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An Alternative formula for the steady-state solution of Markov chains of M/G/1 type and its geometric and subexponential asymptotics

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This paper considers the steady-state solution of Markov chains of M/G/1 type. We first show an alternative formula for the steady-state solution in terms of the sum of the convolutions of certain matrix functions. This formula is closely related to an LCFS discrete-time single-server queue and remedies some theoretical flaws in the conventional M/G/1 paradigm. Besides, it is suitable for studying both the asymptotics of the tail distribution. Employing the Markov key renewal theorem for the light-tailed case, we show the geometric asymptotics of the tail distribution. Further, for the heavy-tailed case, we show a sufficient condition under which the tail distribution has the subexponential asymptotics.

Keywords: Markov chains of M/G/1 type, steady-state solution, geometric asymptotics, subexponential asymptotics.

1 Introduction

In this paper, we consider the steady-state solution of a bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$, where X_n takes a nonnegative integer value and S_n takes a value in $\mathcal{M} = \{1, \ldots, M\}$. We assume that the bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ is of M/G/1 type [13], and its transition probability matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{B}(0) & \boldsymbol{B}(1) & \boldsymbol{B}(2) & \boldsymbol{B}(3) & \cdots \\ \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ $(k = 0, 1, \ldots)$ are $M \times M$ matrices. We define $\boldsymbol{\beta}_A$ and $\boldsymbol{\beta}_B$ as

$$\boldsymbol{\beta}_A = \sum_{k=1}^{\infty} k\boldsymbol{A}(k)\boldsymbol{e}, \qquad \boldsymbol{\beta}_B = \sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e},$$

where $\boldsymbol{e}$ denotes an $M \times 1$ vector whose elements are all equal to one. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote

$$\boldsymbol{A} = \sum_{k=0}^{\infty} \boldsymbol{A}(k), \qquad \boldsymbol{B} = \sum_{k=0}^{\infty} \boldsymbol{B}(k),$$

respectively. Note that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are stochastic matrices. Let $\boldsymbol{\pi}$ denote a $1 \times M$ vector which satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{A}, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1, \tag{2}$$

and we define ρ as

$$\rho = \boldsymbol{\pi}\boldsymbol{\beta}_A.$$

Throughout the paper, we assume the following.

Assumption 1

(a) The bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ *is irreducible,*

(b) $\boldsymbol{A}$ *is irreducible,*

(c) $\boldsymbol{\beta}_B$ *is a finite vector, and*

(d) $\rho < 1$.

Remark 1

The irreducibility of $\boldsymbol{A}$ *implies the uniqueness of* $\boldsymbol{\pi}$ *in (2). Further, the irreducible bivariate Markov chain* $\{(X_n, S_n);\ n = 0, 1, \ldots\}$ *with an irreducible* $\boldsymbol{A}$ *is positive recurrent if and only if Assumption 1 (c) and (d) hold [13]*

We define X and S as generic random variables for X_n and S_n in steady state, respectively. Let $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \ldots)$ denote the steady-state solution of the bivariate Markov chain $\{(X_n, S_n);\ n = 0, 1, \ldots\}$, where $\boldsymbol{x}(k)$ $(k = 0, 1, \ldots)$ denotes a $1 \times M$ vector whose jth element represents $\Pr(X = k, S = j)$. An algorithmic solution method to compute the $\boldsymbol{x}(k)$ is known [13], which is called the M/G/1 paradigm, where an $M \times M$ stochastic matrix $\boldsymbol{G}$ satisfying

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}(k)\boldsymbol{G}^k. \tag{3}$$

plays a central role. Note that $\boldsymbol{G}$ represents the transition probability matrix of the second variable S in the fundamental period, i.e., the (i, j)th element of $\boldsymbol{G}$ represents the probability that the first passage time from state $(k + 1, i)$ to any state in $\{(k, 1), \ldots, (k, M)\}$ ends with state (k, j), and $\boldsymbol{G}$ is given by a minimal nonnegative solution of (3) [13]. Following the M/G/1 paradigm, we obtain the boundary vector $\boldsymbol{x}(0)$ by

$$\boldsymbol{x}(0) = \left[1 + \frac{\boldsymbol{\kappa}\boldsymbol{\beta}_B}{1 - \rho} + \boldsymbol{\kappa}(\boldsymbol{B} - \boldsymbol{I})[\boldsymbol{I} - \boldsymbol{A} + (\boldsymbol{e} - \boldsymbol{\beta}_A)\boldsymbol{g}]^{-1}\boldsymbol{e}\right]^{-1} \boldsymbol{\kappa},$$

where $\boldsymbol{g}$ and $\boldsymbol{\kappa}$ denote the invariant probability row vectors of stochastic matrices $\boldsymbol{G}$ and

$$\boldsymbol{K} = \sum_{k=0}^{\infty} \boldsymbol{B}(k)\boldsymbol{G}^k,$$

respectively, i.e.,

$$\boldsymbol{g} = \boldsymbol{g}\boldsymbol{G}, \quad \boldsymbol{g}\boldsymbol{e} = \boldsymbol{e}, \qquad \boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \quad \boldsymbol{\kappa}\boldsymbol{e} = \boldsymbol{e}.$$

Further $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ is recursively determined by [14]

$$\boldsymbol{x}(k) = \left(\boldsymbol{x}(0)\boldsymbol{V}(k-1) + \sum_{j=1}^{k-1} \boldsymbol{x}_j \boldsymbol{U}(k-j) \right) (\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \tag{4}$$

where for $k = 0, 1, \ldots$,

$$\boldsymbol{U}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-1}, \qquad \boldsymbol{V}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{B}(m)\boldsymbol{G}^{m-k-1}.$$

In the special case of $\boldsymbol{B}_k = \boldsymbol{A}_k$ $(k = 0, 1, \ldots)$, the results become much simpler, yet it is especially important in application because the time-average queue length distribution in the stationary BMAP/G/1 queue is identical to the solution of this special case [17]. Note that in this special case, $\boldsymbol{K} = \boldsymbol{G}$ and $\boldsymbol{x}_0$ is given by

$$\boldsymbol{x}_0 = (1 - \rho)\boldsymbol{g}.$$

In this paper, we first show a closed formula for $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$, which is given in terms of the sum of the convolutions of some matrix functions involving $\boldsymbol{U}(m)$ and $\boldsymbol{V}(m)$. Even though this formula is equivalent to (4) and its derivation is straightforward, it has some advantages. After discussing the connection between this formula and a discrete-time single-server queue under the LCFS (last-come, first-served) queueing discipline, we show that the formula remedies some theoretical flaws in the conventional M/G/1 paradigm. Besides, this formula is suitable for analyzing the asymptotics of the tail distribution of the $\boldsymbol{x}(k)$.

Let $\overline{\boldsymbol{x}}(k)$ denote a $1 \times M$ vector whose jth element represents $\Pr(X > k, S = j)$ in steady state, i.e.,

$$\overline{\boldsymbol{x}}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{x}(m).$$

We then show the geometric asymptotics of the $\overline{\boldsymbol{x}}(k)$, assuming the $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ are light-tailed. More precisely, we derive a sufficient condition under which there exists θ such that

$$\lim_{k \to \infty} \theta^k \overline{\boldsymbol{x}}(k) = \boldsymbol{c}, \tag{5}$$

for some positive vector $\boldsymbol{c}$. This problem was attacked in [1] with the help of the Tauberian theorem [8]. and a necessary condition for the geometric asymptotics (5) was obtained. Recently, Møller studied the geometric asymptotics by studying the cycle maxima in a regenerative cycle [11], and obtained a sufficient condition for the geometric asymptotics (5). See [7], too, which studied a stronger form of the geometric asymptotics of the tail distribution.

The approach we will take is based on the Markov key renewal theorem (e.g., see [4]), which was successfully applied to study the asymptotic tail distribution of risk processes [3, 10] and the virtual waiting time distribution in a single-server queue with Markovian arrival streams

[10]. We follow a line similar to those works to show a sufficient condition for the geometric asymptotics of the $\overline{\boldsymbol{x}}(k)$.

Finally, we consider the subexponential asymptotics of the $\overline{\boldsymbol{x}}(k)$, assuming at least one of the equilibrium distributions of the $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$ is subexponential. That is, we show the asymptotic formula in a form:

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Z}(k)} = \boldsymbol{c},$$

for some subexponential complementary distribution function $\overline{Z}(k)$ and some positive vector $\boldsymbol{c}$. The line of the proof is very similar to that in [18], which shows the subexponential asymptotics of the tail distributions of virtual and actual waiting times in a single-server queue with Markovian arrival streams. The subexponential asymptotics in a more general structured Markov chain was studied in [2], where the Markov chain of M/G/1 type is included as a special case. However, the sufficient condition for the subexponentiality in [2] is more restrictive than ours.

The rest of this paper is divided into three sections. In section 2, we first show an alternative formula for the $\boldsymbol{x}(k)$. We then discuss the connection of the new formula and the LCFS queueing discipline and some implications of the formula. In section 3, we study the geometric asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$, applying the Markov key renewal theorem. Finally, in section 4, we show a subexponential asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$.

2 Alternative formula for the steady-state solution

2.1 Results

In this subsection, we show an alternative formula for $\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ in terms of $\boldsymbol{x}(0)$. Let $\boldsymbol{F}(k)$ $(k = 1, 2, \ldots)$ and $\boldsymbol{E}(k)$ $(k = 0, 1, \ldots)$ denote $M \times M$ matrices which is given by

$$\boldsymbol{F}(k) = \begin{cases} \boldsymbol{O}, & k = 0, \\ \boldsymbol{U}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, & k = 1, 2, \ldots, \end{cases}$$
$$\boldsymbol{E}(k) = \boldsymbol{V}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \qquad k = 0, 1, \ldots.$$

Further we define $\boldsymbol{F}^{(n)}(k)$ $(n = 1, 2, \ldots, k = n-1, n, n+1, \ldots)$ recursively as

$$\boldsymbol{F}(k)^{(1)} = \boldsymbol{E}(k), \qquad k = 0, 1, \ldots,$$

and for $n = 2, 3, \ldots,$

$$\boldsymbol{F}^{(n)}(k) = \sum_{j=n-2}^{k-1} \boldsymbol{F}^{(n-1)}(j)\boldsymbol{F}(k-j), \qquad k = n-1, n, \ldots.$$

Theorem 1
$\boldsymbol{x}(k)$ $(k = 1, 2, \ldots)$ is given by

$$\boldsymbol{x}(k) = \boldsymbol{x}(0) \sum_{n=1}^{k} \boldsymbol{F}^{(n)}(k-1), \qquad k = 1, 2, \ldots. \tag{6}$$

Proof. Note first that (4) is equivalent to

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \sum_{j=1}^{k-1} \boldsymbol{x}(j)\boldsymbol{F}(k-j), \qquad k = 1, 2, \ldots. \tag{7}$$

For $k = 1$, (6) holds because (7) implies $\boldsymbol{x}(1) = \boldsymbol{x}(0)\boldsymbol{E}(0) = \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(0)$. Suppose (6) holds for $j = 1, \ldots, k-1$ ($k \geq 2$), i.e.,

$$\boldsymbol{x}(j) = \boldsymbol{x}(0) \sum_{n=1}^{j} \boldsymbol{F}^{(n)}(j-1), \qquad j = 1, \ldots, k-1. \tag{8}$$

Substituting (8) into (7), we have

$$\begin{aligned}
\boldsymbol{x}(k) &= \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \sum_{j=1}^{k-1} \boldsymbol{x}(0) \sum_{n=1}^{j} \boldsymbol{F}^{(n)}(j-1)\boldsymbol{F}(k-j) \\
&= \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \sum_{j=n}^{k-1} \boldsymbol{F}^{(n)}(j-1)\boldsymbol{F}(k-j) \\
&= \boldsymbol{x}(0)\boldsymbol{E}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \sum_{j=n-1}^{k-2} \boldsymbol{F}^{(n)}(j)\boldsymbol{F}(k-1-j) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=1}^{k-1} \boldsymbol{F}^{(n+1)}(k-1) \\
&= \boldsymbol{x}(0)\boldsymbol{F}^{(1)}(k-1) + \boldsymbol{x}(0) \sum_{n=2}^{k} \boldsymbol{F}^{(n)}(k-1),
\end{aligned}$$

which shows (6) holds for $j = k$, too. □

2.2 Connection between Theorem 1 and LCFS queueing discipline

Even though Theorem 1 was derived only with algebraic manipulations, it is closely related to the LCFS queueing discipline if we regard the first variable X_n in the Markov chain (X_n, S_n) as the queue length in a discrete-time single-server queue. In this subsection, we briefly discuss another derivation of Theorem 1 considering the LCFS queueing discipline.

We consider a discrete-time single-server queue, where service times of all customers are equal to one unit time and the queue length X_n ($n = 0, 1, \ldots$) at time n satisfies

$$X_{n+1} = \max(X_n - 1, 0) + H_{n+1},$$

where H_n denote the number of customers in a batch arriving in the service of the nth served customer. We assume that H_n is modulated by S_n and the transition probability matrix of (X_n, S_n) is given by $\boldsymbol{T}$ in (1). Further we assume that customers are served on an LCFS basis, and the service order of customers in the same batch is random. Let L denote the number of batches in which at least one customer is being in the system in steady state and given $L = l$, $W(n)$ ($n = 1, \ldots, l$) denote the number of *waiting* customers arriving in the nth oldest batch.

For this queue, we define a string $(l; k(1), \ldots, k(l))$ ($l \geq 1$) as the state of the queue, which indicates $L = l$ ($l \geq 1$) and $W(n) = k(n)$ ($n = 1, \ldots, l$). Note that $k(m) \geq 1$ for $m = 1, \ldots, l-1$, whereas $k(l) \geq 0$ because $k(l)$ represents the number of waiting customers in the latest arriving batch and one customer in the latest arriving batch is being served.

