdot)$, $Z(\cdot)$, D_A and D_B in (I) and (II) satisfy the following assumption.

Assumption A

(a) *Set-valued mappings $Y : \Re^n \to \mathcal{P}(\Re^n)$ and $Z : \Re^m \to \mathcal{P}(\Re^m)$ are continuous, and $Y(y)$ and $Z(z)$ are nonempty compact for any $y \in \Re^n$ and $z \in \Re^m$.*

(b) *$D_A \subseteq \Re^{n \times m}$ and $D_B \subseteq \Re^{n \times m}$ are nonempty and compact sets.*

The functions $\tilde{f}_1$ and $\tilde{f}_2$ defined by (2) are well-defined under this assumption. By simple arguments on continuity, we can show that $\tilde{f}_1$ and $\tilde{f}_2$ are continuous everywhere. Furthermore, we have the following lemma on the convexity of $\tilde{f}_1(\cdot, z)$ and $\tilde{f}_2(y, \cdot)$. We omit the proof since it is trivial.

Lemma 1 *Suppose that Assumption* A *holds. Let $\tilde{f}_1$ and $\tilde{f}_2$ be defined by* (2)*. Then, for any fixed $z \in \Re^m$ and $y \in \Re^n$, the functions $\tilde{f}_1(\cdot, z)$ and $\tilde{f}_2(y, \cdot)$ are convex.*

The next lemma is a fundamental result for noncooperative n-person game [2, Theorem 9.1.1].

Lemma 2 *Consider a noncooperative two-person game where cost functions are given by* $\theta_1 : \Re^n \times \Re^m \to \Re$ *and* $\theta_2 : \Re^n \times \Re^m \to \Re$*. Suppose that functions* θ_1 *and* θ_2 *are continuous at any* (y, z)*, and that functions* $\theta_1(\cdot, z)$ *and* $\theta_2(y, \cdot)$ *are convex. Suppose that* S_1 *and* S_2 *are nonempty compact convex sets. Then, the game has a Nash equilibrium.*

By the above lemmas, we obtain the following theorem for the existence of a robust Nash equilibrium of game (1)

Theorem 1 *Suppose that Assumption* A *holds, and that* S_1 *and* S_2 *are nonempty compact convex sets. Then, game* (1) *has a robust Nash equilibrium.*

Proof. By Lemma 1, the functions $\tilde{f}_1(\cdot, z)$ and $\tilde{f}_2(y, \cdot)$ are convex. Moreover, as pointed out earlier, $\tilde{f}_1$ and $\tilde{f}_2$ are continuous everywhere. Therefore, from Lemma 2, game (3) has a Nash equilibrium. This means, by Definition 1, that game (1) has a robust Nash equilibrium. ■

3 SOCCP formulation of robust Nash equilibrium

In this section, we focus on the bimatrix game where each player takes a mixed strategy, that is, $S_1 = \{y \mid y \geq 0,\ e_n^T y = 1\}$ and $S_2 = \{z \mid z \geq 0,\ e_m^T z = 1\}$, and show that the robust Nash equilibrium problem reduces to an SOCCP.

The SOCCP is to find a vector $(\xi, \eta, \zeta) \in \Re^\ell \times \Re^\ell \times \Re^\nu$ satisfying the conditions

$$\mathcal{K} \ni \xi \perp \eta \in \mathcal{K}, \quad G(\xi, \eta, \zeta) = 0, \tag{4}$$

where $G : \Re^\ell \times \Re^\ell \times \Re^\nu \to \Re^\ell \times \Re^\nu$ is a given function, $\xi \perp \eta$ denotes $\xi^T \eta = 0$, and $\mathcal{K}$ is a closed convex cone defined by $\mathcal{K} = \mathcal{K}^{\ell_1} \times \mathcal{K}^{\ell_2} \times \cdots \times \mathcal{K}^{\ell_m}$ with ℓ_j-dimensional second-order cones $\mathcal{K}^{\ell_j} = \{(\zeta_1, \zeta_2) \in \Re \times \Re^{\ell_j - 1} \mid \|\zeta_2\| \leq \zeta_1\}$. Since $\mathcal{K}^1$ is the set of nonnegative reals, the nonlinear complementarity problem (NCP) is a special case of SOCCP (4) with $\mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^1$ and $G(\xi, \eta, \zeta) := F(\xi) - \eta$ for a given function $F : \Re^\ell \to \Re^\ell$.

