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Mean Buffer Contents in Discrete-Time Single-Server Queues with Heterogeneous Markovian Sources

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Abstract. We consider stationary discrete-time single-server queues fed by independent heterogeneous Markovian sources with geometrically distributed idle periods. While being active, each source generates some cells depending on the state of the underlying Markov chain. The purpose of this paper is twofold. One is the derivation of the explicit formula for the mean buffer contents when the underlying Markov chain of each source has finite state space. The other is the derivation of the explicit formula for the mean buffer contents in a queue fed by Markovian autoregressive sources, each of which is characterized completely by the marginal distribution of cells arriving in a slot and its correlation coefficients.

Keywords: Discrete-time queue, Markovian sources, mean buffer contents, Markovian autoregressive sources

1 Introduction

We consider a discrete-time work-conserving single-server queue with K independent heterogeneous sources. Time is divided into slots, and sources can emit some cells to the queue in each slot. Service times of cells are assumed to be constant and equal to one slot.

The purpose of this paper is twofold. One is to derive the explicit formula for the mean buffer contents in a queue with independent heterogeneous Markovian sources with geometrically distributed idle periods. Here we use the word "explicit" to imply that there is no need to solve a system of equations or a fixed-point equation to evaluate the formula, which arises typically in queueing analysis. The other is to derive the explicit expression for the mean buffer contents in a queue with independent Markovian autoregressive sources, each of which is characterized completely by the marginal distribution of cells arriving in a slot and its correlation coefficients.

To obtain the explicit expression for the mean buffer contents, we need some special assumptions. The simplest case is that the numbers of cells arriving from each source are independent and identically distributed (i.i.d.), see, e.g., [15]. The advantage of discrete-time queues over the continuous counterparts is that even when arrivals are correlated, we can derive the explicit formulas for performance measure of interest such as the mean buffer contents. For example, assume that each source is alternately in idle periods during which no cells are generated and in active periods during which exactly one cell is generated at each slot, and that idle and active periods form an alternating renewal process. When both idle and active periods are geometrically distributed, Viterbi derived the explicit formula for the mean buffer contents [17]. This result was generalized by Neuts [14], where active periods follow a discrete phase-type distribution. See [2] also for explicit formulas in other models.

In section 2, we consider the generalization of the result in [14] by allowing multiple cell arrivals from each source during active periods with a discrete phase-type distribution. By doing so, we can model the fluctuation in input rate of each source. Note that any discrete distribution is given by a limit of a sequence of discrete phase-type distributions and the distribution of any bounded discrete random variable is a special case of a family of discrete phase-type distributions. The analytical approach to obtain the explicit expression is very similar to [14]. However, the resulting explicit formula is very useful because it completely reveals the impact of characteristics of heterogeneous sources on the mean buffer contents, and it serves as a unified formula to derive an explicit expression for the mean buffer contents in various queues.

In section 3, we consider a queue fed by independent discrete-time autoregressive sources with heterogeneous characteristics. Discrete-time autoregressive arrival processes have been used to model video traffic [3, 6, 12], and queues with discrete-time autoregressive sources were studied in [8, 9, 10]. In general, a discrete-time autoregressive source is characterized as follows. There exists an increasing sequence of time epochs $\tau(n)$ with $\tau(0) = 0$. Let B_n $(n = 1, 2, \ldots)$ denote the number of cells generated from a discrete-time autoregressive source at the nth slot. Then $B_{\tau(m)+1} = B_{\tau(m)+2} = \cdots = B_{\tau(m+1)} = Y_m$ $(m = 0, 1, \ldots)$, where the Y_m is a sequence of i.i.d. nonnegative random variables.

Let D_m $(m = 1, 2, \ldots)$ denote $\tau(m) - \tau(m-1)$. In [8, 9, 10], the D_m is assumed to be a sequence of geometrically distributed i.i.d. random variables, and the resulting arrival process is called a discrete-time autoregressive process of order one (DAR(1)). In [8] and [9], a discrete-time queue with a single DAR(1) source was studied. In [10], the mean and variance of the buffer contents in a discrete-time queue with independent homogeneous DAR(1) sources were derived.

In this paper, we relax the geometrically distributed assumption on the D_m of DAR(1) in the following way. If $Y_m \geq 1$, we assume that D_m follows a general distribution with mean $\overline{d}$. On the other hand, if $Y_m = 0$, D_m is geometrically distributed with the same mean $\overline{d}$ as D_m given $Y_m \geq 1$. It is easy to see that the marginal distribution of the B_n is equal to the distribution of Y_m. Further, as you will see later, the correlation coefficient $\gamma(m)$ of the B_n is given by the complementary distribution of the equilibrium random variable of D_m given $Y_m \geq 1$. Conversely, given a marginal distribution of cells arriving in a slot and a nonincreasing and convex sequence of nonnegative correlation coefficients of the number of arriving cells, we will show that a generalized autoregressive source with these features is uniquely constructed.

To obtain the explicit expression for the mean buffer contents in the queue with heterogeneous autoregressive sources, we assume that for each source, the conditional distribution D_m given $Y_m \geq 1$ is a discrete phase-type distribution, which we call a Markovian autoregressive source. Then, if the marginal distribution of the B_n of each source has a finite support, the general explicit formula holds. We show that our general formula is also applicable to the unbounded B_n case, by the truncation technique and the monotone convergence theorem.

Throughout the paper, we use the following conventions. For any function $f(z)$ of z, we denote, by $f^{(1)}$ and $f^{(2)}$, the first and the second derivatives, respectively, at $z = 1-$, i.e.,

$$f^{(1)} = \lim_{z \to 1-} \frac{d}{dz} f(z), \qquad f^{(2)} = \lim_{z \to 1-} \frac{d^2}{dz^2} f(z).$$

This convention applies to vector and matrix functions, too. Further, for any nonnegative integer random variable F, $\tilde{F}$ denotes a random variable that follows the equilibrium distribution of F, i.e.,

$$\Pr(\tilde{F} = n) = \frac{\Pr(F > n)}{E[F]}, \qquad n = 0, 1, \ldots.$$

In particular, $E[\tilde{F}] = E[F(F-1)]/(2E[F])$.