Let $\boldsymbol{y}(l; k(1), \ldots, k(l))$ denote a $1 \times M$ vector whose jth element represent the steady-state probability that the queue is in state $(l; k(1), \ldots, k(l))$ and $S = j$. Then, following a very similar reasoning to [19], we can show the following matrix product-form solution to $\boldsymbol{y}(l; k(1), \ldots, k(l))$: for $l \geq 1$,

$$\begin{aligned}
\boldsymbol{y}(l; k(1), \ldots, k(l)) = \boldsymbol{x}(0)[\boldsymbol{V}(k(1)) + \boldsymbol{V}(0)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(1))] \\
\cdot[(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(2))] \cdot \cdots \cdot [(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(l))].
\end{aligned} \tag{9}$$

The brief sketch of the derivation of (9) is given in Appendix A. Note that $\boldsymbol{x}(k)$ $(k \geq 1)$ and $\boldsymbol{y}(l; k(1), \ldots, k(l))$ are related by

$$\begin{aligned}
\boldsymbol{x}(k) &= \sum_{l=1}^{k-1} \sum_{\substack{k(1)+\cdots+k(l)=k-1 \\ x(m)\geq 1 \ (m=1,\ldots,l)}} [\boldsymbol{y}(l; k(1), \ldots, k(l)) + \boldsymbol{y}(l+1; k(1), \ldots, k(l), 0)] \\
&= \boldsymbol{x}(0) \sum_{l=1}^{k-1} \sum_{\substack{k(1)+\cdots+k(l)=k-1 \\ x(m)\geq 1 \ (m=1,\ldots,l)}} \boldsymbol{y}(l; k(1), \ldots, k(l))[\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(0)] \\
&= \boldsymbol{x}(0) \sum_{l=1}^{k-1} \sum_{\substack{k(1)+\cdots+k(l)=k-1 \\ x(m)\geq 1 \ (m=1,\ldots,l)}} \boldsymbol{y}(l; k(1), \ldots, k(l))(\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \boldsymbol{x}(0) \sum_{l=1}^{k-1} \sum_{\substack{k(1)+\cdots+k(l)=k-1 \\ x(m)\geq 1 \ (m=1,\ldots,l)}} [\boldsymbol{V}(k(1)) + \boldsymbol{V}(0)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(1))] \\
&\qquad \cdot [(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(2))] \cdot \cdots \cdot [(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{U}(k(l))](\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \boldsymbol{x}(0) \sum_{l=1}^{k-1} \sum_{\substack{k(1)+\cdots+k(l)=k-1 \\ x(m)\geq 1 \ (m=1,\ldots,l)}} [\boldsymbol{E}(k(1)) + \boldsymbol{E}(0)\boldsymbol{F}(k(1))]\boldsymbol{F}(k(2)) \cdot \cdots \cdot \boldsymbol{F}(k(l)),
\end{aligned}$$

which leads to Theorem 1.

2.3 Implications of Theorem 1

Let $\boldsymbol{x}^*(z)$ denote the vector generating function of the steady-state solution $\boldsymbol{x}(k)$:

$$\boldsymbol{x}^*(z) = \sum_{k=0}^{\infty} \boldsymbol{x}(k) z^k, \qquad |z| \leq 1.$$

It is easy to see that $\boldsymbol{x}^*(z)$ satisfies [13]

$$\boldsymbol{x}^*(z)[z\boldsymbol{I} - \boldsymbol{\Lambda}_A(z)] = \boldsymbol{x}_0(z\boldsymbol{\Lambda}_B(z) - \boldsymbol{\Lambda}_A(z)), \qquad |z| \leq 1, \tag{10}$$

where

$$\boldsymbol{\Lambda}_A(z) = \sum_{k=0}^{\infty} \boldsymbol{A}(k) z^k, \qquad \boldsymbol{\Lambda}_B(z) = \sum_{k=0}^{\infty} \boldsymbol{B}(k) z^k. \tag{11}$$

Note that $z\boldsymbol{I} - \boldsymbol{\Lambda}_A(z)$ is singular for some points z^* in $|z| \leq 1$ (e.g., $z^* = 1$), and therefore for such a z^*, the inverse matrix of $z\boldsymbol{I} - \boldsymbol{\Lambda}_A(z)$ does not exist.

Let $\boldsymbol{\Omega}(z)$ and $\boldsymbol{\Gamma}(z)$ denote

$$\boldsymbol{\Omega}(z) = \sum_{k=0}^{\infty} \boldsymbol{F}(k) z^k, \qquad \boldsymbol{\Gamma}(z) = \sum_{k=0}^{\infty} \boldsymbol{E}(k) z^k,$$

respectively. As we will show in Lemma 13, the spectral radius of $\sum_{k=0}^{\infty} \boldsymbol{F}(k)$ is less than one, so that $\boldsymbol{\Omega}(z)$ is well defined for $|z| < 1$ and $\sum_{n=0}^{\infty} (\boldsymbol{\Omega}(z))^n z^n$ converges to $(\boldsymbol{I} - \boldsymbol{\Omega}(z))^{-1}$ for all $|z| \leq 1$. Note also that $\sum_{k=0}^{\infty} \boldsymbol{E}(k)$ is bounded because if not, the sum of the steady state solution $\boldsymbol{x}(k)$ is unbounded. Thus $\boldsymbol{\Gamma}(z)$ is well defined for $|z| < 1$. It then follows from Theorem 1 that

$$\boldsymbol{x}^*(z) = \boldsymbol{x}_0 \left[\boldsymbol{I} + z\boldsymbol{\Gamma}(z)(\boldsymbol{I} - \boldsymbol{\Omega}(z))^{-1}\right], \qquad |z| \leq 1. \tag{12}$$

Note that (10) and (12) are essentially different expressions for $\boldsymbol{x}^*(z)$.

In [13], a recursive algorithm to compute the factorial moments of the $\boldsymbol{x}(k)$ is given based on (10). Note that in this recursion, the nth ($n = 1, 2, \ldots$) factorial moment of the $\boldsymbol{x}(k)$ is obtained under the assumption that the $n+1$st factorial moment is finite. This feature is recognized as a flaw in the theoretical aspect in the conventional M/G/1 paradigm [13], even though it does not matter in practice.

Based on (12), we can construct an alternative recursion to compute the factorial moments of the $\boldsymbol{x}(k)$, which does not require the finiteness of higher moments. Let $\boldsymbol{x}^{[n]}$ ($n = 1, 2, \ldots$) denote the nth factorial moment of the $\boldsymbol{x}(k)$, and let $\boldsymbol{\Omega}^{[n]}$ and $\boldsymbol{\Gamma}^{[n]}$ ($n = 1, 2, \ldots$) denote the nth factorial moments of the $\boldsymbol{F}(k)$ and the $\boldsymbol{E}(k)$, respectively. With $\boldsymbol{x}^{[0]} = \boldsymbol{x}^*(1)$, we have

$$\boldsymbol{x}^{[n]} = \left[\boldsymbol{x}_0 \left(\boldsymbol{\Gamma}^{[n]} + n\boldsymbol{\Gamma}^{[n-1]} - \boldsymbol{\Omega}^{[n]} \right) + \sum_{m=0}^{n-1} \binom{n}{m} \boldsymbol{x}^{[m]} \boldsymbol{\Omega}^{[n-m]} \right] (\boldsymbol{I} - \boldsymbol{\Omega}(1))^{-1}, \qquad n = 1, 2, \ldots.$$

In the following two sections, we consider the asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$. For this purpose, we rewrite Theorem 1 as follows.

Corollary 2
$\boldsymbol{x}(k)$ ($k = 1, 2, \ldots$) is given by

$$\boldsymbol{x}(k) = \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k-1), \qquad k = 1, 2, \ldots. \tag{13}$$

where the $\boldsymbol{F}^{(n)}(k)$ is recursively given by

$$\boldsymbol{F}^{(1)}(k) = \boldsymbol{E}(k), \qquad k = 0, 1, \ldots,$$

and for $n = 2, 3, \ldots$,

$$\boldsymbol{F}^{(n)}(k) = \sum_{j=0}^{k} \boldsymbol{F}^{(n-1)}(j) \boldsymbol{F}(k-j), \qquad k = 0, 1, \ldots. \tag{14}$$

Proof. Because $\boldsymbol{F}(0) = \boldsymbol{O}$, we have from (14)

$$\boldsymbol{F}^{(n)}(j) = \boldsymbol{O}, \qquad n = 2, 3, \ldots, \ j = 0, \ldots, n-2.$$

Thus Corollary 2 is equivalent to Theorem 1. □

Corollary 2 will serve as the basis for the remaining part of this paper. Note that, with Corollary 2, we have for $k = 0, 1, \ldots$

$$\begin{aligned} \overline{\boldsymbol{x}}(k) &= \sum_{m=k+1}^{\infty} \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(m-1) \\ &= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \sum_{m=k}^{\infty} \boldsymbol{F}^{(n)}(m) \\ &= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k), \end{aligned}$$

where

$$\overline{\boldsymbol{F}^{(n)}}(k) = \sum_{m=k}^{\infty} \boldsymbol{F}^{(n)}(m), \qquad n = 1, 2, \ldots.$$

Remark 2
We are concerned with the asymptotics of the complementary distribution $\overline{\boldsymbol{x}}(k)$. *Because* $\boldsymbol{x}(k) = \overline{\boldsymbol{x}}(k-1) - \overline{\boldsymbol{x}}(k)$, *however, the asymptotic behavior of the* $\boldsymbol{x}(k)$ *is obvious with the result for the* $\overline{\boldsymbol{x}}(k)$.

3 Geometric asymptotics

In this section we derive a sufficient condition under which $\overline{\boldsymbol{x}}(k)$ has a geometric tail. To do so, we first show a Markov renewal equation for a certain matrix function that is closely related to $\sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k)$, and applying the Markov key renewal theorem to it, we obtain the desired result.

3.1 Preliminaries

We define $\widetilde{\boldsymbol{A}}(k)$ and $\widetilde{\boldsymbol{B}}(k)$ $(k = 0, 1, \ldots)$ as nonnegative $M \times M$ matrices given by

$$\widetilde{\boldsymbol{A}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{A}(k)^T \Delta_{\boldsymbol{\pi}}, \qquad \widetilde{\boldsymbol{B}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{B}(k)^T \Delta_{\boldsymbol{\pi}}, \qquad k = 0, 1, \ldots,$$

where $\Delta_{\boldsymbol{x}}$ represents an $M \times M$ diagonal matrix whose (j,j)th element is equal to the jth element of vector $\boldsymbol{x}$, and T represents the transpose operator. Let $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$ denote $M \times M$ matrices given by

$$\widetilde{\boldsymbol{A}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{A}}(k), \qquad \widetilde{\boldsymbol{B}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{B}}(k),$$

respectively. Note that $\widetilde{\boldsymbol{A}}$ is a stochastic matrix satisfying $\boldsymbol{\pi}\widetilde{\boldsymbol{A}} = \boldsymbol{\pi}$. We define $\tilde{\boldsymbol{\alpha}}^A$ and $\tilde{\boldsymbol{\alpha}}^B$ as

$$\tilde{\boldsymbol{\alpha}}^A = \boldsymbol{\pi} \sum_{k=1}^{\infty} k\widetilde{\boldsymbol{A}}(k), \qquad \tilde{\boldsymbol{\alpha}}^B = \boldsymbol{\pi} \sum_{k=1}^{\infty} k\widetilde{\boldsymbol{B}}(k),$$

respectively. We then have

$$\tilde{\boldsymbol{\alpha}}^A \boldsymbol{e} = \rho < 1.$$

We define $\boldsymbol{R}$ as

$$\boldsymbol{R} = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{G}^T \Delta_{\boldsymbol{\pi}}.$$

It then follows from (3) that $\boldsymbol{R}$ satisfies

$$\boldsymbol{R} = \sum_{k=0}^{\infty} \boldsymbol{R}^k \widetilde{\boldsymbol{A}}(k).$$

The maximum eigenvalue of $\boldsymbol{R}$ is equal to one because $\rho < 1$ [12] and its associated left and right eigenvectors are given by $\boldsymbol{\pi}$ and $\tilde{\boldsymbol{g}}^T = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{g}^T$, respectively.

$$\boldsymbol{\pi}\boldsymbol{R} = \boldsymbol{\pi}, \qquad \boldsymbol{R}\tilde{\boldsymbol{g}}^T = \tilde{\boldsymbol{g}}^T.$$

It is easy to see that $\boldsymbol{\pi}\tilde{\boldsymbol{g}}^T = 1$.

We then define $\widetilde{\boldsymbol{F}}(k)$ and $\widetilde{\boldsymbol{E}}(k)$ $(k = 0, 1, \ldots)$ as

$$\widetilde{\boldsymbol{F}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{F}(k)^T \Delta_{\boldsymbol{\pi}}, \qquad \widetilde{\boldsymbol{E}}(k) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{E}(k)^T \Delta_{\boldsymbol{\pi}},$$

respectively. Note that $\widetilde{\boldsymbol{F}}(k)$ and $\widetilde{\boldsymbol{E}}(k)$ are represented by

$$\widetilde{\boldsymbol{F}}(k) = \begin{cases} \boldsymbol{O}, & k=0, \\ (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{U}}(k), & k=1,2,\ldots, \end{cases}$$
$$\widetilde{\boldsymbol{E}}(k) = (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{V}}(k), \quad k=0,1,\ldots,$$

respectively, where for $k=0,1,\ldots$

$$\widetilde{\boldsymbol{U}}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{A}}(m), \qquad \widetilde{\boldsymbol{V}}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{B}}(m).$$

We define $\widetilde{\boldsymbol{F}}$ and $\widetilde{\boldsymbol{E}}$ as

$$\widetilde{\boldsymbol{F}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{F}}(k), \qquad \widetilde{\boldsymbol{E}} = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{E}}(k),$$

respectively.

Lemma 3
$\widetilde{\boldsymbol{F}}$ and $\widetilde{\boldsymbol{E}}$ are given by

$$\widetilde{\boldsymbol{F}} = \boldsymbol{I} - (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}(\boldsymbol{I}-\widetilde{\boldsymbol{A}}+\tilde{\boldsymbol{g}}^T(\boldsymbol{\pi}-\tilde{\boldsymbol{\alpha}}^A)), \tag{15}$$
$$\widetilde{\boldsymbol{E}} = (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\left(\widetilde{\boldsymbol{B}} - \sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k) + \tilde{\boldsymbol{g}}^T\tilde{\boldsymbol{\alpha}}^B\right), \tag{16}$$

respectively.