Consider the bimatrix game where Players 1 and 2 solve the following minimization problems (5) and (6), respectively:

$$\underset{y}{\text{minimize}} \ \ y^T A z \quad \text{subject to} \ \ y \geq 0,\ e_n^T y = 1, \tag{5}$$

$$\underset{z}{\text{minimize}} \ \ y^T B z \quad \text{subject to} \ \ z \geq 0,\ e_m^T z = 1. \tag{6}$$

It is well known that a Nash equilibrium of this game is given as a solution of a mixed linear complementarity problem. In fact, since z and y are fixed in (5) and (6), respectively, both problems are linear programming problems, and their KKT conditions are given by

$$\begin{aligned} &0 \leq y \perp Az + e_n s \geq 0, \quad e_n^T y = 1, \\ &0 \leq z \perp B^T y + e_m t \geq 0, \quad e_m^T z = 1, \end{aligned} \tag{7}$$

where $s \in \Re$ and $t \in \Re$ are Lagrange multipliers associated with the equality constraints in (5) and (6), respectively. Thus, if some (y, z) satisfies the above two KKT conditions simultaneously, then it is a Nash equilibrium of the bimatrix game. The problem of finding such a (y, z) can be further formulated as a linear complementarity problem (LCP) [10].

Now, we consider bimatrix games involving several types of uncertainty, and show that the robust Nash equilibrium problem corresponding to each game reduces to an SOCCP of the form

$$\mathcal{K} \ni M\zeta + q \perp N\zeta + r \in \mathcal{K}, \quad C\zeta = d \tag{8}$$

with variable $\zeta \in \Re^{\ell+\tau}$ and constants $M, N \in \Re^{\ell \times (\ell+\tau)}$, $q, r \in \Re^{\ell}$, $C \in \Re^{\tau \times (\ell+\tau)}$ and $d \in \Re^{\tau}$. Note that, by introducing new variables $\xi \in \Re^{\ell}$ and $\eta \in \Re^{\ell}$, problem (8) reduces to SOCCP (4) with $\nu = \ell+\tau$ and $G : \Re^{3\ell+\tau} \to \Re^{2\ell+\tau}$ defined by

$$G(\xi, \eta, \zeta) := \begin{pmatrix} \xi - M\zeta - q \\ \eta - N\zeta - r \\ C\zeta - d \end{pmatrix}.$$

3.1 Uncertainty in the opponent's strategy

In this subsection, we consider the case where each player estimates the cost matrix exactly but opponent's strategy uncertainly. More specifically, we make the following assumption.

Assumption 1

(a) $Y(y) := \{y + \delta y \in \Re^n \mid \|\delta y\| \leq \rho_y,\ e_n^T \delta y = 0\}$ *and* $Z(z) := \{z + \delta z \in \Re^m \mid \|\delta z\| \leq \rho_z,\ e_m^T \delta z = 0\}$, *where* ρ_y *and* ρ_z *are given positive constants.*

(b) $D_A = \{A\}$ *and* $D_B = \{B\}$, *where* $A \in \Re^{n \times m}$ *and* $B \in \Re^{n \times m}$ *are given constant matrices.*

Here, the conditions $e_n^T \delta y = e_m^T \delta z = 0$ in the definitions of $Y(y)$ and $Z(z)$ are provided so that $e_n^T(y+\delta y) = e_m^T(z+\delta z) = 1$ holds from $e_n^T y = e_m^T z = 1$. Under this assumption, the following theorem holds.