2 Queue with Markovian Sources

2.1 Model description

We consider a discrete-time single-server queue with K independent Markovian sources. Formally, this queue is described as follows. Let $B_{k,n}$ ($k = 1, \ldots, K$, $n = 0, 1, \ldots$) denote the number of cells generated from the kth source at the nth slot. We define X_n ($n = 0, 1, \ldots$) as the amount of buffer contents (i.e., the total number of cells in the system) at the nth slot. We assume that the X_n satisfies

$$X_{n+1} = \min(X_n - 1, 0) + B_{n+1}, \qquad n = 0, 1, \ldots, \tag{1}$$

where

$$B_n = \sum_{k=1}^{K} B_{k,n}, \qquad n = 1, 2, \ldots.$$

The kth ($k = 1, \ldots, K$) source is characterized as follows. There exists an irreducible underlying Markov chain with finite state space $\mathcal{M}_k = \{0, 1, \ldots, M_k\}$, which governs the kth source. To avoid triviality, we assume $M_k \geq 1$. Let $S_{k,n}$ denote the state of the underlying Markov chain for the kth source at the nth slot. We assume:

Assumption 1 $B_{k,n}$ depends only on $S_{k,n-1}$ and $S_{k,n}$.

We then define $b_{k,i,j}(z)$ as

$$b_{k,i,j}(z) = E\left[z^{B_{k,n}} \mid S_{k,n-1} = i, S_{k,n} = j\right], \qquad i, j \in \mathcal{M}_k.$$

We further assume:

Assumption 2

$$S_{k,n} = 0 \Leftrightarrow B_{k,n} = 0, \qquad S_{k,n} \in \mathcal{M}_k^+ \Leftrightarrow B_{k,n} \geq 1,$$

where $\mathcal{M}_k^+$ denotes $\{1, 2, \ldots M_k\}$.

Accordingly, we assume that the transition probability matrix $\boldsymbol{P}_k$ of the underlying Markov chain for the kth source is given by

$$\boldsymbol{P}_k = \begin{pmatrix} p_k & (1 - p_k)\,\boldsymbol{\alpha}_k \\ (\boldsymbol{I} - \boldsymbol{T}_k)\boldsymbol{e} & \boldsymbol{T}_k \end{pmatrix}, \tag{2}$$

where $\boldsymbol{\alpha}_k$ and $\boldsymbol{T}_k$ denote a $1 \times M_k$ probability vector and an $M_k \times M_k$ strictly substochastic matrix, respectively, and $\boldsymbol{e}$ denotes a column vector with an appropriate dimension, whose elements are all equal to one. In what follows, we denote $(\boldsymbol{I} - \boldsymbol{T}_k)\boldsymbol{e}$ by $\boldsymbol{T}_k^\circ$, the jth ($j \in \mathcal{M}_k^+$) element of $\boldsymbol{\alpha}_k$ by $\alpha_{k,j}$, and the (i,j)th ($i, j \in \mathcal{M}_k^+$) element of $\boldsymbol{T}_k$ by $T_{k,i,j}$.

Remark 1 Assumption 2 together with (2) implies the followings: The kth source becomes idle and active alternately, and while being active, it generates some cells at each slot. Idle and active periods constitute an alternating renewal process. The lengths of idle periods are geometrically distributed with probability mass function $(1 - p_k)\{p_k\}^{n-1}$ ($n = 1, 2, \ldots$), and the lengths of active periods follow a discrete phase-type distribution with irreducible representation ($\boldsymbol{\alpha}_k$, $\boldsymbol{T}_k$), i.e., its probability mass function is given by $\boldsymbol{\alpha}_k \boldsymbol{T}_k^{n-1}(\boldsymbol{I} - \boldsymbol{T}_k)\boldsymbol{e}$ ($n = 1, 2, \ldots$).

We define $\boldsymbol{B}_k(z)$ ($k = 1, \ldots, K$) as an $M_k \times M_k$ matrix whose (i,j)th element $B_{k,i,j}(z)$ is given by

$$B_{k,i,j}(z) = P_{k,i,j} b_{k,i,j}(z), \qquad i, j \in \mathcal{M}_k,$$

where $P_{k,i,j}$ denotes the (i,j)th element of $\boldsymbol{P}_k$. It then follows that

$$\boldsymbol{B}_k(z) = \begin{pmatrix} p_k & (1 - p_k)\boldsymbol{\alpha}_k(z) \\ \boldsymbol{T}_k^\circ & \boldsymbol{T}_k(z) \end{pmatrix}, \tag{3}$$

where $\boldsymbol{\alpha}_k(z)$ denotes a $1 \times M_k$ vector whose jth element $\alpha_{k,j}(z)$ is given by

$$\alpha_{k,j}(z) = \alpha_{k,j} b_{k,0,j}(z), \qquad j \in \mathcal{M}_k^+,$$

and $\boldsymbol{T}_k(z)$ denotes an $M_k \times M_k$ matrix whose (i,j)th element $T_{k,i,j}(z)$ is given by

$$T_{k,i,j}(z) = T_{k,i,j} b_{k,i,j}(z), \qquad i, j \in \mathcal{M}_k^+.$$

Therefore the probability law of the $B_{k,n}$ is fully characterized by p_k, $\boldsymbol{\alpha}_k(z)$ and $\boldsymbol{T}_k(z)$.

Let $\boldsymbol{\pi}_k$ denote the steady-state probability vector of the underlying Markov chain for the kth source. Because the underlying Markov chain for each source is assumed to be irreducible, $\boldsymbol{\pi}_k$ is uniquely determined by $\boldsymbol{\pi}_k \boldsymbol{P}_k = \boldsymbol{\pi}_k$ and $\boldsymbol{\pi}_k \boldsymbol{e} = 1$, i.e.,

$$\boldsymbol{\pi}_k = (\pi_{k,0}, \boldsymbol{\pi}_{k,1}),$$

where

$$\begin{aligned} \pi_{k,0} &= \frac{1}{1 + (1-p_k)\boldsymbol{\alpha}_k (\boldsymbol{I} - \boldsymbol{T}_k)^{-1} \boldsymbol{e}}, \\ \boldsymbol{\pi}_{k,1} &= \frac{(1-p_k)\boldsymbol{\alpha}_k (\boldsymbol{I} - \boldsymbol{T}_k)^{-1}}{1 + (1-p_k)\boldsymbol{\alpha}_k (\boldsymbol{I} - \boldsymbol{T}_k)^{-1} \boldsymbol{e}}. \end{aligned}$$

Let ρ_k denote the traffic intensity of the kth source. We then have

$$\begin{aligned} \rho_k &= \boldsymbol{\pi}_k \boldsymbol{B}_k^{(1)} \boldsymbol{e} \\ &= \frac{(1-p_k)\left\{\boldsymbol{\alpha}_k^{(1)} \boldsymbol{e} + \boldsymbol{\alpha}_k (\boldsymbol{I} - \boldsymbol{T}_k)^{-1} \boldsymbol{T}_k^{(1)} \boldsymbol{e}\right\}}{1 + (1-p_k)\boldsymbol{\alpha}_k (\boldsymbol{I} - \boldsymbol{T}_k)^{-1} \boldsymbol{e}}. \end{aligned}$$