The proof of Lemma 3 is given in Appendix B.1.

Lemma 4
$\widetilde{\boldsymbol{F}}$ is a strictly substochastic matrix.

Proof. It is clear that $\widetilde{\boldsymbol{F}}$ is a nonnegative matrix. From (15), we have

$$\begin{aligned}\widetilde{\boldsymbol{F}}\boldsymbol{e} &= \boldsymbol{e} - (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\tilde{\boldsymbol{g}}^T(1-\rho) \\ &= \boldsymbol{e} - (1-\rho)(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\tilde{\boldsymbol{g}}^T \\ &< \boldsymbol{e},\end{aligned}$$

where the second equality comes from $(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\tilde{\boldsymbol{g}}^T = \tilde{\boldsymbol{g}}^T$, and the inequality comes from $\tilde{\boldsymbol{g}}^T > \boldsymbol{0}$, $(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\boldsymbol{e} > \boldsymbol{e}$ and $\rho < 1$. □

We define $\overline{\widetilde{\boldsymbol{F}}}(k)$ and $\overline{\widetilde{\boldsymbol{E}}}(k)$ as

$$\overline{\widetilde{\boldsymbol{F}}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}(m), \qquad \overline{\widetilde{\boldsymbol{E}}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{E}}(m), \qquad k=0,1,\ldots,$$

respectively. Let $\widetilde{\boldsymbol{F}}^{(n)}(k)$ $(n=1,2,\ldots,\ k=0,1,\ldots)$ denote

$$\widetilde{\boldsymbol{F}}^{(n)}(k) = \Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{F}^{(n)}(k)^T\Delta_{\boldsymbol{\pi}}, \qquad n=1,2,\ldots,\ k=0,1,\ldots.$$

We then define $\widetilde{\boldsymbol{Q}}(k)$ $(k = 0, 1, \ldots)$ as

$$\widetilde{\boldsymbol{Q}}(k) = \sum_{n=1}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(k), \qquad k = 0, 1, \ldots,$$

where

$$\overline{\widetilde{\boldsymbol{F}}^{(n)}}(k) = \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m).$$

Lemma 5
$\widetilde{\boldsymbol{Q}}(k)$ $(k = 0, 1, \ldots)$ satisfies

$$\widetilde{\boldsymbol{Q}}(k) = \overline{\widetilde{\boldsymbol{E}}}(k) + \overline{\widetilde{\boldsymbol{F}}}(k+1)(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}} + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j)\widetilde{\boldsymbol{Q}}(k-j). \tag{17}$$

Remark 3
(17) is considered as a Markov renewal equation for $\widetilde{\boldsymbol{Q}}(k)$ with defective semi-Markov kernel $\widetilde{\boldsymbol{F}}(j)$ $(j = 0, 1, \ldots)$ [4].

Proof. By definition, $\widetilde{\boldsymbol{Q}}(0)$ is given by

$$\begin{aligned}
\widetilde{\boldsymbol{Q}}(0) &= \sum_{n=1}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(0) \\
&= \overline{\widetilde{\boldsymbol{F}}^{(1)}}(0) + \sum_{n=2}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(0) \\
&= \widetilde{\boldsymbol{E}} + \sum_{n=2}^{\infty} \widetilde{\boldsymbol{F}}^{n-1}\widetilde{\boldsymbol{E}} \\
&= \widetilde{\boldsymbol{E}} + \widetilde{\boldsymbol{F}}(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}}.
\end{aligned}$$

On the other hand, because $\widetilde{\boldsymbol{F}}(0) = \boldsymbol{O}$, (17) implies for $k = 0$

$$\begin{aligned}
\widetilde{\boldsymbol{Q}}(0) &= \overline{\widetilde{\boldsymbol{E}}}(0) + \overline{\widetilde{\boldsymbol{F}}}(1)(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}} \\
&= \widetilde{\boldsymbol{E}} + \widetilde{\boldsymbol{F}}(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}}
\end{aligned}$$

Thus (17) holds for $k = 0$. For $k = 1, 2, \ldots$, we have

$$\begin{aligned}
\widetilde{\boldsymbol{Q}}(k) &= \overline{\widetilde{\boldsymbol{E}}}(k) + \sum_{n=2}^{\infty} \overline{\widetilde{\boldsymbol{F}}^{(n)}}(k) \\
&= \overline{\widetilde{\boldsymbol{E}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m) \\
&= \overline{\widetilde{\boldsymbol{E}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \sum_{j=0}^{m} \widetilde{\boldsymbol{F}}(j)\widetilde{\boldsymbol{F}}^{(n-1)}(m-j) \\
&= \overline{\widetilde{\boldsymbol{E}}}(k) + \sum_{n=2}^{\infty} \sum_{m=k}^{\infty} \left(\sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j)\widetilde{\boldsymbol{F}}^{(n-1)}(m-j) + \sum_{j=k+1}^{m} \widetilde{\boldsymbol{F}}(j)\widetilde{\boldsymbol{F}}^{(n-1)}(m-j) \right) \\
&= \overline{\widetilde{\boldsymbol{E}}}(k) + \sum_{m=k}^{\infty} \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{m=k+1}^{\infty} \sum_{j=k+1}^{m} \widetilde{\boldsymbol{F}}(j) \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j) \\
&= \overline{\overline{\boldsymbol{E}}}(k) + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{Q}}(k-j) + \sum_{j=k+1}^{\infty} \widetilde{\boldsymbol{F}}(j) \sum_{m=j}^{\infty} \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}^{(n)}(m-j) \\
&= \overline{\overline{\boldsymbol{E}}}(k) + \overline{\overline{\boldsymbol{F}}}(k+1)(\boldsymbol{I} - \widetilde{\boldsymbol{F}})^{-1} \widetilde{\boldsymbol{E}} + \sum_{j=0}^{k} \widetilde{\boldsymbol{F}}(j) \widetilde{\boldsymbol{Q}}(k-j),
\end{aligned}
$$

which completes the proof. □

In the following subsection, we study the geometric asymptotics of the tail distribution $\overline{\boldsymbol{x}}(k)$ by using the Markov key renewal theorem that requires the following additional assumptions.

Assumption 2
$\widetilde{\boldsymbol{F}}$ is an irreducible matrix. Further in the semi-Markov process with the kernel $\widetilde{\boldsymbol{F}}(j)$, the recurrence time of every state $j \in \{1, \ldots, M\}$ is aperiodic.

A sufficient condition for Assumption 2 is either (2-a) $\sum_{k=2}^{\infty} \boldsymbol{A}(k)$ is irreducible and all diagonal elements of $\boldsymbol{A}(2)$ are positive, or (2-b) $\sum_{k=2}^{\infty} \boldsymbol{A}(k)\boldsymbol{e} > 0$ and $\boldsymbol{A}(1)$ is irreducible.

To see those, note first that $\widetilde{\boldsymbol{F}}$ is irreducible if and only if $\sum_{k=0}^{\infty} \boldsymbol{F}(k)$ is irreducible. Note also that the aperiodicity in Assumption 2 holds if all the diagonal elements of $\widetilde{\boldsymbol{F}}(1)$ is positive, and the latter is equivalent to the condition that all the diagonal elements of $\boldsymbol{F}(1)$ is positive. By definition,

$$
\begin{aligned}
\sum_{k=0}^{\infty} \boldsymbol{F}(k) &= \sum_{k=1}^{\infty} \boldsymbol{U}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \left(\sum_{k=2}^{\infty} \boldsymbol{A}(k) + \sum_{k=3}^{\infty} \sum_{m=k}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-k+1} \right) \left(\boldsymbol{I} + \sum_{n=1}^{\infty} \left(\boldsymbol{A}(1) + \sum_{m=2}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-1} \right)^{n} \right), \\
\boldsymbol{F}(1) &= \boldsymbol{U}(1)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \\
&= \left(\boldsymbol{A}(2) + \sum_{m=3}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-2} \right) \left(\boldsymbol{I} + \sum_{n=1}^{\infty} \left(\boldsymbol{A}(1) + \sum_{m=2}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-1} \right)^{n} \right).
\end{aligned}
$$

Thus $\sum_{k=0}^{\infty} \boldsymbol{F}(k)$ has terms $\sum_{k=2}^{\infty} \boldsymbol{A}(k)$ and $\sum_{k=2}^{\infty} \boldsymbol{A}(k) \sum_{n=1}^{\infty} (\boldsymbol{A}(1))^n$, and therefore $\sum_{k=0}^{\infty} \boldsymbol{F}(k)$ is irreducible if either $\sum_{k=2}^{\infty} \boldsymbol{A}(k)$ is irreducible, or $\sum_{k=2}^{\infty} \boldsymbol{A}(k)\boldsymbol{e} > 0$ and $\boldsymbol{A}(1)$ is irreducible. Similarly, all the diagonal elements of $\boldsymbol{F}(1)$ is positive if either all diagonal elements of $\boldsymbol{A}(2)$ is positive, or $[\boldsymbol{A}(2) + \sum_{m=3}^{\infty} \boldsymbol{A}(m) \boldsymbol{G}^{m-2}]\boldsymbol{e} = \sum_{k=2}^{\infty} \boldsymbol{A}(k)\boldsymbol{e} > 0$ and $\boldsymbol{A}(1)$ is irreducible.

Remark 4
Assumption 2 holds if the $\boldsymbol{A}_k$ represents the number of arrivals from a BMAP stream during a service time, because all $\boldsymbol{A}(k)$ are positive matrices in this case. Note that this is exactly what [1] considers.

3.2 Results

Let r_A and r_B denote the convergence radiuses of matrix generating functions $\boldsymbol{\Lambda}_A(z)$ and $\boldsymbol{\Lambda}_B(z)$ given in (11), respectively. Let $\delta(z)$ denote the Perron-Frobenius eigenvalue of $\boldsymbol{\Lambda}_A(z)$, and $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ denote the left and right eigenvectors associated with $\delta(z)$, with normalizing condition $\boldsymbol{u}(z)\boldsymbol{v}(z) = \boldsymbol{u}(z)\boldsymbol{e} = 1$.

Assumption 3
(a) The convergence radiuses r_A of $\boldsymbol{\Lambda}_A(z)$ and r_B of $\boldsymbol{\Lambda}_B(z)$ are greater than one, and

(b) There exists θ such that $1 < \theta < \min(r_A, r_B)$ and $\theta = \delta(\theta)$.

Remark 5
Suppose $r_A > 1$. Then there exists at most one η such that $1 < \eta < r_A$ and $\eta = \delta(\eta)$, because $\delta(z)$ is a convex function of z and $\frac{d}{dz}\delta(z)|_{z=1} = \rho < 1$ [13].

Theorem 6
Under Assumptions 1, 2 and 3,

$$\lim_{k\to\infty} \theta^k \sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k) = \frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta} - 1} \cdot \frac{(\boldsymbol{\Lambda}_B(\theta) - \boldsymbol{K})\boldsymbol{v}(\theta)}{\theta - 1} \cdot \boldsymbol{u}(\theta), \tag{18}$$

and therefore

$$\lim_{k\to\infty} \theta^k \overline{\boldsymbol{x}}(k) = \frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta} - 1} \cdot \frac{\boldsymbol{x}_0(\boldsymbol{\Lambda}_B(\theta) - \boldsymbol{I})\boldsymbol{v}(\theta)}{\theta - 1} \cdot \boldsymbol{u}(\theta). \tag{19}$$

Remark 6
When $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \ldots$, (18) and (19) are reduced to

$$\lim_{k\to\infty} \theta^k \sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k) = \frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta} - 1} \cdot \frac{(\theta\boldsymbol{I} - \boldsymbol{G})\boldsymbol{v}(\theta)}{\theta - 1} \cdot \boldsymbol{u}(\theta),$$

$$\lim_{k\to\infty} \theta^k \overline{\boldsymbol{x}}(k) = \frac{(1-\rho)\boldsymbol{g}\boldsymbol{v}(\theta)}{\frac{d}{dz}\delta(z)|_{z=\theta} - 1} \cdot \boldsymbol{u}(\theta),$$

because $\boldsymbol{\Lambda}_A(\theta)\boldsymbol{v}(\theta) = \delta(\theta)\boldsymbol{v}(\theta) = \theta\boldsymbol{v}(\theta)$ and $\boldsymbol{x}_0 = (1-\rho)\boldsymbol{g}$.

Proof. We define $\widetilde{\boldsymbol{\Lambda}}_A(z)$ and $\widetilde{\boldsymbol{\Lambda}}_B(z)$ as

$$\widetilde{\boldsymbol{\Lambda}}_A(z) = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{A}}(k) z^k, \qquad \widetilde{\boldsymbol{\Lambda}}_B(z) = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{B}}(k) z^k,$$

respectively. Note that $\widetilde{\boldsymbol{\Lambda}}_A(z) = \Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Lambda}_A(z)\Delta_{\boldsymbol{\pi}}$ and $\widetilde{\boldsymbol{\Lambda}}_B(z) = \Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Lambda}_B(z)\Delta_{\boldsymbol{\pi}}$. Note also that $\widetilde{\boldsymbol{\Lambda}}_A(z)$ has the same eigenvalues as $\boldsymbol{\Lambda}_A(z)$. Let $\tilde{\boldsymbol{u}}(z)^T$ and $\tilde{\boldsymbol{v}}(z)^T$ denote

$$\tilde{\boldsymbol{u}}(z)^T = \Delta_{\boldsymbol{\pi}}^{-1}\boldsymbol{u}(z)^T, \qquad \tilde{\boldsymbol{v}}(z)^T = \boldsymbol{v}(z)^T\Delta_{\boldsymbol{\pi}}.$$

We then have

$$\widetilde{\boldsymbol{\Lambda}}_A(z)\tilde{\boldsymbol{u}}(z)^T = \delta(z)\tilde{\boldsymbol{u}}(z)^T, \qquad \tilde{\boldsymbol{v}}(z)^T\widetilde{\boldsymbol{\Lambda}}_A(z) = \delta(z)\tilde{\boldsymbol{v}}(z)^T,$$

and $\tilde{\boldsymbol{v}}(z)^T\tilde{\boldsymbol{u}}(z)^T = 1$.