Theorem 2 *If Assumption 1 holds, then the bimatrix game has a robust Nash equilibrium.*

Proof. It is easily seen that Assumptions 1(a) and 1(b) imply Assumptions A(a) and A(b), respectively. Hence, the theorem readily follows from Theorem 1. ■

We now show that the robust Nash equilibrium problem can be formulated as SOCCP (8) under Assumption 1. Player 1 solves the following minimization problem to determine his/her strategy:

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \max\left\{ y^T A(z + \delta z) \;\middle|\; \|\delta z\| \leq \rho_z,\ e_m^T \delta z = 0 \right\} \\ \text{subject to} \quad & e_n^T y = 1,\ y \geq 0. \end{aligned} \tag{9}$$

Since the projection of vector $A^T y$ onto hyperplane $\pi := \{z \mid e_m^T z = 0\}$ can be represented as $(I_m - m^{-1} e_m e_m^T) A^T y$, the cost function can be written as

$$\begin{aligned} \tilde{f}_1(y, z) &= \max\left\{ y^T A(z + \delta z) \;\middle|\; \|\delta z\| \leq \rho_z,\ e_m^T \delta z = 0 \right\} \\ &= y^T A z + \max\left\{ y^T A \delta z \;\middle|\; \|\delta z\| \leq \rho_z,\ e_m^T \delta z = 0 \right\} \\ &= y^T A z + \rho_z \|\tilde{A}^T y\|, \end{aligned}$$

where $\tilde{A} := A(I_m - m^{-1} e_m e_m^T)$. Hence, by introducing an auxiliary variable $y_0 \in \Re$, problem (9) can be reduced to the following convex minimization problem:

$$\begin{aligned} \underset{y_0, y}{\text{minimize}} \quad & y^T A z + \rho_z y_0 \\ \text{subject to} \quad & \|\tilde{A}^T y\| \leq y_0,\ \ y \geq 0,\ \ e_n^T y = 1. \end{aligned}$$

This is a second-order cone programming problem [1, 21] and its KKT conditions can be written as the following SOCCP:

$$\mathcal{K}^{m+1} \ni \begin{pmatrix} \lambda_0 \\ \lambda \end{pmatrix} \perp \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y_0 \\ y \end{pmatrix} \in \mathcal{K}^{m+1},$$
$$\Re^n_+ \ni y \perp Az - \tilde{A}\lambda + e_n s \in \Re^n_+, \quad e_n^T y = 1, \quad \lambda_0 = \rho_z,$$

where $\lambda \in \Re^m$ and $s \in \Re$ are Lagrange multipliers, and $\lambda_0 \in \Re$ is an auxiliary variable. In a similar manner, the KKT conditions for Player 2 can be written as

$$\mathcal{K}^{n+1} \ni \begin{pmatrix} \mu_0 \\ \mu \end{pmatrix} \perp \begin{pmatrix} 1 & 0 \\ 0 & \tilde{B} \end{pmatrix} \begin{pmatrix} z_0 \\ z \end{pmatrix} \in \mathcal{K}^{n+1},$$
$$\Re^m_+ \ni z \perp B^T y - \tilde{B}^T \mu + e_m t \in \Re^m_+, \quad e_m^T z = 1, \quad \mu_0 = \rho_y,$$

where $\mu \in \Re^n$ and $t \in \Re$ are Lagrange multipliers, and $\mu_0 \in \Re$ is an auxiliary variable. Consequently, the problem to find (y, z) satisfying the above two KKT conditions simultaneously can be reformulated as SOCCP (8) with $\ell = 2m + 2n + 2$, $\tau = 4$, $\mathcal{K} = \mathcal{K}^{n+1} \times \mathcal{K}^{m+1} \times \Re^m_+ \times \Re^n_+$,

$$\zeta = \begin{pmatrix} y_0 \\ y \\ z_0 \\ z \\ \lambda_0 \\ \lambda \\ \mu_0 \\ \mu \\ s \\ t \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{A}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{B} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & -\tilde{A} & 0 & 0 & e_n & 0 \\ 0 & B^T & 0 & 0 & 0 & 0 & 0 & -\tilde{B}^T & 0 & e_m \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & e_n^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_m^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \\ \rho_z \\ \rho_y \end{pmatrix}.$$

3.2 Component-wise uncertainty in the cost matrices

In the following three subsections 3.2, 3.3 and 3.4, we consider the case where each player estimates the opponent's strategy exactly but his/her own cost matrix uncertainly. In this subsection, we particularly focus on the case where the uncertainty in each cost matrix occurs component-wise independently. That is, we make the following assumption.

Assumption 2

(a) $Y(y) = \{y\}$ *and* $Z(z) = \{z\}$.