We define $\delta_k(z)$ $(k = 1, 2, \ldots, K)$ as the Perron-Frobenius eigenvalue of $\boldsymbol{B}_k(z)$, and let $\boldsymbol{u}_k(z)$ and $\boldsymbol{v}_k(z)$ denote the left and right eigenvectors of $\boldsymbol{B}_k(z)$ associated with $\delta_k(z)$. We assume that the normalizing equations, $\boldsymbol{u}_k(z)\boldsymbol{e} = 1$ and $\boldsymbol{u}_k(z)\boldsymbol{v}_k(z) = 1$, hold, so that $\boldsymbol{u}_k(z)$ and $\boldsymbol{v}_k(z)$ are uniquely determined. Note here that [13]

$$\boldsymbol{u}_k(1-) = \boldsymbol{\pi}_k, \qquad \boldsymbol{v}_k(1-) = \boldsymbol{e}, \tag{4}$$

and

$$\delta_k^{(1)} = \rho_k. \tag{5}$$

In the rest of this paper, we assume

Assumption 3

$$\rho = \sum_{k=1}^{K} \rho_k < 1, \tag{6}$$

and the system is in steady state.

2.2 Mean Buffer Contents

Because of (1) and Assumption 1, the multivariate process $\{(X_n,\, S_{1,n},\, S_{2,n},\, \ldots, S_{K,n});\, n = 0, 1, \ldots\}$ forms a Markov chain. We list $\{S_{1,n}, S_{2,n}, \ldots, S_{K,n}\}$ in lexicographic order and denote it by S_n. Note that S_n takes a value in $\mathcal{M} = \{0, 1, \ldots, M-1\}$, where $M = \prod_{k=1}^{K}(1 + M_k)$. It then follows from (1) that the bivariate process $\{(X_n, S_n); n = 0, 1, \ldots\}$ forms a Markov chain of M/G/1 type. To see this, we define $\boldsymbol{B}(z)$ as

$$\boldsymbol{B}(z) = \boldsymbol{B}_1(z) \otimes \boldsymbol{B}_2(z) \otimes \cdots \otimes \boldsymbol{B}_K(z),$$

where $\otimes$ denotes the Kronecker product [4]. The transition probability matrix of the irreducible Markov chain $\{(X_n, S_n); n = 0, 1, \ldots\}$ then takes the form:

$$\begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \cdots \\ \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_0 & \boldsymbol{B}_1 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where B_m denotes the coefficient matrix of z^m in $\boldsymbol{B}(z)$. Note that the irreducible Markov chain $\{(X_n, S_n); n = 0, 1, \ldots\}$ is positive recurrent when (6) holds [13].

Let X and S denote generic random variables representing the buffer contents X_n and the state S_n of the product underlying Markov chain, respectively, in steady state. We define $\boldsymbol{x}(z)$ as a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $x_j(z)$ represents

$$x_j(z) = E\left[z^X 1_{\{S=j\}}\right],$$

where $1_{\{\chi\}}$ denotes an indicator function of event χ. It follows from the general theory of Markov chains of M/G/1 type [13], we have

$$\boldsymbol{x}(z)[\boldsymbol{I} - \boldsymbol{B}(z)] = (1-\rho)(z-1)\frac{\boldsymbol{x}(0)}{\boldsymbol{x}(0)\boldsymbol{e}}\boldsymbol{B}(z). \tag{7}$$

Note that $\boldsymbol{x}(0)/\boldsymbol{x}(0)\boldsymbol{e}$ denotes the conditional probability vector of S given $X = 0$. Owing to Assumption 2, $X_n = 0$ only if $S_{k,n} = 0$ for all $k = 1, 2, \ldots, K$. Thus we have

$$\frac{\boldsymbol{x}(0)}{\boldsymbol{x}(0)\boldsymbol{e}} = (1, 0, 0, \ldots, 0).$$

In what follows, the $1 \times M$ vector $(1, 0, 0, \ldots, 0)$ is denoted by $\boldsymbol{g}$.

Let $\delta(z)$ denote the Perron-Frobenius eigenvalue of $\boldsymbol{B}(z)$, and let $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ denote the left and right eigenvectors of $\boldsymbol{B}(z)$ associated with $\delta(z)$, where we normalize $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ to be $\boldsymbol{u}(z)\boldsymbol{e} = \boldsymbol{u}(z)\boldsymbol{v}(z) = 1$. Note here that

$$\begin{aligned}\delta(z) &= \prod_{k=1}^{K} \delta_k(z),\\ \boldsymbol{u}(z) &= \boldsymbol{u}_1(z) \otimes \boldsymbol{u}_2(z) \otimes \cdots \otimes \boldsymbol{u}_K(z),\\ \boldsymbol{v}(z) &= \boldsymbol{v}_1(z) \otimes \boldsymbol{v}_2(z) \otimes \cdots \otimes \boldsymbol{v}_K(z).\end{aligned}$$

Postmultiplying both sides of (7) by $\boldsymbol{v}(z)$, differentiating them twice with respect to z and taking the limit $z \to 1-$, we obtain

$$E[X] = \rho + \frac{\delta^{(2)}}{2(1-\rho)} + \boldsymbol{g}\boldsymbol{v}^{(1)} - \boldsymbol{u}(1-)\boldsymbol{v}^{(1)},$$

where we use (4) and (5). It is easy to see $\boldsymbol{u}(1-)\boldsymbol{v}^{(1)} = 0$ because $\boldsymbol{u}(z)\boldsymbol{e} = \boldsymbol{u}(z)\boldsymbol{v}(z) = 1$. We denote the jth ($j \in \mathcal{M}_k$) element of $\boldsymbol{v}_k(z)$ by $v_{k,j}(z)$. We then have

$$E[X] = \rho + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \delta_k^{(2)} + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k(\rho - \rho_k) + \sum_{k=1}^{K} v_{k,0}^{(1)}. \tag{8}$$

Let A_k denote a generic random variable representing the total number of cells arriving from the kth source during a typical active period. Also, let C_k denote a generic random variable representing the length of a typical active period of the kth source.

Lemma 1 $\delta_k^{(2)}$ and $v_{k,0}^{(1)}$ ($k = 1, \ldots, K$) are given by

$$\delta_k^{(2)} = 2\rho_k E[\tilde{A}_k] - 2\rho_k \boldsymbol{\pi}_{k,1}\boldsymbol{e}\frac{E[C_k A_k]}{E[C_k]} + 2\rho_k^2 \boldsymbol{\pi}_{k,1}\boldsymbol{e}\left(1 + E[\tilde{C}_k]\right), \tag{9}$$

$$v_{k,0}^{(1)} = \rho_k \boldsymbol{\pi}_{k,1}\boldsymbol{e}\left(1 + E[\tilde{C}_k]\right) - \boldsymbol{\pi}_{k,1}\boldsymbol{e}E[A_k(\tilde{C}_k)], \tag{10}$$

respectively, where $A_k(\tilde{C}_k)$ denotes a generic random variable representing the total number of cells arriving from the kth source during the forward recurrence time of a typical active period.