Let $\widetilde{\boldsymbol{\Omega}}(z)$ and $\widetilde{\boldsymbol{\Gamma}}(z)$ denote

$$\widetilde{\boldsymbol{\Omega}}(z) = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{F}}(k) z^k, \qquad \widetilde{\boldsymbol{\Gamma}}(z) = \sum_{k=0}^{\infty} \widetilde{\boldsymbol{E}}(k) z^k.$$

Lemma 7
For $1 < z < r_A$, $\widetilde{\boldsymbol{\Omega}}(z)$ *is given by*

$$\widetilde{\boldsymbol{\Omega}}(z) = \boldsymbol{I} + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \boldsymbol{R}/z)^{-1}(\tilde{\boldsymbol{\Lambda}}_A(z)/z - \boldsymbol{I}). \tag{20}$$

Also, for $1 < z < r_B$, $\tilde{\boldsymbol{\Gamma}}(z)$ *is given by*

$$\tilde{\boldsymbol{\Gamma}}(z) = (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \boldsymbol{R}/z)^{-1}\frac{1}{z}\left(\tilde{\boldsymbol{\Lambda}}_B(z) - \sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)\right). \tag{21}$$

The proof of Lemma 7 is given in Appendix B.2.

We define $\widehat{\boldsymbol{\Omega}}(z)$ as

$$\widehat{\boldsymbol{\Omega}}(z) = \sum_{k=0}^{\infty}\widehat{\boldsymbol{F}}(k)z^k,$$

where $\widehat{\boldsymbol{F}}(k)$ $(k = 0, 1, \ldots)$ is given by

$$\widehat{\boldsymbol{F}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{F}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad k = 0, 1, \ldots.$$

Note that

$$\widehat{\boldsymbol{\Omega}}(z) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{\Omega}}(z)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

Thus we have

$$\begin{aligned}
\widehat{\boldsymbol{\Omega}}(\theta)\boldsymbol{e} &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{k=0}^{\infty}\theta^k\widetilde{\boldsymbol{F}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}\boldsymbol{e} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{\Omega}}(\theta)\tilde{\boldsymbol{u}}(\theta) \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\tilde{\boldsymbol{u}}(\theta)^T + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \boldsymbol{R}/\theta)^{-1}\left(\tilde{\boldsymbol{\Lambda}}_A(\theta)\tilde{\boldsymbol{u}}(\theta)^T - \theta\tilde{\boldsymbol{u}}(\theta)^T\right)/\theta\right) \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\tilde{\boldsymbol{u}}(\theta)^T \\
&= \boldsymbol{e},
\end{aligned} \tag{22}$$

so that $\widehat{\boldsymbol{\Omega}}(\theta)$ is a stochastic matrix. Let $\hat{\boldsymbol{\nu}}(\theta)$ denote

$$\hat{\boldsymbol{\nu}}(\theta) = \frac{\tilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}}{c(\hat{\boldsymbol{\nu}}(\theta))},$$

where

$$c(\hat{\boldsymbol{\nu}}(\theta)) = \tilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))\tilde{\boldsymbol{u}}(\theta)^T.$$

Note that $\hat{\boldsymbol{\nu}}(\theta)\boldsymbol{e} = 1$ and

$$\hat{\boldsymbol{\nu}}(\theta)\widehat{\boldsymbol{\Omega}}(\theta) = \hat{\boldsymbol{\nu}}(\theta).$$

Let $\widehat{\boldsymbol{Q}}(k)$ denote

$$\widehat{\boldsymbol{Q}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{Q}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

It then follows from Lemma 5 that

$$\widehat{\boldsymbol{Q}}(k) = \overline{\widehat{\boldsymbol{E}}}(k) + \overline{\widehat{\boldsymbol{F}}}(k+1)(\boldsymbol{I} - \widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} + \sum_{j=0}^{k}\widehat{\boldsymbol{F}}(j)\widehat{\boldsymbol{Q}}(k-j), \tag{23}$$

where

$$\overline{\widehat{\boldsymbol{F}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\overline{\widetilde{\boldsymbol{F}}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad \overline{\widehat{\boldsymbol{E}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\overline{\widetilde{\boldsymbol{E}}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T},$$
$$\widehat{\boldsymbol{F}} = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{F}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}, \qquad \widehat{\boldsymbol{E}} = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{E}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

Multiplying both sides of (23) by θ^k, we obtain

$$\theta^k\widehat{\boldsymbol{Q}}(k) = \theta^k\overline{\widehat{\boldsymbol{E}}}(k) + \theta^k\overline{\widehat{\boldsymbol{F}}}(k+1)(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} + \sum_{j=0}^{k}\theta^j\widehat{\boldsymbol{F}}(j)\cdot\theta^{k-j}\widehat{\boldsymbol{Q}}(k-j). \tag{24}$$

Recall that $\widehat{\boldsymbol{\Omega}}(\theta) = \sum_{k=0}^{\infty}\theta^k\widehat{\boldsymbol{F}}(k)$ is a stochastic matrix (see (22)). Thus (24) is considered as a Markov renewal equation with proper semi-Markov kernel $\theta^k\widehat{\boldsymbol{F}}(k)$ $(k=0,1,\ldots)$ [4]. We shall apply the Markov key renewal theorem to (24). In Appendix C, we summarize the results for discrete-time Markov renewal processes which will be used.

Form Proposition 18 in Appendix C, the conditions we need here are (i) the irreducibility of $\widehat{\boldsymbol{F}}$, (ii) the aperiodicity of the recurrence time of every state in the semi-Markov process driven by the $\widehat{\boldsymbol{F}}(k)$, and (iii) the directly Riemann integrability [4, 15] of $\theta^k\overline{\widehat{\boldsymbol{E}}}(k)$ and $\theta^k\overline{\widehat{\boldsymbol{F}}}(k+1)$. Because of Assumption 2, $\widetilde{\boldsymbol{F}}$ is irreducible, so is $\widehat{\boldsymbol{F}}(k)$. Also, Assumption 2 implies that the recurrence time of every state in the semi-Markov process driven by $\widehat{\boldsymbol{F}}(k)$ is aperiodic. For the third condition, we first note that $\overline{\widehat{\boldsymbol{E}}}(k)$ and $\overline{\widehat{\boldsymbol{F}}}(k)$ are nonincreasing functions. Thus from Lemma 6.1.4. in [15], what we need is the finiteness of $\sum_{k=0}^{\infty}\theta^k\overline{\widehat{\boldsymbol{E}}}(k)$ and $\sum_{k=0}^{\infty}\theta^k\overline{\widehat{\boldsymbol{F}}}(k+1)$.

Lemma 8

$$\sum_{k=0}^{\infty}\theta^k\overline{\widehat{\boldsymbol{E}}}(k) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{\theta}{\theta-1}\widetilde{\boldsymbol{\Gamma}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T},$$
$$\sum_{k=0}^{\infty}\theta^k\overline{\widehat{\boldsymbol{F}}}(k+1) = \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{1}{\theta-1}\widetilde{\boldsymbol{\Omega}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{F}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

The proof of Lemma 8 is given in Appendix B.3.

Thus we can apply Proposition 18 to (24). To obtain the final result, we need the following lemma.

Lemma 9

$$\widehat{\boldsymbol{\nu}}(\theta)\sum_{k=1}^{\infty}k\theta^k\widehat{\boldsymbol{F}}(k)\boldsymbol{e} = \frac{1}{c(\widehat{\boldsymbol{\nu}}(\theta))}\left(\frac{d}{dz}\delta(z)|_{z=\theta} - 1\right).$$

The proof of Lemma 9 is given in Appendix B.4.

Therefore applying Proposition 18 to (24) and using Lemmas 8 and 9, we obtain

$$\lim_{k\to\infty}\theta^k\widehat{\boldsymbol{Q}}(k) = \frac{\boldsymbol{e}c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)}{\left(\frac{d}{dz}\delta(z)|_{z=\theta}-1\right)}$$
$$\cdot\left[\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{\theta}{\theta-1}\widetilde{\boldsymbol{\Gamma}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}\right.$$
$$\left.+\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{1}{\theta-1}\widetilde{\boldsymbol{\Omega}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{F}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}}\right].$$

By definition, we have

$$\sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k) = \left(\Delta_{\boldsymbol{\pi}} \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \widehat{\boldsymbol{Q}}(k) \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \Delta^{-1}_{\boldsymbol{\pi}}\right)^T.$$

Thus we obtain

$$\begin{aligned}
\lim_{k\to\infty} \theta^k \sum_{n=1}^{\infty} \overline{\boldsymbol{F}^{(n)}}(k) = \Bigg(& \Delta_{\boldsymbol{\pi}} \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \boldsymbol{e} \cdot \frac{c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)}{\left(\frac{d}{dz}\delta(z)|_{z=\theta} - 1\right)} \\
& \cdot \left[\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{\theta}{\theta-1}\widetilde{\boldsymbol{\Gamma}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \right. \\
& \left. + \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{1}{\theta-1}\widetilde{\boldsymbol{\Omega}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{F}}\right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} (\boldsymbol{I} - \widehat{\boldsymbol{F}})^{-1} \widehat{\boldsymbol{E}}\right] \\
& \cdot \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \Delta^{-1}_{\boldsymbol{\pi}} \Bigg)^T. \qquad (25)
\end{aligned}$$

We evaluate the right hand side of (25) term by term. From (16) and (21), we have

$$\begin{aligned}
c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\theta\widetilde{\boldsymbol{\Gamma}}(\theta) &= \widetilde{\boldsymbol{v}}(\theta)^T \left(\widetilde{\boldsymbol{\Lambda}}_B(\theta) - \sum_{k=0}^{\infty} \boldsymbol{R}^k \widetilde{\boldsymbol{B}}(k)\right), \\
c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{E}} &= \widetilde{\boldsymbol{v}}(\theta)^T (\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \boldsymbol{R} + \widetilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1} \\
&\quad \cdot \left(\widetilde{\boldsymbol{B}} - \sum_{k=0}^{\infty} \boldsymbol{R}^k \widetilde{\boldsymbol{B}}(k) + \widetilde{\boldsymbol{g}}^T \widetilde{\boldsymbol{\alpha}}_B\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \left(\frac{\theta}{\theta-1}\widetilde{\boldsymbol{\Gamma}}(\theta) - \frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \frac{\widetilde{\boldsymbol{v}}(\theta)^T}{\theta-1} \left[\left(\widetilde{\boldsymbol{\Lambda}}_B(\theta) - \sum_{k=0}^{\infty} \boldsymbol{R}^k \widetilde{\boldsymbol{B}}(k)\right) \right. \\
&\quad \left. - (\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \boldsymbol{R} + \widetilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1} \left(\widetilde{\boldsymbol{B}} - \sum_{k=0}^{\infty} \boldsymbol{R}^k \widetilde{\boldsymbol{B}}(k) + \widetilde{\boldsymbol{g}}^T \widetilde{\boldsymbol{\alpha}}_B\right)\right] \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}. \qquad (26)
\end{aligned}$$

Also, from (15) and (20), we have

$$\begin{aligned}
& c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{\Omega}}(\theta) \\
&\quad = \widetilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0)) + \widetilde{\boldsymbol{v}}(\theta)^T(\widetilde{\boldsymbol{\Lambda}}(\theta)/\theta - \boldsymbol{I}) \\
&\quad = \widetilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0)), \\
& c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\widetilde{\boldsymbol{F}} \\
&\quad = \widetilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0)) \\
&\qquad - \widetilde{\boldsymbol{v}}(\theta)^T(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \boldsymbol{R} + \widetilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{g}}^T(\boldsymbol{\pi} - \widetilde{\boldsymbol{\alpha}}_A)),
\end{aligned}$$

and therefore

$$\begin{aligned}
& c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T} \frac{\widetilde{\boldsymbol{\Omega}}(\theta) - \widetilde{\boldsymbol{F}}}{\theta-1} \\
&\quad = \frac{\widetilde{\boldsymbol{v}}(\theta)^T}{\theta-1}(\boldsymbol{I} - \boldsymbol{R}/\theta)(\boldsymbol{I} - \boldsymbol{R} + \widetilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}(\boldsymbol{I} - \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{g}}^T(\boldsymbol{\pi} - \widetilde{\boldsymbol{\alpha}}_A)). \qquad (27)
\end{aligned}$$

Further Lemma 3 implies

$$\begin{aligned}
&\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} \\
&\quad=(\boldsymbol{I}-\widetilde{\boldsymbol{F}})^{-1}\widetilde{\boldsymbol{E}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&\quad=(\boldsymbol{I}-\widetilde{\boldsymbol{A}}+\tilde{\boldsymbol{g}}^T(\boldsymbol{\pi}-\tilde{\boldsymbol{\alpha}}_A))^{-1}(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))\widetilde{\boldsymbol{E}}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&\quad=(\boldsymbol{I}-\widetilde{\boldsymbol{A}}+\tilde{\boldsymbol{g}}^T(\boldsymbol{\pi}-\tilde{\boldsymbol{\alpha}}_A))^{-1}\left(\widetilde{\boldsymbol{B}}-\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)+\tilde{\boldsymbol{g}}^T\tilde{\boldsymbol{\alpha}}_B\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}. \qquad (28)
\end{aligned}$$

It then follows from (27) and (28) that

$$\begin{aligned}
&c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\frac{\widetilde{\boldsymbol{\Omega}}(\theta)-\widetilde{\boldsymbol{F}}}{\theta-1}\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}} \\
&\quad=\frac{\widetilde{\boldsymbol{v}}(\theta)^T}{\theta-1}(\boldsymbol{I}-\boldsymbol{R}/\theta)(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})^{-1}\left(\widetilde{\boldsymbol{B}}-\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)+\tilde{\boldsymbol{g}}^T\tilde{\boldsymbol{\alpha}}_B\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}. \qquad (29)
\end{aligned}$$