(b) $D_A := \{A + \delta A \in \Re^{n\times m} \,|\, |\delta A_{ij}| \leq (\Gamma_A)_{ij} \ (i = 1,\dots,n,\ j = 1,\dots,m)\}$ *and* $D_B := \{B + \delta B \in \Re^{n\times m} \,|\, |\delta B_{ij}| \leq (\Gamma_B)_{ij} \ (i = 1,\dots,n,\ j = 1,\dots,m)\}$ *with given constant matrices* $A \in \Re^{n\times m}$ *and* $B \in \Re^{n\times m}$ *and positive constant matrices* $\Gamma_A \in \Re^{n\times m}$ *and* $\Gamma_B \in \Re^{n\times m}$.

Under this assumption, we have the following theorem. We omit the proof, since it is similar to that of Theorem 2.

Theorem 3 *If Assumption 2 holds, then the bimatrix game has a robust Nash equilibrium.*

From Assumption 2, together with the constraints $y \geq 0$ and $z \geq 0$, the cost function $\tilde{f}_1$ can be represented as

$$\begin{aligned}
\tilde{f}_1(y,z) &= \max\left\{ y^T \hat{A} z \;\middle|\; \hat{A} \in D_A \right\} \\
&= y^T A z + \max_{|\delta A_{ij}| \leq (\Gamma_A)_{ij}} \sum_{i=1}^{n}\sum_{j=1}^{m} \delta A_{ij} y_i z_j \\
&= y^T A z + \sum_{i=1}^{n}\sum_{j=1}^{m} (\Gamma_A)_{ij} y_i z_j \\
&= y^T (A + \Gamma_A) z.
\end{aligned}$$

Analogously, we have $\tilde{f}_2(y,z) = y^T(B + \Gamma_B)z$. Hence, the robust Nash equilibrium problem is simply the problem of finding a Nash equilibrium of the bimatrix game with cost matrices $A+\Gamma_A$ and $B+\Gamma_B$. This problem reduces to the MCP (7) with A and B replaced by $A+\Gamma_A$ and $B+\Gamma_B$, respectively.

3.3 Column/row-wise uncertainty in the cost matrices

In this subsection, we focus on the case where the uncertainty in matrices A and B respectively occur row-wise independently and column-wise independently. That is, we make the following assumption.

Assumption 3

(a) $Y(y) = \{y\}$ *and* $Z(z) = \{z\}$.

(b) $D_A := \{A + \delta A \,|\, \|\delta A^c_j\| \leq (\gamma_A)_j \ (j = 1,\dots,m)\}$ *and* $D_B := \{B + \delta B \,|\, \|\delta B^r_i\| \leq (\gamma_B)_i \ (i = 1,\dots,n)\}$ *with given constant matrices* $A \in \Re^{n\times m}$ *and* $B \in \Re^{n\times m}$ *and positive constant vectors* $\gamma_A \in \Re^m$ *and* $\gamma_B \in \Re^n$.

This assumption implies that the degree of uncertainty in each player's cost depends on the opponent's each pure strategy. Under this assumption, we have the following theorem. We omit the proof, since it is similar to that of Theorem 2.

Theorem 4 *If Assumption 3 holds, then the bimatrix game has a robust Nash equilibrium.*

Next, we formulate the Nash equilibrium problem as an SOCCP. From Assumption 3, we have

$$\begin{aligned}
\tilde{f}_1(y,z) &= \max_{\hat{A}\in D_A} y^T \hat{A} z \\
&= y^T A z + \max_{\|\delta A^c_j\| \leq (\gamma_A)_j} \sum_{j=1}^{m} z_j \, y^T \delta A^c_j \\
&= y^T A z + \sum_{j=1}^{m} z_j \|y\| (\gamma_A)_j \\
&= y^T A z + \gamma_A^T z \|y\|
\end{aligned}$$

and $\gamma_A^T z \geq 0$. Hence, by introducing an auxiliary variable $y_0 \in \Re$, Player 1's problem can be written as

$$\begin{array}{ll} \underset{y_0,\, y}{\text{minimize}} & y^T A z + (\gamma_A^T z) y_0 \\ \text{subject to} & \|y\| \leq y_0, \;\; y \geq 0, \;\; e_n^T y = 1. \end{array}$$