Because the proof of Lemma 1 is straightforward, it is given in Appendix A. Substituting (9) and (10) into (8) and rearranging terms yield the following theorem.

Theorem 1 Under Assumptions 1 to 3, the mean buffer contents $E[X]$ is given by

$$E[X] = \rho + \frac{1}{1-\rho} \sum_{k=1}^{K} \rho_k E[\tilde{A}_k] + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k(\rho - \rho_k)$$

$$
+ \frac{1}{1-\rho} \sum_{k=1}^{K} \rho_k (1-\rho+\rho_k) \boldsymbol{\pi}_{k,1} \boldsymbol{e} \left(1 + E[\tilde{C}_k]\right)
$$
$$
- \frac{1}{1-\rho} \sum_{k=1}^{K} \rho_k \boldsymbol{\pi}_{k,1} \boldsymbol{e} \frac{E[C_k A_k]}{E[C_k]} - \sum_{k=1}^{K} \boldsymbol{\pi}_{k,1} \boldsymbol{e} E[A_k(\tilde{C}_k)]. \qquad (11)
$$

Theorem 1 can be regarded as a unified formula to derive explicit expressions for the mean buffer contents in specific queueing models with finite sources, which is demonstrate below.

Example 1 *Sources with i.i.d. arrivals*: Suppose the $B_{k,n}$ $(k = 1, \ldots, K)$ is an i.i.d. sequence with the probability mass function $b_k(m)$, i.e.,

$$
\boldsymbol{B}_k(z) = \begin{pmatrix} b_k(0) & E[z^{B_k}] - b_k(0) \\ b_k(0) & E[z^{B_k}] - b_k(0) \end{pmatrix},
$$

where B_k denotes a generic random variable for $B_{k,n}$. We denote the mean buffer contents in this case by $E[X_{\Sigma \mathrm{GI/G/1}}]$. Note that $\rho_k = E[B_k]$ and $\boldsymbol{\pi}_{k,1}\boldsymbol{e} = 1 - b_k(0)$. Further, it is easy to see

$$
E[\tilde{A}_k] = E[\tilde{B}_k] + \frac{\rho_k}{b_k(0)}, \qquad E[A_k(\tilde{C}_k)] = \frac{\rho_k}{b_k(0)},
$$
$$
E[\tilde{C}_k] = \frac{1 - b_k(0)}{b_k(0)}, \qquad \frac{E[C_k A_k]}{E[C_k]} = \frac{\rho_k(2 - b_k(0))}{b_k(0)(1 - b_k(0))},
$$

and therefore we have

$$
E[X_{\Sigma \mathrm{GI/G/1}}] = \rho + \frac{1}{1-\rho} \sum_{k=1}^{K} \rho_k E[\tilde{B}_k] + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k (\rho - \rho_k), \qquad (12)
$$

which coincides with the well-known result [15].

Example 2 *Sources with i.i.d. arrivals in active periods*: Suppose the number of cells arriving from the kth source during active periods is independent of the state of the underlying Markov chain, i.e., $\boldsymbol{\alpha}_k(z) = E[z^{B_k^+}]\boldsymbol{\alpha}_k$ and $\boldsymbol{T}_k(z) = E[z^{B_k^+}]\boldsymbol{T}_k$ for all $k = 1, 2, \ldots, K$, where B_k^+ denotes the conditional random variable of B_k given $B_k \geq 1$. The special case of homogeneous sources with geometric active periods was studied in [1]. Note that

$$
\boldsymbol{\pi}_{k,1}\boldsymbol{e} = \frac{\rho_k}{E[B_k^+]}, \qquad E[\tilde{A}_k] = E[\tilde{B}_k^+] + E[B_k^+]E[\tilde{C}_k],
$$
$$
E[A_k(\tilde{C}_k)] = E[B_k^+]E[\tilde{C}_k], \qquad \frac{E[C_k A_k]}{E[C_k]} = 2E[B_k^+]E[\tilde{C}_k] + E[B_k^+],
$$

and therefore we have

$$
E[X] = \rho + \frac{1}{1-\rho} \sum_{k=1}^{K} \rho_k \left(E[B_k^+] - 1 + \rho - \rho_k\right) \left(1 - \frac{\rho_k}{E[B_k^+]}\right) E[\tilde{C}_k]
$$
$$
+ \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k \left[2E[\tilde{B}_k^+] + 1 - 2\rho_k - (1 - \rho + \rho_k)\left(1 - \frac{2\rho_k}{E[B_k^+]}\right)\right].
$$

Example 3 *Sources with constant arrivals in active periods*: We specialize the model in Example 2 to be $B_k^+ = m_k$ for all $k = 1, 2, \ldots, K$, where m_k is a positive integer. Because $E[\tilde{B}_k^+] = (m_k - 1)/2$ and $E[B_k^+] = m_k$, we have

$$E[X] = \rho + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k (m_k - 1 + \rho - \rho_k) \cdot \left[2\left(1 - \frac{\rho_k}{m_k}\right) E[\tilde{C}_k] + 1 - \frac{2\rho_k}{m_k} \right]. \qquad (13)$$

The special case of $K = 1$ with geometric active periods was studied in [2]. Further when $m_k = 1$ for all $k = 1, 2, \ldots, K$, the above formula is reduced to be

$$E[X] = \rho + \frac{1}{2(1-\rho)} \sum_{k=1}^{K} \rho_k (\rho - \rho_k) \left[2(1-\rho_k) E[\tilde{C}_k] + 1 - 2\rho_k \right],$$

which is equivalent to the result in [14], because $\gamma(k)$ in [14] is given by $1 - [(1-\rho_k)(1 + E[\tilde{C}_k])]^{-1}$ in our notations.

3 Queue with Markovian Autoregressive Sources

3.1 Generalized discrete-time autoregressive process

For the kth ($k = 1, \ldots, K$) sources, we first consider the sequence of random variables $B_{k,n}$ ($n = 0, 1, \ldots$) such that

$$B_{k,n+1} = I_{k,n} B_{k,n} + (1 - I_{k,n}) Z_{k,n}, \qquad n = 0, 1, \ldots,$$

where the $Z_{k,n}$ is a sequence of nonnegative i.i.d. random variables and the $I_{k,n}$ is a sequence of i.i.d. Bernoulli random variables, i.e., $\Pr(I_{k,n} = 1) = 1 - \Pr(I_{k,n} = 0) = q_k$ for all $n = 0, 1, \ldots$. The B_n is called a discrete-time autoregressive process of order one [3, 6, 8, 9, 10, 12].