Therefore, with (26) and (29), we obtain

$$\begin{aligned}
&c(\widehat{\boldsymbol{\nu}}(\theta))\widehat{\boldsymbol{\nu}}(\theta)\left[\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{\theta}{\theta-1}\widetilde{\boldsymbol{\Gamma}}(\theta)-\frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}\right. \\
&\qquad\left.+\Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{1}{\theta-1}\widetilde{\boldsymbol{\Omega}}(\theta)-\frac{1}{\theta-1}\widetilde{\boldsymbol{F}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}(\boldsymbol{I}-\widehat{\boldsymbol{F}})^{-1}\widehat{\boldsymbol{E}}\right] \\
&\quad=\frac{\widetilde{\boldsymbol{v}}(\theta)^T}{\theta-1}\left(\tilde{\boldsymbol{\Lambda}}_B(\theta)-\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}. \qquad (30)
\end{aligned}$$

Thus substituting (30) into (25), we have

$$\begin{aligned}
\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k) &= \left(\Delta_{\boldsymbol{\pi}}\tilde{\boldsymbol{u}}(\theta)^T\cdot\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\right. \\
&\qquad\left.\cdot\frac{\widetilde{\boldsymbol{v}}(\theta)^T}{\theta-1}\left(\tilde{\boldsymbol{\Lambda}}_B(\theta)-\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)\right)\Delta_{\boldsymbol{\pi}}^{-1}\right)^T \\
&=\Delta_{\boldsymbol{\pi}}^{-1}\left(\Delta_{\boldsymbol{\pi}}\boldsymbol{\Lambda}_B(\theta)\Delta_{\boldsymbol{\pi}}^{-1}-\Delta_{\boldsymbol{\pi}}\sum_{k=0}^{\infty}\boldsymbol{B}(k)\boldsymbol{G}^k\Delta_{\boldsymbol{\pi}}^{-1}\right) \\
&\qquad\cdot\frac{\Delta_{\boldsymbol{\pi}}\boldsymbol{v}(\theta)\Delta_{\boldsymbol{\pi}}^{-1}}{1-\theta}\Delta_{\boldsymbol{\pi}}\boldsymbol{u}(\theta)\Delta_{\boldsymbol{\pi}}^{-1}\cdot\Delta_{\pi}\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1} \\
&=(\boldsymbol{\Lambda}_B(\theta)-\boldsymbol{K})\frac{\boldsymbol{v}(\theta)}{1-\theta}\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\boldsymbol{u}(\theta).
\end{aligned}$$

Further

$$\begin{aligned}
\lim_{k\to\infty}\theta^k\sum_{n=1}^{\infty}\overline{\boldsymbol{x}}(k) &= \lim_{k\to\infty}\theta^k\boldsymbol{x}_0\sum_{n=1}^{\infty}\overline{\boldsymbol{F}^{(n)}}(k) \\
&=\boldsymbol{x}_0(\boldsymbol{\Lambda}_B(\theta)-\boldsymbol{I})\frac{\boldsymbol{v}(\theta)}{1-\theta}\frac{1}{\frac{d}{dz}\delta(z)|_{z=\theta}-1}\boldsymbol{u}(\theta),
\end{aligned}$$

because $\boldsymbol{x}_0\boldsymbol{K}=\boldsymbol{x}_0$. □

4 Subexponential asymptotics

In this section, we show the subexponential asymptotics of the steady-state solution $\boldsymbol{x}(k)$.

4.1 Preliminaries

For any nonnegative discrete random variable Y with finite mean y, let $Y(k)$ $(k = 0,1,\ldots)$ denote the mass function of Y, i.e., $Y(k) = \Pr(Y = k)$. Further let $\overline{Y}(k)$ denote the complementary distribution function of $Y(k)$, i.e.,

$$\overline{Y}(k) = 1 - \sum_{m=0}^{k} Y(m).$$

Further we define Y_e as the equilibrium random variable associated with Y. Note that the equilibrium mass function $Y_e(k) = \Pr(Y_e = k)$ is given by

$$Y_e(k) = y^{-1}\overline{Y}(k).$$

Let $\overline{Y_e}(k)$ denote the complementary distribution function of $Y_e(k)$:

$$\overline{Y_e}(k) = 1 - \sum_{m=0}^{k} Y_e(m).$$

A mass function $Y(k)$ and the corresponding discrete random variable Y are called heavy-tailed if $\overline{Y}(k) > 0$ for all $k = 0,1,\ldots$ and

$$\lim_{k\to\infty} \frac{\overline{Y}(k+m)}{\overline{Y}(k)} = 1, \qquad \text{for all } m = 1,2,\ldots. \tag{31}$$

Let $Y^{(n)}(k)$ denote the n-fold convolution of $Y(k)$ with itself, i.e.,

$$Y^{(n)}(k) = \sum_{l=0}^{k} Y(k-l)Y^{(n-1)}(l), \qquad n = 2,3,\ldots,$$

with $Y^{(1)}(k) = Y(l)$, and $\overline{Y^{(n)}}(k)$ denote

$$\overline{Y^{(n)}}(k) = 1 - \sum_{m=0}^{k} Y^{(n)}(m).$$

Then the heavy-tailed $Y(k)$ (and the corresponding random variable Y) is called subexponential if

$$\lim_{k\to\infty} \frac{\overline{Y^{(2)}}(k)}{\overline{Y}(k)} = 2.$$

Readers are referred to [6, 9, 16] for details. In what follows, we provide some lemmas that we need in the sequel.

Lemma 10 (Corollary 3.3 in [16])
For any heavy-tailed equilibrium distribution function $Y_e(k)$,

$$\lim_{k\to\infty} \frac{\overline{Y}(k)}{\overline{Y_e}(k)} = 0.$$

Lemma 11 (Theorem 2.7 in [5])
Suppose $Y(k)$ is subexponential and $Z(k)$ is any distribution function such that

$$\lim_{k\to\infty} \frac{\overline{Z}(k)}{\overline{Y}(k)} = c < \infty.$$

If $c > 0$, $Z(k)$ is subexponential.

Lemma 12 (Lemma 10 in [9])
Let Y denote a subexponential random variable and let Z_i's ($i = 1, \ldots, n$) be independent random variables. Suppose

$$\lim_{k\to\infty} \frac{\Pr(Z_i > k)}{\Pr(Y > k)} = c_i \in [0, \infty), \qquad i = 1, \ldots, n.$$

(a). The following equality holds.

$$\lim_{k\to\infty} \frac{\Pr(Z_1 + \cdots + Z_n > k)}{\Pr(Y > k)} = \sum_{i=1}^{n} c_i. \tag{32}$$

(b) Let c^* denote $\max(c_1, \ldots, c_n)$. Then for each $\epsilon > 0$, there exists some constant $N = N(\epsilon, c^*) < \infty$ such that

$$\frac{\Pr(Z_1 + \cdots + Z_n > k)}{\Pr(Y > k)} \le N(1+\epsilon)^n, \qquad \text{for all } k = 0, 1, \ldots,$$

where N is independent of n.

Remark 7
All the results in Lemmas 10 to 12 were originally stated for general random variables. The statements here are adaptations to discrete random variables.

We define $\boldsymbol{F}$ and $\boldsymbol{E}$ as

$$\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{F}(k), \qquad \boldsymbol{E} = \sum_{k=0}^{\infty} \boldsymbol{E}(k).$$

Let $F_{i,j}$ and $E_{i,j}$ denote the (i,j)th elements of $\boldsymbol{F}$ and $\boldsymbol{E}$, respectively.

Lemma 13
(a). $\boldsymbol{F}$ is a nonnegative matrix whose spectral radius is less than one.

(b). Let $\boldsymbol{\pi}^*$ denote a probability vector whose jth element represents $\Pr(S = j)$ in steady state. Then $\boldsymbol{\pi}^*$ and $\boldsymbol{x}(0)$ are related by

$$\boldsymbol{\pi}^* = \boldsymbol{x}(0) \left(\boldsymbol{I} + \boldsymbol{E}(\boldsymbol{I} - \boldsymbol{F})^{-1} \right). \tag{33}$$

(c). Let $F_{i,j}(k)$ and $E_{i,j}(k)$ denote the (i,j)th elements of $\boldsymbol{F}(k)$ and $\boldsymbol{E}(k)$, respectively. If $F_{i,j} > 0$ for some $i, j \in \mathcal{M}$, $F_{i,j}(k)/F_{i,j}$ is a mass function for a proper nonnegative random variable. Similarly, if $E_{i,j} > 0$ for some $i, j \in \mathcal{M}$, $E_{i,j}(k)/E_{i,j}$ is a mass function for a proper nonnegative random variable.

(d) $\boldsymbol{\pi}$ and $\boldsymbol{F}$ are related by

$$\boldsymbol{\pi}(\boldsymbol{I} - \boldsymbol{F}) = (1 - \rho)\boldsymbol{g} \left(\boldsymbol{I} - \boldsymbol{U}(0) \right)^{-1}.$$

The proof of Lemma 13 is given in Appendix B.5.

4.2 Results

Let $\overline{\boldsymbol{A}}(k)$ and $\overline{\boldsymbol{B}}(k)$ $(k = 0, 1, \ldots)$ denote

$$\overline{\boldsymbol{A}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{A}(l), \qquad \overline{\boldsymbol{B}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{B}(l).$$

Assumption 4
There exists a random variable Y with finite mean y such that

(a) the equilibrium random variable Y_e of Y is subexponential,

(b) for $\overline{Y}(k) = \Pr(Y > x)$, the limits

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}(k)} = \boldsymbol{C}^A/y, \qquad \lim_{k\to\infty} \frac{\overline{\boldsymbol{B}}(k)}{\overline{Y}(k)} = \boldsymbol{C}^B/y,$$

exist, where $\boldsymbol{C}^A$ and $\boldsymbol{C}^B$ are finite nonnegative matrices, and

(c) $\boldsymbol{C}^A + \boldsymbol{C}^B \neq \boldsymbol{O}$.

Note that the subexponentiality of Y_e implies $\Pr(Y > k) > 0$ for all $k = 0, 1, \ldots$.

Remark 8
Suppose Assumption 4 holds and the (i,j)th element $C^A_{i,j}$ of $\boldsymbol{C}^A$ is positive for some $i, j \in \mathcal{M}$. Let $A_{i,j}(k)$ and $A_{i,j}$ denote the (i,j)th elements of $\boldsymbol{A}(k)$ and $\boldsymbol{A}$, respectively. Then $a_{i,j}(k) = A_{i,j}(k)/A_{i,j}$ is a probability mass function of a proper random variable $J^A(i,j)$ representing the amount of a jump given the state transition from state i $(i > 0)$ to state j. It can be shown that

$$\lim_{k\to\infty} \frac{\Pr(J^A_e(i,j) > k)}{\Pr(Y_e > k)} = \frac{yC^A_{i,j}}{\sum_{k=1}^{\infty} k a_{i,j}(k)}.$$

Thus Lemma 11 implies that the equilibrium distribution of the amount $J^A(i,j)$ of the conditional jump is subexponential. The same discussion is applicable to $\boldsymbol{B}(k)$, too. Note that the subexponentiality of the equilibrium random variable $J^A_e(i,j)$ implies neither the subexponentiality nor the heavy-tailed property of the original random variable $J^A(i,j)$ [16]. The subexponentiality of Y is assumed in [2], so that our assumption is weaker than in [2], while [2] studies a more general structured Markov chain than ours.

Let $\overline{Y_e}(k)$ denote the complementary equilibrium distribution function associated with Y:

$$\overline{Y_e}(k) = y^{-1} \sum_{m=0}^{\infty} \overline{Y}(k+m+1).$$

Theorem 14
Under Assumptions 1 and 4, $\overline{\boldsymbol{x}}(k)$ satisfies

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \frac{\boldsymbol{x}_0 \boldsymbol{C}^B \boldsymbol{e} + (\boldsymbol{\pi}^* - \boldsymbol{x}_0)\boldsymbol{C}^A \boldsymbol{e}}{1-\rho} \cdot \boldsymbol{\pi}.$$

Remark 9
Owing to Lemma 11, Theorem 14 implies that the $\boldsymbol{x}(k)$ is subexponential because $Y_e(k)$ is so.

Remark 10
When $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \ldots$, $\boldsymbol{C}^B = \boldsymbol{C}^A$ and $\boldsymbol{\pi}^* = \boldsymbol{\pi}$, so that

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \frac{\boldsymbol{\pi}\boldsymbol{C}^A\boldsymbol{e}}{1-\rho} \cdot \boldsymbol{\pi}.$$

Proof. Let $\overline{\boldsymbol{F}}(k)$ and $\overline{\boldsymbol{E}}(k)$ denote

$$\overline{\boldsymbol{F}}(k) = \sum_{l=k}^{\infty} \boldsymbol{F}(l), \qquad \overline{\boldsymbol{E}}(k) = \sum_{l=k}^{\infty} \boldsymbol{E}(l),$$

respectively. Note that $\overline{\boldsymbol{E}}(k) = \overline{\boldsymbol{F}^{(1)}}(k)$.

Lemma 15
Under Assumption 4, we have

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}}(k)}{\overline{Y}_e(k)} = \boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}, \qquad \lim_{k\to\infty} \frac{\overline{\boldsymbol{E}}(k)}{\overline{Y}_e(k)} = \boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}.$$

The proof of Lemma 15 is given in Appendix B.6.

Lemma 16
Under Assumption 4,

$$\lim_{k\to\infty} \frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)} = \boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1} + \sum_{m=1}^{n-1} \boldsymbol{E}\boldsymbol{F}^{m-1}\boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1-m}. \tag{34}$$

Further for each $\epsilon > 0$ and $n = 1, 2, \ldots$, there exists some constant $N < \infty$ such that

$$\frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)} \le N(1+\epsilon)^n\boldsymbol{E}\boldsymbol{F}^{n-1}, \qquad \textit{for all} \quad k \ge 0, \tag{35}$$

where N is independent of n.