This is a second-order cone programming problem, and its KKT conditions are given by

$$\mathcal{K}^{n+1} \ni \begin{pmatrix} y_0 \\ y \end{pmatrix} \perp \begin{pmatrix} \gamma_A^T z \\ Az + e_n s - \lambda \end{pmatrix} \in \mathcal{K}^{n+1}$$
$$\Re_+^n \ni \lambda \perp y \in \Re_+^n, \;\; e_n^T y = 1$$

with Lagrange multipliers $\lambda \in \Re^n$ and $s \in \Re$.

In a similar way, the KKT conditions for Player 2's minimization problem are given by

$$\mathcal{K}^{m+1} \ni \begin{pmatrix} z_0 \\ z \end{pmatrix} \perp \begin{pmatrix} \gamma_B^T y \\ B^T y + e_m t - \mu \end{pmatrix} \in \mathcal{K}^{m+1}$$
$$\Re_+^m \ni \mu \perp z \in \Re_+^m, \;\; e_m^T z = 1,$$

where $\mu \in \Re^m$ and $t \in \Re$ are Lagrange multipliers. Combining the above two KKT conditions, we obtain SOCCP (8) with $\ell = 2m + 2n + 2$, $\tau = 2$, $\mathcal{K} := \mathcal{K}^{n+1} \times \mathcal{K}^{m+1} \times \Re_+^n \times \Re_+^m$,

$$\zeta = \begin{pmatrix} y_0 \\ y \\ z_0 \\ z \\ \lambda \\ \mu \\ s \\ t \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 0 & 0 & \gamma_A^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & -I_n & 0 & e_n & 0 \\ 0 & \gamma_B^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B^T & 0 & 0 & 0 & -I_m & 0 & e_m \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & e_n^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_m^T & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3.4 General uncertainty in the cost matrices

In this subsection, we consider the general case of uncertainty in each cost matrix. That is, we make the following assumption.

Assumption 4

(a) *$Y(y) = \{y\}$ and $Z(z) = \{z\}$.*

(b) *$D_A := \{A + \delta A \in \Re^{n \times m} \mid \|\delta A\|_F \leq \rho_A\}$ and $D_B := \{B + \delta B \in \Re^{n \times m} \mid \|\delta B\|_F \leq \rho_B\}$ with given constant matrices $A \in \Re^{n \times m}$ and $B \in \Re^{n \times m}$ and positive scalars ρ_A and ρ_B.*

Under this assumption, we also have the following theorem.

Theorem 5 *If Assumption 4 holds, then the bimatrix game has a robust Nash equilibrium.*

Next, we consider the SOCCP formulation of the game. First note that

$$\tilde{f}(y,z) = \max\{y^T \hat{A} z \mid \hat{A} \in D_A\} = y^T A z + \max_{\|\delta A\|_F \le \rho_A} y^T (\delta A) z.$$

Moreover, we have

$$\max_{\|\delta A\|_F \le \rho_A} y^T (\delta A) z \;=\; \max_{\|\delta A\|_F \le \rho_A} (z \otimes y)^T \mathrm{vec}\,(\delta A) \;=\; \|z \otimes y\| \rho_A \;=\; \rho_A \|y\| \|z\|,$$

where vec$(\cdot)$ denotes the vec operator that creates an nm-dimensional vector $((p_1^c)^T, \ldots, (p_m^c)^T)^T$ from a matrix $P \in \Re^{n \times m}$ with column vectors $p_1^c, \ldots, p_m^c$, and $\otimes$ denotes Kronecker product (see Sections 4.2 and 4.3 in [19]). Hence, by introducing an auxiliary variable $y_0 \in \Re$, Player 1's minimization problem reduces to the following problem:

$$\begin{array}{ll} \underset{y_0,\, y}{\text{minimize}} & y^T A z + \rho_A \|z\| y_0 \\ \text{subject to} & \|y\| \le y_0, \quad e_n^T y = 1, \quad y \ge 0. \end{array}$$