Yet another view of the $B_{k,n}$ is as follows. There exists an increasing sequence of time epochs $\tau_k(m)$ ($m = 0, 1, \ldots$) with $\tau_k(0) = 0$. Then $B_{k,\tau(m)+1} = B_{k,\tau(m)+2} = \cdots = B_{k,\tau(m+1)} = Y_{k,m}$ ($m = 0, 1, \ldots$), where the $Y_{k,m}$ is a sequence of nonnegative random variables with the same marginal distribution as the $Z_{k,n}$, and the $D_{k,m} = \tau_k(m) - \tau_k(m-1)$ is a sequence of i.i.d. geometric random variables, i.e., $\Pr(D_{k,m} = l) = (1 - q_k) q_k^{l-1}$ for all $m = 1, 2, \ldots$.

We now generalize the geometrically distributed assumption on the $D_{k,m}$ in the following way. If $Y_{k,m} \geq 1$, $D_{k,m}$ is distributed according to a general distribution with mean $\overline{d}_k$. On the other hand, if $Y_{k,m} = 0$, $D_{k,m}$ is geometrically distributed with the same mean $\overline{d}_k$ as $D_{k,m}$ given $Y_m \geq 1$. We call this arrival process a generalized discrete-time autoregressive process.

The generalized discrete-time autoregressive process has the following features [16]. Because $E[D_{k,m}] = \overline{d}_k$ for all $m = 1, 2, \ldots$, the marginal distribution of the $B_{k,n}$ is identical to that of the $Y_{k,m}$. Further, for n such that $\tau_k(m) < n \le \tau_k(m+1)$,

$$\begin{aligned}
E[B_{k,n}B_{k,n+l}] &= \Pr(B_{k,n} = 0)E[B_{k,n}B_{k,n+l} \mid B_{k,n} = 0] \\
&\qquad + \Pr(B_{k,n} \ge 1)E[B_{k,n}B_{k,n+l} \mid B_{k,n} \ge 1] \\
&= \Pr(Y_{k,m} \ge 1)\Big(\Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1) \\
&\qquad\qquad \cdot E[B_{k,n}B_{k,n+l} \mid Y_{k,m} \ge 1, \tilde{D}_{k,m} \ge l] \\
&\qquad + \{1 - \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)\} \\
&\qquad\qquad \cdot E[B_{k,n}B_{k,n+l} \mid Y_{k,m} \ge 1, \tilde{D}_{k,m} < l]\Big) \\
&= \Pr(Y_{k,m} \ge 1)\Big(\Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)E[Y_k^2 \mid Y_k \ge 1] \\
&\qquad + \{1 - \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)\}E[Y_k \mid Y_k \ge 1]E[Y_k]\Big) \\
&= \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)E[Y_k^2] \\
&\qquad + \{1 - \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)\}E[Y_k]^2 \\
&= \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1)\mathrm{Var}[Y_k] + E[Y_k]^2,
\end{aligned}$$

where Y_k denotes a generic random variable for $Y_{k,m}$. Thus the correlation coefficient $\gamma_k(l)$ of the $B_{k,n}$ at lag l is given by

$$\gamma_k(l) = \frac{\mathrm{Cov}[B_{k,n}, B_{k,n+l}]}{\mathrm{Var}[B_k]} = \Pr(\tilde{D}_{k,m} \ge l \mid Y_{k,m} \ge 1) \ge 0, \tag{14}$$

for all $l = 1, 2, \ldots$, because the $B_{k,n}$ and the $Y_{k,m}$ have the same marginal distribution. (14) implies that correlation coefficients of the $B_{k,n}$ are nonnegative and determined completely by the conditional distribution of $D_{k,m}$ given $Y_{k,m} \ge 1$. Further they are independent of the conditional distribution of $D_{k,m}$ given $Y_{k,m} = 0$ and the marginal distribution of the $B_{k,n}$.

Conversely, it follows from (14) that the conditional probability mass function of $D_{k,m}$ given $Y_{k,m} \ge 1$ is uniquely determined by $\gamma_k(l)$, i.e., with $\gamma_k(0) = 1$,

$$\Pr(D_{k,m} = n \mid Y_{k,m} \ge 1) = \frac{\gamma_k(n-1) - 2\gamma_k(n) + \gamma_k(n+1)}{1 - \gamma_k(1)},$$

for $n = 1, 2, \ldots$, if the $\gamma_k(l)$ is a nonnegative, nonincreasing and convex sequence. Note here that

$$E[D_{k,m} \mid Y_{k,m} \ge 1] = \frac{1}{1 - \gamma_k(1)}, \tag{15}$$

$$E[\tilde{D}_k] = \sum_{l=1}^{\infty} \gamma_k(l). \tag{16}$$

Thus the geometric distribution of $D_{k,m}$ given $Y_{k,m} = 0$ is also determined uniquely to be

$$\Pr(D_{k,m} = n \mid Y_{k,m} = 0) = \{1 - \gamma_k(1)\}\gamma_k^{n-1}(1), \qquad n = 1, 2, \ldots,$$

because $E[D_{k,m} \mid Y_{k,m} = 0] = E[D_{k,m} \mid Y_{k,m} \geq 1]$. Therefore, once the marginal distribution of the number of arriving cells and its correlation coefficients are given, the generalized discrete-time autoregressive process with those features is uniquely determined, if correlation coefficients are nonnegative and constitute a nonincreasing and convex sequence.

Remark 2 The arrival process with a decreasing and convex sequence of nonnegative correlation coefficients can be modeled by the $M/G/\infty$ input process, too [11]. However, the marginal distribution in the $M/G/\infty$ input process is restricted to a family of Poisson distributions. Thus the $M/G/\infty$ input process is less flexible than the generalized autoregressive process. In particular, the $M/G/\infty$ input process cannot represent the bounded marginal distribution, which naturally arises in telecommunication applications because any transmission channel has finite capacity. In this sense, the generalized discrete-time autoregressive source may be more suitable for teletraffic modeling.