Proof. We define random variables $H^F_{i,j}$ and $H^E_{i,j}$ $(i, j \in \mathcal{M})$ whose mass functions $H^F_{i,j}(k)$ and $H^E_{i,j}(k)$ are given by

$$H^F_{i,j}(k) = \begin{cases} \dfrac{F_{i,j}(k)}{F_{i,j}}, & \text{if } F_{i,j} > 0, \\ 1, & \text{otherwise,} \end{cases} \qquad H^E_{i,j}(k) = \begin{cases} \dfrac{E_{i,j}(k)}{E_{i,j}}, & \text{if } E_{i,j} > 0, \\ 1, & \text{otherwise,} \end{cases}$$

Recall that $H^F_{i,j}(k)$ and $H^E_{i,j}(k)$ are mass functions for proper random variables (see lemma 13 (c)). We define $\overline{H^F_{i,j}}(k)$ and $\overline{H^E_{i,j}}(k)$ as

$$\overline{H^F_{i,j}}(k) = \sum_{m=k+1}^{\infty} H^F_{i,j}(m), \qquad \overline{H^E_{i,j}}(k) = \sum_{m=k+1}^{\infty} H^E_{i,j}(m),$$

It then follows from Lemma 15 that for $i,j=1,\ldots,M$,

$$\lim_{k\to\infty}\frac{\overline{H^F_{i,j}}(k)}{\overline{Y_e}(k)}=D^F_{i,j},\qquad \lim_{k\to\infty}\frac{\overline{H^E_{i,j}}(k)}{\overline{Y_e}(k)}=D^E_{i,j},$$

where

$$D^F_{i,j}=\begin{cases}\displaystyle\sum_{l=1}^{M}\sum_{m=1}^{M}\frac{C^A_{i,l}g_m[(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{m,j}}{F_{i,j}}, & \text{if } F_{i,j}\neq 0\\ 0, & \text{otherwise,}\end{cases}$$

$$D^E_{i,j}=\begin{cases}\displaystyle\sum_{l=1}^{M}\sum_{m=1}^{M}\frac{C^B_{i,l}g_m[(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{m,j}}{E_{i,j}}, & \text{if } E_{i,j}\neq 0\\ 0, & \text{otherwise,}\end{cases}$$

Thus we have

$$\frac{\left[\overline{\boldsymbol{F}^{(n)}}(k)\right]_{i,j}}{\overline{Y_e}(k)}=\sum_{l_1\in\mathcal{M}}\cdots\sum_{l_{n-1}\in\mathcal{M}}E_{i,l_1}F_{l_1,l_2}\cdots F_{l_{n-2},l_{n-1}}F_{l_{n-1},j}\cdot\frac{\Pr(H^E_{i,l_1}+H^F_{l_1,l_2}+\cdots+H^F_{l_{n-2},l_{n-1}}+H^F_{l_{n-1},j}>k)}{\overline{Y_e}(k)}.\tag{36}$$

Applying Lemma 12 (a) to (36), we have

$$\lim_{k\to\infty}\frac{\left[\overline{\boldsymbol{F}^{(n)}}(k)\right]_{i,j}}{\overline{Y_e}(k)}=\sum_{l_1\in\mathcal{M}}\cdots\sum_{l_{n-1}\in\mathcal{M}}E_{i,l_1}F_{l_1,l_2}\cdots F_{l_{n-2},l_{n-1}}F_{l_{n-1},j}\cdot\left(D^E_{i,l_1}+D^F_{l_1,l_2}+\cdots+D^F_{l_{n-2},l_{n-1}}+D^F_{l_{n-1},j}\right).$$

Therefore, noting for all $i,j\in\mathcal{M}$

$$F_{i,j}D^F_{i,j}=[\boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{i,j},\qquad E_{i,j}D^E_{i,j}=[\boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}]_{i,j},$$

we have (34).

On the other hand, the second part of Lemma 16 is obtained, by applying Lemma 12 (b) to (36), where N depends only on ϵ and $H^*=\max(D^F_{i,j},D^E_{i,j};i,j\in\mathcal{M})$, and independent of n. □

Because the spectral radius of nonnegative matrix $\boldsymbol{F}$ is less than one (Lemma 13 (a)), we can choose ϵ in such a way that the spectral radius of $(1+\epsilon)\boldsymbol{F}$ is less than one. It then follows from (35) and bounded convergence theorem that

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)}=\boldsymbol{x}_0\sum_{n=1}^{\infty}\lim_{k\to\infty}\frac{\overline{\boldsymbol{F}^{(n)}}(k)}{\overline{Y_e}(k)}.$$

Then applying (34) to the above equation, we obtain

$$\lim_{k\to\infty}\frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)}=\boldsymbol{x}_0\boldsymbol{C}^B\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}(\boldsymbol{I}-\boldsymbol{F})^{-1}+\boldsymbol{x}_0\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\boldsymbol{E}\boldsymbol{F}^{m-1}\boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\boldsymbol{F}^{n-1-m}$$

$$
\begin{aligned}
&= \boldsymbol{x}_0 \Big[\boldsymbol{C}^B \boldsymbol{e}\boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} (\boldsymbol{I} - \boldsymbol{F})^{-1} \\
&\qquad + \sum_{m=1}^{\infty} \boldsymbol{E} \boldsymbol{F}^{m-1} \boldsymbol{C}^A \boldsymbol{e}\boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \sum_{n=m+1}^{\infty} \boldsymbol{F}^{n-1-m} \Big] \\
&= \boldsymbol{x}_0 \Big[\boldsymbol{C}^B \boldsymbol{e}\boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} (\boldsymbol{I} - \boldsymbol{F})^{-1} \\
&\qquad + \boldsymbol{E} (\boldsymbol{I} - \boldsymbol{F})^{-1} \boldsymbol{C}^A \boldsymbol{e}\boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} (\boldsymbol{I} - \boldsymbol{F})^{-1} \Big] \\
&= \boldsymbol{x}_0 \left[\boldsymbol{C}^B + \boldsymbol{E} (\boldsymbol{I} - \boldsymbol{F})^{-1} \boldsymbol{C}^A \right] \boldsymbol{e}\boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} (\boldsymbol{I} - \boldsymbol{F})^{-1} \\
&= \left[\boldsymbol{x}_0 \left(\boldsymbol{C}^B - \boldsymbol{C}^A \right) + \boldsymbol{x}_0 \left(\boldsymbol{I} + \boldsymbol{E} (\boldsymbol{I} - \boldsymbol{F})^{-1} \right) \boldsymbol{C}^A \right] \boldsymbol{e} \\
&\qquad \cdot \boldsymbol{g} (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} (\boldsymbol{I} - \boldsymbol{F})^{-1}.
\end{aligned}
$$

Finally, applying Lemma 13 (b) and (d) to the above equation, we obtain

$$
\lim_{k \to \infty} \frac{\overline{\boldsymbol{x}}(k)}{\overline{Y_e}(k)} = \left[\boldsymbol{x}_0 \left(\boldsymbol{C}^B - \boldsymbol{C}^A \right) + \boldsymbol{\pi}^* \boldsymbol{C}^A \right] \boldsymbol{e} \frac{\boldsymbol{\pi}}{1 - \rho},
$$

which completes the proof. □

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A Outline of the derivation of matrix product-form solution (9)

Suppose the queue is in state $\boldsymbol{y}(l; k(1), \ldots, k(l))$ ($l \geq 2$). Then, immediately before the arrival of the last arriving batch, the queue was in either state $\boldsymbol{y}(l-1; k(1), \ldots, k(l-1))$ or state $\boldsymbol{y}(l-1; k(1), \ldots, k(l-1), 0)$, and the batch size of the last arriving batch is greater than $k(l)$. Thus according to the same approach as in [19], we obtain

$$
\boldsymbol{y}(l; k(1), \ldots, k(l)) = [\boldsymbol{y}(l-1; k(1), \ldots, k(l-1)) + \boldsymbol{y}(l-1; k(1), \ldots, k(l-1), 0)] \boldsymbol{U}(k(l)).
$$

In the above equation, let $k(l) = 0$ and solving it for $\boldsymbol{y}(l; k(1), \ldots, k(l-1), 0)$, we have

$$
\boldsymbol{y}(l; k(1), \ldots, k(l-1), 0) = \boldsymbol{y}(l-1; k(1), \ldots, k(l-1)) [(\boldsymbol{I} - \boldsymbol{U}(0))^{-1} - \boldsymbol{I}],
$$

so that

$$
\boldsymbol{y}(l; k(1), \ldots, k(l)) = \boldsymbol{y}(l-1; k(1), \ldots, k(l-1)) (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \boldsymbol{U}(k(l)), \qquad l = 2, 3, \ldots. \tag{37}
$$

Similarly, we can show

$$
\boldsymbol{y}(1; k(1)) = \boldsymbol{x}(0) \boldsymbol{V}(k(1)) + \boldsymbol{y}(1; 0) \boldsymbol{U}(k(1)),
$$

and letting $k(1) = 0$ and solving the resulting equation for $\boldsymbol{y}(1; 0)$, we have

$$
\boldsymbol{y}(1; 0) = \boldsymbol{x}(0) \boldsymbol{V}(0) (\boldsymbol{I} - \boldsymbol{U}(0))^{-1},
$$

so that

$$
\boldsymbol{y}(1; k(1)) = \boldsymbol{x}(0) [\boldsymbol{V}(k(1)) + \boldsymbol{V}(0) (\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \boldsymbol{U}(k(1))]. \tag{38}
$$

With (37) and (38), we obtain (9).

B Proofs of Lemmas

B.1 Proof of Lemma 3

By definition, we have

$$\begin{aligned}\widetilde{\boldsymbol{F}} &= \sum_{k=1}^{\infty}(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=k+1}^{\infty}\boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{A}}(m)\\ &= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=1}^{\infty}\widetilde{\boldsymbol{A}}(m+1+k)\\ &= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=0}^{\infty}\widetilde{\boldsymbol{A}}(m+1+k)\\ &\qquad -(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\widetilde{\boldsymbol{A}}(m+1).\end{aligned}$$

Recall that $\widetilde{\boldsymbol{U}}(0)=\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\widetilde{\boldsymbol{A}}(m+1)$. Thus we have

$$\widetilde{\boldsymbol{F}} = (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=0}^{\infty}\widetilde{\boldsymbol{A}}(m+1+k)+\boldsymbol{I}-(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}. \tag{39}$$

Note here that

$$\begin{aligned}(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=0}^{\infty}\widetilde{\boldsymbol{A}}(m+1+k) &= (\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=m+1}^{\infty}\widetilde{\boldsymbol{A}}(k)\\ &= (\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})\sum_{k=1}^{\infty}\left(\sum_{m=0}^{k-1}\boldsymbol{R}\right)\widetilde{\boldsymbol{A}}(k)\\ &= \sum_{k=1}^{\infty}(\boldsymbol{I}-\boldsymbol{R}^{k}+k\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})\widetilde{\boldsymbol{A}}(k)\\ &= \widetilde{\boldsymbol{A}}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\tilde{\boldsymbol{\alpha}}^{A}.\end{aligned}$$

Thus noting $\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi}$ is nonsingular, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=0}^{\infty}\widetilde{\boldsymbol{A}}(m+1+k)\\ &\quad= (\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})^{-1}(\widetilde{\boldsymbol{A}}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\tilde{\boldsymbol{\alpha}}^{A})\\ &\quad= (\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})^{-1}\{(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})-(\boldsymbol{I}-\widetilde{\boldsymbol{A}}+\tilde{\boldsymbol{g}}^{T}(\boldsymbol{\pi}-\tilde{\boldsymbol{\alpha}}^{A}))\}\\ &\quad= \boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^{T}\boldsymbol{\pi})^{-1}(\boldsymbol{I}-\widetilde{\boldsymbol{A}}+\tilde{\boldsymbol{g}}^{T}(\boldsymbol{\pi}-\tilde{\boldsymbol{\alpha}}^{A})).\end{aligned} \tag{40}$$

Substituting (40) into (39), we obtain (15).

On the other hand, we have

$$\begin{aligned}\widetilde{\boldsymbol{E}} &= \sum_{k=0}^{\infty}(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{V}}(k)\\ &= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}\boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{B}}(m)\\ &= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=1}^{\infty}\left(\sum_{m=0}^{k-1}\boldsymbol{R}^{m}\right)\widetilde{\boldsymbol{B}}(k).\end{aligned}$$

Note here that

$$(\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})\sum_{m=0}^{k-1}\boldsymbol{R}^m\widetilde{\boldsymbol{B}}(k)=\sum_{k=1}^{\infty}(\boldsymbol{I}-\boldsymbol{R}^k+k\tilde{\boldsymbol{g}}^T\boldsymbol{\pi})\widetilde{\boldsymbol{B}}(k)$$
$$=\widetilde{\boldsymbol{B}}-\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)+\tilde{\boldsymbol{g}}^T\tilde{\boldsymbol{\alpha}}^B.$$

Thus noting $\boldsymbol{I}-\boldsymbol{R}+\tilde{\boldsymbol{g}}^T\boldsymbol{\pi}$ is nonsingular, we obtain (16). □

B.2 Proof of Lemma 7

Because $\widetilde{\boldsymbol{F}}(0)=\boldsymbol{O}$, we have

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}(z)&=\sum_{k=1}^{\infty}(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{U}}(k)z^k\\&=\sum_{k=0}^{\infty}(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{U}}(k)z^k-(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{U}}(0)\\&=(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}\boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{A}}(m)z^k+\boldsymbol{I}-(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\\&=(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^m\sum_{k=m+1}^{\infty}\widetilde{\boldsymbol{A}}(k)z^{k-m-1}+\boldsymbol{I}-(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}.\end{aligned}\tag{41}$$

Note here that

$$\sum_{m=0}^{\infty}\boldsymbol{R}^m\sum_{k=m+1}^{\infty}\widetilde{\boldsymbol{A}}(k)z^{k-m-1}=\sum_{k=1}^{\infty}\left(\sum_{m=0}^{k-1}\boldsymbol{R}^m\right)\widetilde{\boldsymbol{A}}(k)z^{k-m-1}$$
$$=\sum_{k=1}^{\infty}\left(\sum_{m=0}^{k-1}(\boldsymbol{R}/z)^m\right)\widetilde{\boldsymbol{A}}(k)z^{k-1}.$$

Because $z>1$ and the maximum eigenvalue of $\boldsymbol{R}$ is equal to one, $(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}$ exists. Thus

$$\sum_{m=0}^{\infty}\boldsymbol{R}^m\sum_{k=m+1}^{\infty}\widetilde{\boldsymbol{A}}(k)z^{k-m-1}\tag{42}$$
$$\begin{aligned}&=\sum_{k=1}^{\infty}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\left(\boldsymbol{I}-(\boldsymbol{R}/z)^k\right)\widetilde{\boldsymbol{A}}(k)z^{k-1}\\&=(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\left(\sum_{k=1}^{\infty}\widetilde{\boldsymbol{A}}(k)z^{k-1}-\sum_{k=1}^{\infty}(\boldsymbol{R}/z)^k\widetilde{\boldsymbol{A}}(k)/z\right)\\&=(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\left((\tilde{\boldsymbol{\Lambda}}_A(z)-\widetilde{\boldsymbol{A}}(0))/z-(\boldsymbol{R}-\widetilde{\boldsymbol{A}}(0))/z\right)\\&=(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}(\tilde{\boldsymbol{\Lambda}}_A(z)-\boldsymbol{R})/z\\&=(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\tilde{\boldsymbol{\Lambda}}_A(z)/z+\boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}.\end{aligned}\tag{43}$$

Substituting (43) into (41), we obtain

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}(z)&=(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\tilde{\boldsymbol{\Lambda}}_A(z)/z-(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}+\boldsymbol{I}\\&=(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}\left(\tilde{\boldsymbol{\Lambda}}_A(z)/z-\boldsymbol{I}\right)+\boldsymbol{I},\end{aligned}$$

from which (20) follows.