Again, this is a second-order cone programming problem, and its KKT conditions are given by

$$\begin{array}{c} \mathcal{K}^{n+1} \ni \begin{pmatrix} y_0 \\ y \end{pmatrix} \perp \begin{pmatrix} \rho_A \|z\| \\ Az + e_n s - \lambda \end{pmatrix} \in \mathcal{K}^{n+1}, \\ \Re_+^n \ni \lambda \perp y \in \Re_+^n \quad e_n^T y = 1, \end{array}$$

where $\lambda \in \Re^n$ and $s \in \Re$ are Lagrange multipliers. Moreover, we can show that this SOCCP is rewritten as the following SOCCP:

$$\begin{array}{l} \mathcal{K}^{n+1} \ni \begin{pmatrix} y_0 \\ y \end{pmatrix} \perp \begin{pmatrix} \rho_A z_1 \\ Az + e_n s - \lambda \end{pmatrix} \in \mathcal{K}^{n+1}, \quad e_n^T y = 1, \\ \mathcal{K}^{m+1} \ni \begin{pmatrix} z_1 \\ z \end{pmatrix} \perp \begin{pmatrix} y_0 \\ u \end{pmatrix} \in \mathcal{K}^{m+1}, \quad \Re_+^n \ni \lambda \perp y \in \Re_+^n \end{array} \tag{10}$$

with an auxiliary variable $u \in \Re^m$. To see this, it suffices to notice that the complementarity condition

$$\mathcal{K}^{m+1} \ni \begin{pmatrix} z_1 \\ z \end{pmatrix} \perp \begin{pmatrix} y_0 \\ u \end{pmatrix} \in \mathcal{K}^{m+1} \tag{11}$$

in (10) implies $\|z\| = z_1$. This fact can be verified as follows: On one hand, $\binom{z_1}{z} \in \mathcal{K}^{m+1}$ implies $\|z\| \le z_1$. On the other hand, it holds that $0 = z_1 y_0 + z^T u \ge z_1 y_0 - \|z\| \|u\| \ge z_1 y_0 - \|z\| y_0$, where the equality follows from the perpendicularity in (11), the first inequality follows from the Cauchy-Schwarz inequality, and the last inequality follows from the condition $\binom{y_0}{u} \in \mathcal{K}^{m+1}$ in (11). Moreover, $e_n^T y = 1$ and $\binom{y_0}{y} \in \mathcal{K}^{n+1}$ imply $y_0 > 0$. Hence, we have $\|z\| \ge z_1$.

In a similar way, the KKT conditions for Player 2's problem are given by

$$\begin{array}{l} \mathcal{K}^{m+1} \ni \begin{pmatrix} z_0 \\ z \end{pmatrix} \perp \begin{pmatrix} \rho_B y_1 \\ B^T y + e_m t - \mu \end{pmatrix} \in \mathcal{K}^{m+1}, \quad e_m^T z = 1, \\ \mathcal{K}^{n+1} \ni \begin{pmatrix} y_1 \\ y \end{pmatrix} \perp \begin{pmatrix} z_0 \\ v \end{pmatrix} \in \mathcal{K}^{n+1}, \quad \Re_+^m \ni \mu \perp z \in \Re_+^m, \end{array}$$

where $t \in \Re$, $\mu \in \Re^m$ and $v \in \Re^n$ are auxiliary variables. Combining the above two KKT conditions, we obtain SOCCP (8) with $\ell = 3m + 3n + 4$, $\tau = 2$, $\mathcal{K} = \mathcal{K}^{n+1} \times \mathcal{K}^{n+1} \times \mathcal{K}^{m+1} \times \mathcal{K}^{m+1} \times \Re^n_+ \times \Re^m_+$,

$$\zeta = \begin{pmatrix} y_0 \\ y \\ y_1 \\ v \\ z_0 \\ z \\ z_1 \\ u \\ \lambda \\ \mu \\ s \\ t \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \rho_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & -I_n & 0 & e_n & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_n & 0 & e_m \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & e_n^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_m^T & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As shown thus far, the robust Nash equilibrium problems under Assumptions 1, 3 and 4 are transformed into SOCCPs, while the robust Nash equilibrium problem under Assumption 2 reduces to an LCP. This is a natural consequence of the fact that the Euclidean norm is used in Assumptions 1, 3 and 4 to describe the uncertainty and the absolute value is used in Assumption 2.