3.2 Mean buffer contents

To obtain the explicit formula for the mean buffer contents in a queue with discrete-time autoregressive sources, we assume that $D_{k,m}$ given $Y_{k,m} \geq 1$ follows a discrete phase-type distribution with irreducible representation $(\boldsymbol{\eta}_k, \boldsymbol{R}_k)$, which we call a Markovian autoregressive source. Note that for $n = 1, 2, \ldots$,

$$\begin{aligned}\Pr(D_{k,m} = n \mid Y_{k,m} \geq 1) &= \boldsymbol{\eta}_k \boldsymbol{R}_k^{n-1} \boldsymbol{R}_k^{\circ}, \\ \Pr(D_{k,m} = n \mid Y_{k,m} = 1) &= (1 - q_k) q_k^{n-1},\end{aligned}$$

where $\boldsymbol{R}_k^{\circ} = (\boldsymbol{I} - \boldsymbol{R}_k)\boldsymbol{e}$ and q_k is given by

$$q_k = 1 - \frac{1}{\boldsymbol{\eta}_k (\boldsymbol{I} - \boldsymbol{R}_k)^{-1} \boldsymbol{e}}.$$

The kth source is then characterized fully by $(\boldsymbol{\eta}_k, \boldsymbol{R}_k)$ and the marginal probability mass function $b_k(n) = \Pr(B_k = n)$ $(n = 0, 1, \ldots)$ of the number B_k of cells arriving in a slot. Note that $\rho_k = E[B_k]$. Let D_k denote a generic random variable for $D_{k,m}$ given $Y_{k,m} \geq 1$.

Theorem 2 If Assumption 3 holds, the mean buffer contents is given by

$$\begin{aligned} E[X] = E[X_{\Sigma\mathrm{GI/G/1}}] &+ \frac{1}{1-\rho}\sum_{k=1}^{K}\rho_k(2E[\tilde{B}_k]+\rho-\rho_k)E[\tilde{D}_k] \\ &+ \frac{1}{1-\rho}\sum_{k=1}^{K}\rho_k\left[(1-\rho+\rho_k)\{1-b_k(0)\}-\rho_k\right] \\ &\cdot\left(E[\tilde{D}_k]-E[D_k]+1\right), \end{aligned} \tag{17}$$

where $E[X_{\Sigma\mathrm{GI/G/1}}]$ is given in (12).

Before showing the proof of Theorem 2, we provide some remarks.

Remark 3 Suppose that all sources are discrete autoregressive sources of order one, i.e., D_k is geometrically distributed with mean $(1-q_k)^{-1}$. Then $E[\tilde{D}_k] = E[D_k]-1$, so that

$$\begin{aligned} E[X] &= E[X_{\Sigma\mathrm{GI/G/1}}] + \frac{1}{1-\rho}\sum_{k=1}^{K}\rho_k(2E[\tilde{B}_k]+\rho-\rho_k)\frac{q_k}{1-q_k} \\ &= \rho + \frac{1}{2(1-\rho)}\sum_{k=1}^{K}\rho_k\left(2E[\tilde{B}_k]+\rho-\rho_k\right)\frac{1+q_k}{1-q_k}. \end{aligned} \tag{18}$$

Further, if all sources are homogeneous, (18) is reduced to be

$$E[X] = \rho + \frac{\rho}{2(1-\rho)}\left(2E[\tilde{B}]+\frac{K-1}{K}\rho\right)\frac{1+q}{1-q},$$

which is equivalent to the result in [10].

Remark 4 The on-off source model in Example 3 is a special case of a Markovian autoregressive source. Namely, $b_k(n) = 0$ for all n other than $n = 0, m_k$. Thus

$$\begin{aligned} E[\tilde{B}_k] = \frac{m_k-1}{2}, \quad b_k(0) = 1-\frac{\rho_k}{m_k}, \quad E[C_k] = \frac{m_k}{m_k-\rho_k}, \\ E[D_k] = (1-\rho_k)E[C_k], \quad E[\tilde{D}_k] = E[\tilde{C}_k]-\rho_k E[C_k], \end{aligned}$$

and substituting these into (17) and rearranging terms, we can readily obtain (13).

Remark 5 It follows from (15) and (16) that (17) is rewritten to be

$$\begin{aligned} E[X] = E[X_{\Sigma\mathrm{GI/G/1}}] &+ \frac{1}{1-\rho}\sum_{k=1}^{K}\rho_k(2E[\tilde{B}_k]+\rho-\rho_k)\Gamma_k \\ &+ \frac{1}{1-\rho}\sum_{k=1}^{K}\rho_k\left[(1-\rho+\rho_k)\{1-b_k(0)\}-\rho_k\right]\left(\Gamma_k-\frac{\gamma_k(1)}{1-\gamma_k(1)}\right), \end{aligned}$$

where

$$\Gamma_k = \sum_{l=1}^{\infty} \gamma_k(l).$$

Thus the mean buffer contents is determined by the first two moments of the marginal distribution of cells arriving in a slot, correlation coefficient at lag one, and the infinite sum of correlation coefficients.

Proof of Theorem 2. When B_k is bounded by n_k^*, i.e., $b_k(n) = 0$ for all $n > n_k^*$, the matrix probability generating function $\boldsymbol{B}_k(z)$, corresponding to (3), for the number of cells arriving from the kth source in a slot is given by

$$\begin{pmatrix} q_k + \overline{q}_k b_k(0) & z\overline{q}_k b_k(1)\boldsymbol{\eta}_k & z^2\overline{q}_k b_k(2)\boldsymbol{\eta}_k & \cdots & z^{n_k^*}\overline{q}_k b_k(n_k^*)\boldsymbol{\eta}_k \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{S}_k(1) & z^2\boldsymbol{\beta}_k(2) & \cdots & z^{n_k^*}\boldsymbol{\beta}_k(n_k^*) \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{\beta}_k(1) & z^2\boldsymbol{S}_k(2) & \cdots & z^{n_k^*}\boldsymbol{\beta}_k(n_k^*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{\beta}_k(1) & z^2\boldsymbol{\beta}_k(2) & \cdots & z^{n_k^*}\boldsymbol{S}_k(n_k^*) \end{pmatrix},$$

where $\overline{q}_k = 1 - q_k$ and for $l = 1, 2, \ldots,$

$$\boldsymbol{\beta}_k(l) = b_k(l)\boldsymbol{R}_k^\circ\boldsymbol{\eta}_k, \qquad \boldsymbol{S}_k(l) = \boldsymbol{R}_k + b_k(l)\boldsymbol{R}_k^\circ\boldsymbol{\eta}_k.$$

Thus Theorem 1 is applicable directly to this bounded case.