Similarly, we have

$$\begin{aligned}
\tilde{\boldsymbol{\Gamma}}(z) &= \sum_{k=0}^{\infty}(\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\widetilde{\boldsymbol{V}}(k)z^k \\
&= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}\boldsymbol{R}^{m-k-1}\widetilde{\boldsymbol{B}}(m)z^k \\
&= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{m=0}^{\infty}\boldsymbol{R}^{m}\sum_{k=m+1}^{\infty}\widetilde{\boldsymbol{B}}(k)z^{k-m-1} \\
&= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=1}^{\infty}\left(\sum_{m=0}^{k-1}(\boldsymbol{R}/z)^m\right)\widetilde{\boldsymbol{B}}(k)z^{k-1} \\
&= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}\sum_{k=1}^{\infty}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1}(\boldsymbol{I}-(\boldsymbol{R}/z)^k)\widetilde{\boldsymbol{B}}(k)z^{k-1} \\
&= (\boldsymbol{I}-\widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I}-\boldsymbol{R}/z)^{-1} \\
&\qquad\cdot\left[\frac{1}{z}(\tilde{\boldsymbol{\Lambda}}_B(z)-\widetilde{\boldsymbol{B}}(0))-\frac{1}{z}\left(\sum_{k=0}^{\infty}\boldsymbol{R}^k\widetilde{\boldsymbol{B}}(k)-\widetilde{\boldsymbol{B}}(0)\right)\right],
\end{aligned}$$

from which (21) follows. □

B.3 Proof of Lemma 8

Note that

$$\begin{aligned}
\sum_{k=0}^{\infty}\theta^k\overline{\widetilde{\boldsymbol{E}}}(k) &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{k=0}^{\infty}\theta^k\overline{\widetilde{\boldsymbol{E}}}(k)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{k=0}^{\infty}\theta^k\sum_{m=k}^{\infty}\widetilde{\boldsymbol{E}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\theta^k\widetilde{\boldsymbol{E}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{m=0}^{\infty}\frac{\theta^{m+1}-1}{\theta-1}\widetilde{\boldsymbol{E}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{\theta}{\theta-1}\tilde{\boldsymbol{\Gamma}}(\theta)-\frac{1}{\theta-1}\widetilde{\boldsymbol{E}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty}\theta^k\overline{\widetilde{\boldsymbol{F}}}(k+1) &= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{k=0}^{\infty}\theta^k\overline{\widetilde{\boldsymbol{F}}}(k+1)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\frac{1}{\theta}\sum_{k=1}^{\infty}\theta^k\sum_{m=k}^{\infty}\widetilde{\boldsymbol{F}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\frac{1}{\theta}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\theta^k\widetilde{\boldsymbol{F}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\sum_{m=1}^{\infty}\frac{\theta^m-1}{\theta-1}\widetilde{\boldsymbol{F}}(m)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \\
&= \Delta^{-1}_{\widetilde{\boldsymbol{u}}(\theta)^T}\left(\frac{1}{\theta-1}\widetilde{\boldsymbol{\Omega}}(\theta)-\frac{1}{\theta-1}\widetilde{\boldsymbol{F}}\right)\Delta_{\widetilde{\boldsymbol{u}}(\theta)^T},
\end{aligned}$$

which completes the proof. □

B.4 Proof of Lemma 9

Recall that

$$\sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k) = \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} \sum_{k=1}^{\infty} k\theta^k \widetilde{\boldsymbol{F}}(k) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}.$$

It follows from Lemma 7 that

$$\begin{aligned}\sum_{k=1}^{\infty} k\theta^k \widetilde{\boldsymbol{F}}(k) &= \theta \frac{d}{dz} \widetilde{\boldsymbol{\Omega}}(z)|_{z=\theta} \\ &= (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \boldsymbol{R}/\theta)^{-1}(\boldsymbol{R}/\theta)(\boldsymbol{I} - \boldsymbol{R}/\theta)^{-1}(\tilde{\boldsymbol{\Lambda}}_A(\theta)/\theta - \boldsymbol{I}) \\ &\quad + (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1}(\boldsymbol{I} - \boldsymbol{R}/\theta)^{-1} \left(\frac{d}{dz} \tilde{\boldsymbol{\Lambda}}_A(z)|_{z=\theta} - \tilde{\boldsymbol{\Lambda}}_A(\theta)/\theta \right).\end{aligned}$$

Using

$$(\tilde{\boldsymbol{\Lambda}}_A(\theta)/\theta - \boldsymbol{I}) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \boldsymbol{e} = (\tilde{\boldsymbol{\Lambda}}_A(\theta)/\theta - \boldsymbol{I}) \widetilde{\boldsymbol{u}}(\theta)^T = \boldsymbol{0},$$

we have

$$\begin{aligned}\widehat{\boldsymbol{\nu}}(\theta) \sum_{k=1}^{\infty} k\theta^k \widehat{\boldsymbol{F}}(k) \boldsymbol{e} &= \widehat{\boldsymbol{\nu}}(\theta) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T}^{-1} (\boldsymbol{I} - \widetilde{\boldsymbol{U}}(0))^{-1} (\boldsymbol{I} - \boldsymbol{R}/\theta)^{-1} \\ &\qquad \cdot \left(\frac{d}{dz} \tilde{\boldsymbol{\Lambda}}_A(z)|_{z=\theta} - \tilde{\boldsymbol{\Lambda}}_A(\theta)/\theta \right) \Delta_{\widetilde{\boldsymbol{u}}(\theta)^T} \boldsymbol{e} \\ &= \frac{\widetilde{\boldsymbol{v}}(\theta)^T}{c(\widehat{\boldsymbol{\nu}}(\theta))} \left(\frac{d}{dz} \tilde{\boldsymbol{\Lambda}}_A(z)|_{z=\theta} - 1 \right) \widetilde{\boldsymbol{u}}(\theta)^T. \end{aligned} \tag{44}$$

Note here that $\widetilde{\boldsymbol{v}}(\theta)^T \widetilde{\boldsymbol{u}}(\theta)^T = 1$ and therefore [13]

$$\widetilde{\boldsymbol{v}}(\theta)^T \frac{d}{dz} \tilde{\boldsymbol{\Lambda}}_A(z)|_{z=\theta} \widetilde{\boldsymbol{u}}(\theta)^T = \frac{d}{dz} \delta(z)|_{z=\theta}. \tag{45}$$

Substituting (45) into (44) yields the lemma. □

B.5 Proof of Lemma 13

We start with part (a). The nonnegativity of $\boldsymbol{F}$ is clear. Note that

$$\begin{aligned}\sum_{k=1}^{\infty} \boldsymbol{x}(k) &= \sum_{k=1}^{\infty} \boldsymbol{x}(0) \sum_{n=1}^{\infty} \boldsymbol{F}^{(n)}(k-1) \\ &= \boldsymbol{x}(0) \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \boldsymbol{F}^{(n)}(k)\end{aligned}$$

By definition,

$$\sum_{k=0}^{\infty} \boldsymbol{F}^{(1)}(k) = \boldsymbol{E},$$

and for $n = 2, 3, \ldots,$

$$\begin{aligned}\sum_{k=0}^{\infty} \boldsymbol{F}^{(n)}(k) &= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \boldsymbol{F}^{(n-1)}(m) \boldsymbol{F}(k-m) \\ &= \sum_{m=0}^{\infty} \boldsymbol{F}^{(n-1)}(m) \sum_{k=m}^{\infty} \boldsymbol{F}(k-m) \\ &= \boldsymbol{E}\boldsymbol{F}^{n-1}, \qquad n = 2, 3, \ldots.\end{aligned}$$

Thus by induction,

$$\sum_{k=1}^{\infty} \boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{E} \sum_{n=0}^{\infty} \boldsymbol{F}^n. \tag{46}$$

Because the bivariate Markov chain is positive recurrent, the infinite sum on the right hand side should converge to $(\boldsymbol{I}-\boldsymbol{F})^{-1}$, and the convergent is equivalent to the statement of part (a). Part (b) comes from

$$\begin{aligned}\boldsymbol{\pi}^* &= \sum_{k=0}^{\infty} \boldsymbol{x}(k) \\ &= \boldsymbol{x}(0) + \boldsymbol{x}(0)\boldsymbol{E}(\boldsymbol{I}-\boldsymbol{F})^{-1},\end{aligned}$$

where we use (46). Part (c) for $F_{i,j}(k)/F_{i,j}$ comes from the fact that $F_{i,j}(k)$ is nonnegative and by definition

$$\sum_{k=0}^{\infty} \frac{F_{i,j}(k)}{F_{i,j}} = 1,$$

if $F_{i,j} > 0$. Part (c) for $E_{i,j}(k)/E_{i,j}$ can be shown in the same way. Finally, we prove part (d). By definition, we have

$$\begin{aligned}\boldsymbol{F} &= \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-1} \left(\boldsymbol{I} - \sum_{n=1}^{\infty} \boldsymbol{A}(n)\boldsymbol{G}^{n-1}\right)^{-1} \\ &= \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-1} \left(\boldsymbol{I} - \sum_{n=1}^{\infty} \boldsymbol{A}(n)\boldsymbol{G}^{n-1}\right)^{-1} \\ &\qquad - \sum_{m=1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-1} \left(\boldsymbol{I} - \sum_{n=1}^{\infty} \boldsymbol{A}(n)\boldsymbol{G}^{n-1}\right)^{-1} \\ &= \sum_{m=1}^{\infty} \boldsymbol{A}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k \left(\boldsymbol{I} - \boldsymbol{U}(0)\right)^{-1} + \boldsymbol{I} - \left(\boldsymbol{I} - \boldsymbol{U}(0)\right)^{-1}.\end{aligned} \tag{47}$$

Note here that

$$\begin{aligned}\sum_{m=1}^{\infty} \boldsymbol{A}(m) \sum_{k=0}^{m-1} \boldsymbol{G}^k(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g}) &= \sum_{m=1}^{\infty} \boldsymbol{A}(m)(\boldsymbol{I}-\boldsymbol{G}+m\boldsymbol{e}\boldsymbol{g}) \\ &= \boldsymbol{A}-\boldsymbol{G}+\boldsymbol{\beta}_A\boldsymbol{g},\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \boldsymbol{A}(n) \sum_{k=0}^{n-1} \boldsymbol{G}^n &= (\boldsymbol{A}-\boldsymbol{G}+\boldsymbol{\beta}_A\boldsymbol{g})(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1} \\ &= \{(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g}) - (\boldsymbol{I}-\boldsymbol{A}+(\boldsymbol{e}-\boldsymbol{\beta}_A)\boldsymbol{g})\}(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1} \\ &= \boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{A}+(\boldsymbol{e}-\boldsymbol{\beta}_A)\boldsymbol{g})(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1},\end{aligned} \tag{48}$$

because $\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g}$ is nonsingular [13]. Substituting (48) into (47), we have

$$\boldsymbol{F} = \boldsymbol{I} - (\boldsymbol{I}-\boldsymbol{A}+(\boldsymbol{e}-\boldsymbol{\beta}_A)\boldsymbol{g})(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1}\left(\boldsymbol{I}-\boldsymbol{U}(0)\right)^{-1}. \tag{49}$$

Pre-multiplying both sides of (49) by $\boldsymbol{\pi}$, and noting $\boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{\pi}$ and $\boldsymbol{g}(\boldsymbol{I}-\boldsymbol{G}+\boldsymbol{e}\boldsymbol{g})^{-1} = \boldsymbol{g}$, we have part (d). □

B.6 Proof of Lemma 15

By definition, we have

$$\begin{aligned}
\overline{\boldsymbol{F}}(k) &= \sum_{l=k}^{\infty} \boldsymbol{U}(l)(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \boldsymbol{U}(k+l)(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \sum_{m=k+l+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-l-1}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \boldsymbol{A}(k+l+1+m)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1} \\
&= \sum_{m=0}^{\infty} \overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}.
\end{aligned}$$

Similarly we have

$$\overline{\boldsymbol{E}}(k) = \sum_{m=0}^{\infty} \overline{\boldsymbol{B}}(m+k)\boldsymbol{G}^{m}(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}$$

In what follows, we shall show

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} = \boldsymbol{C}^{A}\boldsymbol{e}\boldsymbol{g},$$

which is equivalent to the first equation of the lemma. We note that for any integer $n > 0$

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} &= \sum_{m=0}^{n-1} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} \\
&\le \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}_e(k)} \sum_{m=0}^{n-1} \boldsymbol{G}^{m} + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} \\
&= \frac{\overline{Y}(k)}{\overline{Y}_e(k)} \cdot \frac{\overline{\boldsymbol{A}}(k)}{\overline{Y}(k)} \sum_{m=0}^{n-1} \boldsymbol{G}^{m} + \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)}.
\end{aligned}$$

Therefore, we obtain for any $n > 0$

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} &\le 0 \cdot \frac{\boldsymbol{C}^{A}}{y} \sum_{m=0}^{n-1} \boldsymbol{G}^{m} + \limsup_{k\to\infty} \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)} \\
&= \limsup_{k\to\infty} \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^{m}}{\overline{Y}_e(k)}, \qquad (50)
\end{aligned}$$

where we use Lemma 10.