4 Numerical examples of robust Nash equilibria

In the previous section, we have shown that some robust Nash equilibrium problems for bimatrix games reduce to SOCCPs. In this section, we present some numerical examples for the robust Nash equilibria. Several methods have been proposed for solving SOCCPs. Among them, one of the most popular approaches is to reformulate the SOCCP as an equivalent nondifferentiable minimization problem and solve it by Newton-type method combined with a smoothing technique [6, 9, 13, 18]. In our numerical experiments, we use an algorithm based on the methods proposed in [18].

4.1 Uncertainty in opponent's strategy

We first study the case where only the opponents' strategies involve uncertainty, that is, Assumption 1 holds.

We consider the bimatrix game with cost matrices:

$$A_1 = \begin{pmatrix} -1 & -9 & 11 \\ 10 & -1 & 4 \\ 3 & 10 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -5 & -4 & -8 \\ -1 & 0 & 5 \\ 3 & 1 & 4 \end{pmatrix}. \tag{12}$$

The Nash equilibrium of the game is given by $\overline{y} = (0.4815,\ 0.1852,\ 0.3333)$ and $\overline{z} = (0.1699,\ 0.2628,\ 0.5673)$. Robust Nash equilibria $(\overline{y}^r, \overline{z}^r)$ for various values of (ρ_y, ρ_z) are shown in Table 1, where $f_i(\overline{y}^r, \overline{z}^r)$ denotes the cost value of each player $i = 1, 2$ at a robust Nash equilibrium. From the table, we see that robust Nash equilibria $(\overline{y}^r, \overline{z}^r)$ approach the Nash equilibrium $(\overline{y}, \overline{z})$ as both ρ_y and ρ_z tend to 0. Note that Players 1 and 2 estimate the opponents' strategies more precisely as ρ_z and ρ_y become smaller. However, the costs of both players for $(\rho_y, \rho_z) = (0.5, 0.5)$ are smaller than those for $(\rho_y, \rho_z) = (0.01, 0.01)$. This implies that an equilibrium may not necessarily be favorable for either of the players even if the estimation is more precise.

Next, we consider the bimatrix game with cost matrices:

$$A_2 = \begin{pmatrix} 5 & 7 & 8 \\ 2 & 3 & 0 \\ -1 & -3 & -2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 8 & 2 & -7 \\ 5 & 3 & -3 \\ 9 & 1 & -4 \end{pmatrix}. \tag{13}$$

In this case, the Nash equilibrium comprises $\overline{y} = (0, 0, 1)$ and $\overline{z} = (0, 0, 1)$, and for any pair $(\rho_y, \rho_z) \in \{0.5, 0.1, 0.01\} \times \{0.5, 0.1, 0.01\}$, robust Nash equilibrium $(\overline{y}^r, \overline{z}^r)$ remains the same, that is, $\overline{y}^r = \overline{y}$ and $\overline{z}^r = \overline{z}$. From this result, we may expect that, if the Nash equilibrium is a pure strategy, then the robust Nash equilibrium remains unchanged even if there is uncertainty to some extent.

4.2 Uncertainty in cost matrices

We next study the general case where the players' cost matrices involve uncertainty, that is, Assumption 4 holds.

First we consider the bimatrix game with the cost matrices A_1 and B_1 defined by (12). Robust Nash equilibria for various values of (ρ_A, ρ_B) are shown in Table 2, where $f_i(\overline{y}^r, \overline{z}^r)$ denotes the cost value of robust Nash equilibrium. As in the previous case, we see from the table that precise estimation for cost matrices does not necessarily reduce the cost at an equilibrium.

Next we consider the bimatrix game with cost matrices A_2 and B_2 defined by (13). Robust Nash equilibria for various values of (ρ_A, ρ_B) are shown in Table 3, which reveals that $\overline{y}^r = \overline{y}$ and $\overline{z}^r = \overline{z}$ hold when ρ_A and ρ_B are sufficiently small. We also see from the table that precise estimation for cost matrices does not always result in the reduction of the players' costs at an equilibrium.