To utilize Theorem 1, note first that

$$\begin{aligned} &E\left[z^{A_k}\omega^{C_k}\right] \\ &\quad = \sum_{n=1}^{\infty} \frac{b_k(n)}{1-b_k(0)} E\left[z^{nD_k}\omega^{D_k}\right] \sum_{i=0}^{\infty} \left(\sum_{n=1}^{\infty} b_k(n) E\left[z^{nD_k}\omega^{D_k}\right]\right)^i b_k(0) \\ &\quad = \frac{b_k(0)}{1-b_k(0)} \cdot \frac{E\left[z^{B_kD_k}\omega^{D_k}\right] - b_k(0)E\left[\omega^{D_k}\right]}{1 - E\left[z^{B_kD_k}\omega^{D_k}\right] + b_k(0)E\left[\omega^{D_k}\right]}. \end{aligned}$$

Therefore taking appropriate partial derivatives of both sides of the above equation, taking the limits $z \to 1-$ and $\omega \to 1-$, and noting that B_k and D_k are independent, we obtain

$$\begin{aligned} E[C_k] &= \frac{E[D_k]}{b_k(0)}, \\ E[\tilde{C}] &= E[\tilde{D}] + \frac{1-b_k(0)}{b_k(0)} E[D_k], \\ E[A_k] &= \frac{\rho_k E[D_k]}{b_k(0)(1-b_k(0))}, \\ E[\tilde{A}_k] &= 2E[\tilde{B}_k]E[\tilde{D}_k] + E[\tilde{B}_k] + E[\tilde{D}_k] + \frac{\rho_k}{b_k(0)} E[D_k], \\ \frac{E[C_kA_k]}{E[C_k]} &= \frac{2\rho_k}{b_k(0)} E[D_k] + \frac{2\rho_k}{1-b_k(0)} E[\tilde{D}_k] + \frac{\rho_k}{1-b_k(0)}. \end{aligned}$$

Further, it is easy to see that

$$\begin{aligned}E[A_k(\tilde{C}_k)] &= \frac{E[B_k]}{1-b_0}E[\tilde{D}_k] + \frac{E[B_k]}{1-b_0}\cdot\frac{1-b_k(0)}{b_k(0)}E[D_k] \\ &= \frac{\rho_k}{1-b_k(0)}E[\tilde{C}_k].\end{aligned}$$

Therefore substituting those into (11) and rearranging terms, we obtain (17).

Next we assume that at least one of B_k $(k = 1,\ldots,K)$ is unbounded. In this case, we consider a sequence of queues with K independent Markovian autoregressive sources. Recall that the Markovian autoregressive process associated with the kth source is generated by sequences of the $D_{k,m}$ and $Y_{k,m}$. Thus, given a sample path of the $D_{k,m}$ and $Y_{k,m}$ for the kth source, we consider the associated kth sources in the lth queue, whose dynamics is determined by the $D_{k,m}$ and $Y_{k,m}(l)$, where $Y_{k,m}(l) = \min(Y_{k,m}, l)$ $(m = 1,2,\ldots)$. Note here that for all $l = 1,2,\ldots$

$$B_{k,n}(l+1) \geq B_{k,n}(l), \qquad n = 1,2,\ldots, \tag{19}$$

on a given sample path, where $B_{k,n}(l)$ denotes the number of cells arriving from the kth source in the lth queue at the nth slot.

It is clear that the kth source in the lth $(l = 1,2,\ldots)$ queue is characterized by the following matrix generating function $\boldsymbol{B}_k(l,z)$ for the number of cells arriving in a slot:

$$\begin{pmatrix} q_k + \overline{q}_k b_k(0) & z\overline{q}_k b_k(1)\boldsymbol{\eta}_k & z^2\overline{q}_k b_k(2)\boldsymbol{\eta}_k & \cdots & z^l\overline{q}_k \overline{b}_k(l)\boldsymbol{\eta}_k \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{S}_k(1) & z^2\boldsymbol{\beta}_k(2) & \cdots & z^l\overline{\boldsymbol{\beta}}_k(l) \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{\beta}_k(1) & z^2\boldsymbol{S}_k(2) & \cdots & z^l\overline{\boldsymbol{\beta}}_k(l) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_k(0)\boldsymbol{R}_k^\circ & z\boldsymbol{\beta}_k(1) & z^2\boldsymbol{\beta}_k(2) & \cdots & z^l\overline{\boldsymbol{S}}_k(l) \end{pmatrix},$$

where $\overline{b}_k(l) = \Pr(Y_k \geq l)$, $\overline{\boldsymbol{\beta}}_k(l) = \overline{b}_k(l)\boldsymbol{R}_k^\circ\boldsymbol{\eta}_k$ and $\overline{\boldsymbol{S}}_k(l) = \boldsymbol{R}_k + \overline{\boldsymbol{\beta}}_k(l)$. Theorem 2 is then applicable to the lth queue.

Let $X_n(l)$ denote the number of cells in the lth queue at the nth slot. We assume here that $X_0 = X_0(l)$ for all $l = 1,2,\ldots$. It then follows from (1) and (19) that

$$X_n(l+1) \geq X_n(l),$$

for all $n = 0,1,\ldots$ and for all $l = 1,2,\ldots$. Let $X(l)$ denote the stationary number of cells in the lth queue. Because $X(l) = \lim_{n\to\infty} X_n(l)$, $X = \lim_{n\to\infty} X_n$ and $\lim_{l\to infty} X(l) = X$, the monotone convergence theorem leads to

$$\lim_{l\to\infty} E[X(l)] = E\left[\lim_{l\to\infty} X(l)\right] = E[X].$$

Thus Theorem 2 holds for unbounded B_k, too, which completes the proof.

4 Concluding Remarks

We derived an explicit formula for the mean buffer contents in a stationary discrete-time single-server queues fed by independent Markovian sources with heterogeneous characteristics. The essential assumption was that idle periods of each source are geometrically distributed, and from which we can obtain the boundary vector $\boldsymbol{x}(0)/\boldsymbol{x}(0)\boldsymbol{e}$ explicitly. There exists another class of discrete-time queues with finite sources in which an explicit formula can be obtained, i.e., queues with independent periodic sources (e.g., see [5, 7] and references therein).

Further, to obtain the explicit formula, arrivals from each source are assumed to be governed by a finite-state Markov chain. This assumption can be relaxed in some cases, as we showed in section 3 for queues with Markovian autoregressive sources. Another example in which the general formula holds is a discrete-time queue with independent greedy sources, which was introduced in [2]. A greedy source is characterized by for all $i = 0, 1, \dots,$

$$E\left[z^{B_{n+1}} \mid B_n = i\right] = 1 - r + rz^i E[z^G],$$

where $0 < r < 1$ and G denotes a nonnegative random variable. The dynamics of a greedy source is modeled by a Markov chain with infinite states. Even so, we can show that the general formula is applicable to the queue with independent greedy sources, by the truncation technique as in section 3.

There are two interesting limiting cases: (i) active periods follow a general distribution and (ii) the number of sources goes to infinity under an appropriate normalizing condition. Note that, in each case, it is easy to construct a sequence of queues satisfying Assumptions 1 to 3, whose limits is given by the limiting case. Thus, owing to Fatou's lemma, the general formula gives the upper bounds of these cases. We are working on these limiting cases in depth and the result will be reported somewhere.