Note here that $\boldsymbol{G}$ is a stochastic matrix with an invariant probability vector $\boldsymbol{g}$, so that

$$\lim_{m\to\infty} \boldsymbol{G}^{m} = \boldsymbol{e}\boldsymbol{g}.$$

Thus for any $\epsilon > 0$, we can choose an integer n^* ($n^* \ge 1$) so that

$$\boldsymbol{G}^{m} \le (1+\epsilon)\boldsymbol{e}\boldsymbol{g}, \qquad \text{for all } m \ge n^*. \qquad (51)$$

It then follows from (51) that

$$\sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \le (1+\epsilon) \sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{eg}}{\overline{Y_e}(k)} \le (1+\epsilon) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)\boldsymbol{eg}}{\overline{Y_e}(k)}.$$

To proceed further, we need the following lemma.

Lemma 17
Under Assumption 4, we have

$$\lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} = \boldsymbol{C}^A, \qquad \lim_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{B}}(k+m+1)}{\overline{Y_e}(k)} = \boldsymbol{C}^B.$$

Proof. Assumption 4 (b) implies that for any $\epsilon > 0$, there exists $k^* > 0$ enough large so that

$$\frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}(k+m+1)} \le (1+\epsilon)\frac{\boldsymbol{C}^A}{y} \qquad \text{for all } k \ge k^*, m \ge 0,$$

from which it follows that for all $k \ge k^*$,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y}(k+m+1)} \cdot \frac{\overline{Y}(k+m+1)}{\overline{Y_e}(k)} \\ &\le (1+\epsilon)\frac{\boldsymbol{C}^A}{y} \sum_{m=0}^{\infty} \frac{\overline{Y}(k+m+1)}{\overline{Y_e}(k)} \\ &= (1+\epsilon)\boldsymbol{C}^A.\end{aligned}$$

Since ϵ is arbitrary, we have

$$\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} \le \boldsymbol{C}^A. \tag{52}$$

On the other hand, the lower bound is obtained as follows. We first note that

$$\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} \ge \sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)}, \qquad \text{for all } j = 0, 1, \ldots.$$

Thus it suffices to show that

$$\liminf_{k\to\infty} \sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} \ge \boldsymbol{C}^A, \qquad \text{for some } j \ge 0. \tag{53}$$

Note that

$$\begin{aligned}\sum_{m=j}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1+j)}{\overline{Y_e}(k)} \\ &= \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1+j)}{\overline{Y}(k+m+1+j)} \cdot \frac{\overline{Y}(k+m+1+j)}{\overline{Y_e}(k)}.\end{aligned} \tag{54}$$

It follows from Assumption 4 (b) that for any $\epsilon > 0$, there exists $j^* > 0$ such that

$$\frac{\overline{\boldsymbol{A}}(k+j^*)}{\overline{Y}(k+j^*)} \geq (1-\epsilon)\frac{\boldsymbol{C}^A}{y}, \qquad \text{for all } k \geq 0. \tag{55}$$

Thus using (54) and (55), we obtain that

$$\begin{aligned}\sum_{m=j^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m+1)}{\overline{Y_e}(k)} &\geq (1-\epsilon)\boldsymbol{C}^A \sum_{m=0}^{\infty} \frac{\overline{Y}(k+m+1+j^*)}{y\overline{Y_e}(k)} \\ &= (1-\epsilon)\boldsymbol{C}^A \frac{\overline{Y_e}(k+j^*)}{\overline{Y_e}(k)}. \end{aligned} \tag{56}$$

Since ϵ is arbitrary, (53) immediately follows from (56) and the heavy-tailed property (31), which completes the proof for the first equation. The second one can be proved exactly in the same way, so that it is omitted. □

Thus using (50) and Lemma 17, we have

$$\limsup_{k\to\infty} \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \leq \limsup_{k\to\infty} \sum_{m=n^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \leq \boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}.$$

On the other hand, the lower bound is obtained as follows. We first note that

$$\sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \geq \sum_{m=n}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)}, \qquad \text{for all } n = 0, 1, \ldots,$$

and for any $\epsilon_1 > 0$, there exists an integer m^* large enough so that

$$\boldsymbol{G}^m \geq (1-\epsilon_1)\boldsymbol{e}\boldsymbol{g}, \qquad \text{for all } \ m \geq m^*+1.$$

Therefore it suffices to show that

$$\liminf_{k\to\infty} \sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \geq \boldsymbol{C}^A\boldsymbol{e}\boldsymbol{g}, \tag{57}$$

Note here that

$$\begin{aligned}\sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} &\geq (1-\epsilon_1) \sum_{m=m^*+1}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{e}\boldsymbol{g}}{\overline{Y_e}(k)} \\ &= (1-\epsilon_1) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{e}\boldsymbol{g}}{\overline{Y_e}(k)} \\ &= (1-\epsilon_1) \sum_{m=0}^{\infty} \frac{\overline{Y_e}(k+m^*)}{\overline{Y_e}(k)} \cdot \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{e}\boldsymbol{g}}{\overline{Y_e}(k+m^*)}. \end{aligned}$$

Further for any $\epsilon_2 > 0$, there exists k^* large enough so that

$$\frac{\overline{Y_e}(k+m^*)}{\overline{Y_e}(k)} \geq 1-\epsilon_2, \qquad \text{for all } k \geq k^*,$$

because of the heavy-tailed property (31). Thus we have for all $k > k^*$

$$\sum_{m=m^*}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m)\boldsymbol{G}^m}{\overline{Y_e}(k)} \geq (1-\epsilon_1)(1-\epsilon_2) \sum_{m=0}^{\infty} \frac{\overline{\boldsymbol{A}}(k+m^*+m+1)\boldsymbol{e}\boldsymbol{g}}{\overline{Y_e}(k+m^*)}. \tag{58}$$

Because ϵ_1 and ϵ_2 are arbitrary, (57) follows from Lemma 17 and (58), which completes the proof for the first equation. The proof of the second equation is exactly the same as the above, so that it is omitted. □

C Key renewal theorem for discrete-time Markov renewal processes

The materials in this appendix is completely in parallel to [4]. Thus for the detailed discussions, see [4]. Consider a renewal equation

$$q(k) = w(k) + \sum_{m=0}^{k} f(m)q(k-m),$$

where $f(m)$ denotes a mass function for a nonnegative random variable. Let $f^{(0)}(0) = 1$, $f^{(0)}(k) = 0$ for $k = 1, 2, \ldots$, and

$$f^{(n+1)}(k) = \sum_{m=0}^{k} f(m)f^{(n)}(k-m), \qquad n = 0, 1, \ldots.$$

Then from (9.2.3) in [4]

$$q(k) = \sum_{m=0}^{k} z(m)g(k-m),$$

where

$$z(k) = \sum_{n=0}^{\infty} f^{(n)}(k).$$

Assume that $w(k)$ is directly Riemann integrable and $z(k)$ is aperiodic. Then from (9.2.10) in [4]

$$\lim_{k\to\infty} q(k) = \lim_{k\to\infty} \sum_{m=0}^{k} z(m)w(k-m) = \frac{1}{\sum_{m=1}^{\infty} mf(m)} \sum_{m=0}^{\infty} w(m). \tag{59}$$

We now consider a Markov renewal equation

$$\boldsymbol{Q}(k) = \boldsymbol{W}(k) + \sum_{m=0}^{k} \boldsymbol{F}(m)\boldsymbol{Q}(k-m),$$

where $\boldsymbol{Q}(k)$, $\boldsymbol{W}(k)$ and $\boldsymbol{F}(m)$ are nonnegative $M \times M$ matrices. Further we assume that $\boldsymbol{F} = \sum_{k=0}^{\infty} \boldsymbol{F}(k)$ is an irreducible stochastic matrix. Let $\boldsymbol{F}^{(0)}(0) = \boldsymbol{I}$ and $\boldsymbol{F}^{(0)}(k) = \boldsymbol{O}$ for all $k = 1, 2, \ldots$, and

$$\boldsymbol{F}^{(n+1)}(k) = \sum_{m=0}^{k} \boldsymbol{F}(m)\boldsymbol{F}^{(n)}(k-m), \qquad n = 1, 2, \ldots.$$

Then from (10.3.9) in [4]

$$\boldsymbol{Q}(k) = \sum_{m=0}^{k} \boldsymbol{Z}(m)\boldsymbol{W}(m-k)$$

where

$$\boldsymbol{Z}(k) = \sum_{n=0}^{\infty} \boldsymbol{F}^{(n)}(k)$$

Consider the semi-Markov process whose kernel is given by $\boldsymbol{F}(k)$ $(k = 0, 1, \ldots)$. Let $\boldsymbol{H}(k)$ denote an $M \times M$ matrix whose (i, j)th element $H(i, j, k)$ represents: for $i \neq j$, the probability

that the first passage time from state i to state j is equal to k, and for $i = j$, the probability that the recurrence time of state i is equal to k. We define $H^{<n>}(j,j,k)$ as

$$H^{<0>}(j,j,k) = \begin{cases} 1, & k = 0, \\ 0, & k = 1,2,\ldots, \end{cases} \qquad H^{<n+1>}(j,j,k) = \sum_{m=0}^{k} H(j,j,m) H^{<n>}(j,j,k-m).$$

Then, from (3.2.8) and (3.2.9) in [4], we have

$$[\boldsymbol{Z}(k)]_{i,j} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} H^{<n>}(j,j,k), & i = j, \\ \displaystyle\sum_{m=0}^{k} H(i,j,m)[\boldsymbol{Z}(k-m)]_{j,j}, & i \neq j. \end{cases}$$

Assuming $\boldsymbol{F}$ is irreducible, let $\boldsymbol{\nu} = (\nu(1),\ldots,\nu(M))$ be

$$\boldsymbol{\nu}\boldsymbol{F} = \boldsymbol{\nu}, \qquad \boldsymbol{\nu}\boldsymbol{e} = 1,$$

Also, let $\boldsymbol{\eta}$ denote a $1 \times M$ vector whose jth element $\eta(j)$ represents the inverse of the mean recurrence time of state j in $\boldsymbol{F}$. Note that

$$\boldsymbol{\eta} = \frac{\boldsymbol{\nu}}{\boldsymbol{\nu} \displaystyle\sum_{m=1}^{\infty} m\boldsymbol{F}(m)\boldsymbol{e}}.$$

Proposition 18
Assume that $\boldsymbol{W}(k)$ is directly Riemann integrable, $\boldsymbol{F}$ is an irreducible stochastic matrix, and the recurrence time $[\boldsymbol{Z}(k)]_{j,j}$ of every state j is aperiodic. Then

$$\lim_{k\to\infty} \boldsymbol{Q}(k) = \frac{\boldsymbol{e}\boldsymbol{\nu} \displaystyle\sum_{m=0}^{\infty} \boldsymbol{W}(m)}{\boldsymbol{\nu} \displaystyle\sum_{m=1}^{\infty} m\boldsymbol{F}(m)\boldsymbol{e}}.$$

Proof. Because

$$\lim_{k\to\infty} \boldsymbol{Q}(k) = \lim_{k\to\infty} \sum_{m=0}^{k} \boldsymbol{Z}(m)\boldsymbol{W}(k-m),$$

we have

$$\lim_{k\to\infty} [\boldsymbol{Q}(k)]_{i,j} = \sum_{l=1}^{M} \lim_{k\to\infty} \sum_{m=0}^{k} [\boldsymbol{Z}(m)]_{i,l} [\boldsymbol{W}(k-m)]_{l,j}.$$

For $i \neq l$, we have

$$\begin{aligned} \sum_{m=0}^{k} [\boldsymbol{Z}(m)]_{i,l} [\boldsymbol{W}(k-m)]_{l,j} &= \sum_{m=0}^{k} \sum_{n=0}^{m} H(i,l,m-n)[\boldsymbol{Z}(n)]_{l,l} [\boldsymbol{W}(k-m)]_{l,j} \\ &= \sum_{n=0}^{k} [\boldsymbol{Z}(n)]_{l,l} \sum_{m=n}^{k} H(i,l,m-n)[\boldsymbol{W}(k-m)]_{l,j} \\ &= \sum_{n=0}^{k} [\boldsymbol{Z}(n)]_{l,l} \sum_{m=0}^{k-n} H(i,l,k-n-m)[\boldsymbol{W}(m)]_{l,j}. \end{aligned} \tag{60}$$

Because $\boldsymbol{W}(m)$ is directly Riemann integrable, by (9.2.16) (d) in [4], the convolution of $H(i,l,k)$ and $[\boldsymbol{W}(m)]_{l,j}$ is also directly Riemann integrable. Thus applying (59) to (60), we obtain

$$\begin{aligned}\lim_{k\to\infty}\sum_{m=0}^{k}[\boldsymbol{Z}(m)]_{i,l}[\boldsymbol{W}(k-m)]_{l,j} &= \eta(l)\sum_{m=0}^{\infty}\sum_{n=0}^{m}H(i,l,m-n)[\boldsymbol{W}(n)]_{l,j}\\ &= \eta(l)\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}H(i,l,m-n)[\boldsymbol{W}(n)]_{l,j}\\ &= \eta(l)\sum_{n=0}^{\infty}[\boldsymbol{W}(n)]_{l,j},\end{aligned}$$

where we use the aperiodicity of $[\boldsymbol{Z}(m)]_{l,l}$ for the existence of the limit and $\sum_{m=0}^{\infty}H(i,l,m)=1$ which comes from the irreducibility of $\boldsymbol{F}$. Similarly, for $i=l$, we have

$$\lim_{k\to\infty}\sum_{m=0}^{k}[\boldsymbol{Z}(m)]_{l,l}[\boldsymbol{W}(k-m)]_{l,j} = \eta(l)\sum_{m=0}^{\infty}[\boldsymbol{W}(k-m)]_{l,j}.$$

The rest of the proof is routine, so that we omit it. □

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