5 Concluding remarks

In this paper, we have defined the concept of robust Nash equilibrium, and studied a sufficient condition for its existence. Moreover, we have shown that some robust Nash equilibrium problems can be reformulated as SOCCPs. To investigate the behavior of robust Nash equilibria, we have carried out some numerical examples.

Our study is still in the infancy, and many issues remain to be addressed. (1) One is to extend the concept of robust Nash equilibrium to the general N-person game. For the 2-person bimatrix game studied in this paper, it is sufficient to consider the uncertainty in the cost matrices and the opponent's strategy. To discuss general N-person games, more complicated structure should be dealt with. (2) Another issue is to find other sufficient conditions for the existence of robust Nash equilibria. For instance, it may be possible to consider the existence of robust Nash equilibria without assuming the boundedness of strategy sets. (3) Theoretical study on the relation between Nash equilibrium and robust Nash equilibrium is also worthwhile. For example, it is not known whether the uniqueness of Nash equilibrium is inherited to robust Nash equilibrium. (4) In this paper, we have formulated several robust Nash equilibrium problems as SOCCPs. However, we have only considered the cases where either the cost matrices or the opponent's strategy is uncertain for each player. It seems interesting to study the case where both of them are uncertain, or the structure of uncertainty is more complicated. (5) In our numerical experiments, we employed an existing algorithm for solving SOCCPs. But, there is room for improvement of solution methods. It may be useful to develop a specialized method for solving robust Nash equilibrium problems.

Table 1: Robust Nash equilibria for various values of ρ_y and ρ_z

ρ_y	ρ_z	$\overline{y}^r$	$\overline{z}^r$	$f_1(\overline{y}^r, \overline{z}^r)$	$f_2(\overline{y}^r, \overline{z}^r)$
0.01	0.01	(0.4896, 0.1814, 0.3290)	(0.1702, 0.2697, 0.5601)	3.650	−1.668
0.1	0.1	(0.5630, 0.1482, 0.2888)	(0.1758, 0.3304, 0.4938)	3.039	−2.305
0.1	0.5	(0.5621, 0.1560, 0.2819)	(0.1948, 0.6032, 0.2019)	0.345	−2.122
0.5	0.1	(0.8891, 0.0011, 0.1098)	(0.1812, 0.3272, 0.4916)	2.506	−5.152
0.5	0.5	(0.8840, 0.0432, 0.0729)	(0.2129, 0.5929, 0.1942)	−2.424	−4.232

Table 2: Robust Nash equilibria for various values of ρ_A and ρ_B (cost matrices A_1 and B_1)

ρ_A	ρ_B	$\overline{y}^r$	$\overline{z}^r$	$f_1(\overline{y}^r, \overline{z}^r)$	$f_2(\overline{y}^r, \overline{z}^r)$
0.1	0.1	(0.4841, 0.1797, 0.3362)	(0.1721, 0.2623, 0.5656)	3.700	−1.615
1	1	(0.5097, 0.1376, 0.3527)	(0.1969, 0.2552, 0.5479)	3.640	−1.835
1	10	(1.0000, 0.0000, 0.0000)	(0.2931, 0.2326, 0.4743)	2.830	−6.190
10	1	(0.5083, 0.1950, 0.2967)	(0.3497, 0.2453, 0.4050)	3.074	−1.843
10	10	(0.5934, 0.1961, 0.2105)	(0.3326, 0.3002, 0.3672)	2.396	−2.565

Table 3: Robust Nash equilibria for various values of ρ_A and ρ_B (cost matrices A_2 and B_2)

ρ_A	ρ_B	$\overline{y}^r$	$\overline{z}^r$	$f_1(\overline{y}^r, \overline{z}^r)$	$f_2(\overline{y}^r, \overline{z}^r)$
0.1	0.1	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	−2.000	−4.000
1	1	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	−2.000	−4.000
1	10	(0.0000, 0.0000, 1.0000)	(0.0000, 0.3110, 0.6890)	−2.311	−2.445
10	1	(0.0000, 0.4286, 0.5714)	(0.0000, 0.0000, 1.0000)	−1.143	−3.571
10	10	(0.0000, 0.3783, 0.6217)	(0.0000, 0.1935, 0.8065)	−1.144	−2.581

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