A Proof of Lemma 1

Let

$$\boldsymbol{u}_k(z) = (u_{k,0}(z), \boldsymbol{u}_{k,1}(z)), \qquad \boldsymbol{v}_k(z) = \begin{pmatrix} v_{k,0}(z) \\ \boldsymbol{v}_{k,1}(z) \end{pmatrix}.$$

We then have

$$\begin{aligned} u_{k,0}(z)p_k + \boldsymbol{u}_{k,1}(z)\boldsymbol{T}_k^{\circ} &= \delta_k(z)u_{k,0}(z), \\ (1-p_k)u_{k,0}(z)\boldsymbol{\alpha}_k(z) + \boldsymbol{u}_{k,1}(z)\boldsymbol{T}_k(z) &= \delta_k(z)\boldsymbol{u}_{k,1}(z), \\ u_{k,0}(z) + \boldsymbol{u}_{k,1}(z)\boldsymbol{e} &= 1, \end{aligned}$$

from which it follows that

$$u_{k,0}(z)\left(1+(1-p_k)\boldsymbol{\alpha}_k(z)\left[\delta_k(z)\boldsymbol{I}-\boldsymbol{T}_k(z)\right]^{-1}\boldsymbol{e}\right)=1, \tag{20}$$

$$\boldsymbol{u}_{k,1}(z)=(1-p_k)u_{k,0}(z)\boldsymbol{\alpha}_k(z)\left[\delta_k(z)\boldsymbol{I}-\boldsymbol{T}_k(z)\right]^{-1}, \tag{21}$$

$$\delta_k(z)=p_k+(1-p_k)\boldsymbol{\alpha}_k(z)\left[\delta_k(z)\boldsymbol{I}-\boldsymbol{T}_k(z)\right]^{-1}\boldsymbol{T}_k^{\circ}. \tag{22}$$

Also, by definition, we have

$$\begin{aligned}\boldsymbol{T}_k^{\circ}v_{k,0}(z)+\boldsymbol{T}_k(z)\boldsymbol{v}_{k,1}(z)&=\delta_k(z)\boldsymbol{v}_{k,1}(z),\\ u_{k,0}(z)v_{k,0}(z)+\boldsymbol{u}_{k,1}(z)\boldsymbol{v}_{k,1}(z)&=1,\end{aligned}$$

from which it follows that

$$u_{k,0}(z)v_{k,0}(z)\left(1+(1-p_k)\boldsymbol{\alpha}_k(z)\left[\delta_k(z)\boldsymbol{I}-\boldsymbol{T}_k(z)\right]^{-2}\boldsymbol{T}_k^{\circ}\right)=1. \tag{23}$$

Note here that

$$E\left[z^{A_k}\omega^{C_k}\right]=\omega\boldsymbol{\alpha}_k(z)\left[\boldsymbol{I}-\omega\boldsymbol{T}_k(z)\right]^{-1}\boldsymbol{T}_k^{\circ}.$$

Thus (22) is rewritten to be

$$\delta_k(z)=p_k+(1-p_k)E\left[z^{A_k}\delta_k(z)^{-C_k}\right]. \tag{24}$$

Differentiating both sides of (24) and taking the limit $z\to 1-$, we have

$$\delta_k^{(1)}=(1-p_k)\left(E[A_k]-E[C_k]\delta_k^{(1)}\right),$$

from which (5) follows, because

$$\rho_k=\frac{(1-p_k)E[A_k]}{1+(1-p_k)E[C_k]}.$$

Further, differentiating both sides of (24) twice and taking the limit $z\to 1-$, we have

$$\begin{aligned}\delta_k^{(2)}=(1-p_k)\Big(&E[A_k(A_k-1)]-2E[C_kA_k]\delta^{(1)}\\ &+E[C_k(C_k+1)]\left(\delta_k^{(1)}\right)^2-E[C_k]\delta_k^{(2)}\Big),\end{aligned}$$

from which (9) follows.

Next, differentiating both sides of (20) and taking the limit $z\to 1-$, we have

$$\begin{aligned}u_{k,0}^{(1)}&\left[1+(1-p_k)\boldsymbol{\alpha}_k(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{e}\right]+u_{k,0}(1-)(1-p_k)\boldsymbol{\alpha}_k^{(1)}(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{e}\\ &-u_{k,0}(1-)(1-p_k)\boldsymbol{\alpha}_k(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}(\delta_k^{(1)}\boldsymbol{I}-\boldsymbol{T}_k^{(1)})(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{e}=0.\end{aligned}$$

Also, differentiating both sides of (23), taking the limit $z \to 1-$, and utilizing the above equation, we obtain

$$v_{k,0}^{(1)} = u_{k,0}(1-)(1-p_k)\boldsymbol{\alpha}_k(\boldsymbol{I}-\boldsymbol{T}_k)^{-2}\left(\rho_k\boldsymbol{I}-\boldsymbol{T}_k^{(1)}\right)\boldsymbol{e},$$

and with (21), the above equation is rewritten to be

$$v_{k,0}^{(1)} = \rho_k\boldsymbol{\pi}_{k,1}(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{e} - \boldsymbol{\pi}_{k,1}(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{T}_k^{(1)}\boldsymbol{e}, \qquad (25)$$

where we use $\boldsymbol{u}_{k,1}(1-) = \boldsymbol{\pi}_{k,1}$.

On the other hand, because

$$E\left[\omega^{\tilde{C}_k}\right] = \frac{\boldsymbol{\pi}_{k,1}}{\boldsymbol{\pi}_{k,1}\boldsymbol{e}}[\boldsymbol{I}-\omega\boldsymbol{T}_k]^{-1}\boldsymbol{T}_k^{\circ},$$

$$E\left[z^{A_k(\tilde{C}_k)}\right] = \frac{\boldsymbol{\pi}_{k,1}}{\boldsymbol{\pi}_{k,1}\boldsymbol{e}}[\boldsymbol{I}-\boldsymbol{T}_k(z)]^{-1}\boldsymbol{T}_k^{\circ},$$

it is easy to see that

$$E[\tilde{C}_k] = \frac{\boldsymbol{\pi}_{k,1}}{\boldsymbol{\pi}_{k,1}\boldsymbol{e}}(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{e} - 1, \qquad (26)$$

$$E[A_k(\tilde{C}_k)] = \frac{\boldsymbol{\pi}_{k,1}}{\boldsymbol{\pi}_{k,1}\boldsymbol{e}}(\boldsymbol{I}-\boldsymbol{T}_k)^{-1}\boldsymbol{T}_k^{(1)}\boldsymbol{e}. \qquad (27)$$

Thus, (10) follows from (25), (26) and (27).

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